Beginning and Intermediate Algebra
MTH 095

Available for free download at: http://wallace.ccfaculty.org/book/book.html

Edited 2017 by Tillamook Bay Community College

by Tyler Wallace

ISBN #978-1-4583-7768-5

Special thanks to: My beautiful wife, Nicole Wallace

who spent countless hours typing problems and

my two wonderful kids for their patience and

support during this project

Another thanks goes to the faculty reviewers who reviewed this text: Donna Brown, Michelle Sherwood, Ron Wallace, and Barbara Whitney

One last thanks to the student reviewers of the text: Eloisa Butler, Norma Cabanas, Irene Chavez, Anna Dahlke, Kelly Diguilio, Camden Eckhart, Brad Evers, Lisa Garza, Nickie Hampshire, Melissa Hanson, Adriana Hernandez, Tiffany Isaacson, Maria Martinez, Brandon Platt, Tim Ries, Lorissa Smith, Nadine Svopa, Cayleen Trautman, and Erin White

Table of Contents

Beginning and Intermediate Algebra Student Solutions Manual

Complete worked solutions to odd problems

Chapter 4 : Systems of Equations

4.1

Systems of Equations - Graphing

Objective: Solve systems of equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works in both equations. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in both equations

Example 165.

Show (2,1) is the solution to the system $\begin{array}{l} 3x - y = 5 \\ x + y = 3 \end{array}$

$$\begin{array}{rl}
(2,1) & \text{Identify } x \text{ and } y \text{ from the orderd pair} \\
x = 2, y = 1 & \text{Plug these values into each equation} \\
 & \\
3(2) - (1) = 5 & \text{First equation} \\
6 - 1 = 5 & \text{Evaluate} \\
5 = 5 & \text{True} \\
 & \\
(2) + (1) = 3 & \text{Second equation, evaluate} \\
3 = 3 & \text{True}
\end{array}$$

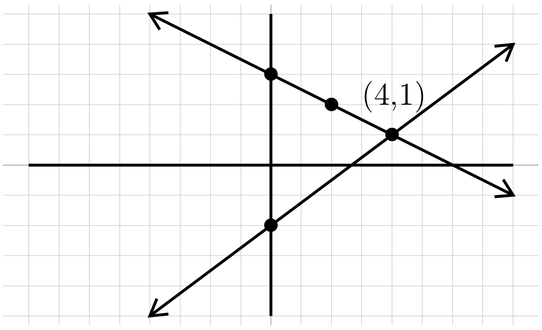
As we found a true statement for both equations we know (2,1) is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this lesson we will be working to find this point given the equations. It seems to follow that if we use points to describe the solution, we can use graphs to find the solutions.

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is a solution for both lines, this would be where the lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 166.

$$\begin{aligned} y &= -\frac{1}{2}x + 3 \\ y &= \frac{3}{4}x - 2 \end{aligned} \qquad \text{To graph we identify slopes and } y - \text{intercepts}$$

$$\begin{aligned} \text{First: } m &= -\frac{1}{2}, b = 3 \\ \text{Second: } m &= \frac{3}{4}, b = -2 \end{aligned} \qquad \text{Now we can graph both lines on the same plane.}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, (4,1)

(4,1) Our Solution

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can idenfity the slope and y-intercept.

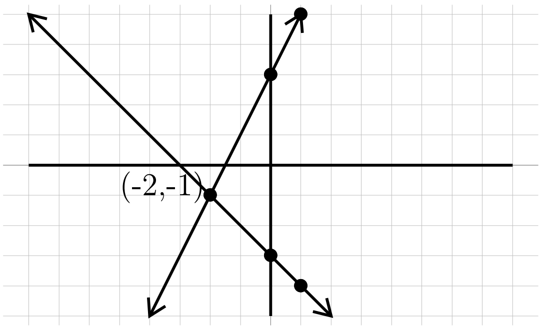
Example 167.

$$\begin{aligned} 6x - 3y &= -9 \\ 2x + 2y &= -6 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{ccc|ccc|l}
6x & -3y = & -9 & 2x & +2y = & -6 & \\
-6x & & -6x & -2x & & -2x & \text{Subtract } x \text{ terms} \\
\hline
 & -3y = & -6x - 9 & & 2y = & -2x - 6 & \text{Put } x \text{ terms first} \\
 & \overline{-3} & \overline{-3} \ \overline{-3} & & \overline{2} & \overline{2} \ \overline{2} & \text{Divide by coefficient of } y \\
 & y = & 2x + 3 & & y = & -x - 3 & \text{Identify slope and } y - \text{intercepts}
\end{array}$$

$$\text{First: } m = \frac{2}{1}, b = 3$$
$$\text{Second: } m = -\frac{1}{1}, b = -3$$

Now we can graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Remember a negative slope is downhill!

Find the intersection point, $(-2, -1)$

$(-2, -1)$ Our Solution

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two example.

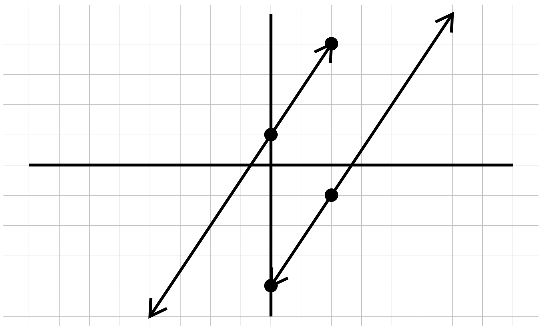
Example 168.

$$y = \frac{3}{2}x - 4$$
$$y = \frac{3}{2}x + 1$$

Identify slope and $y-$intercept of each equation

$$\text{First: } m = \frac{3}{2}, b = -4$$
$$\text{Second: } m = \frac{3}{2}, b = 1$$

Now we can graph both equations on the same plane



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

The two lines do not intersect! They are parallel! If the lines do not intersect we know that there is no point that works in both equations, there is no solution

$\varnothing$ No Solution

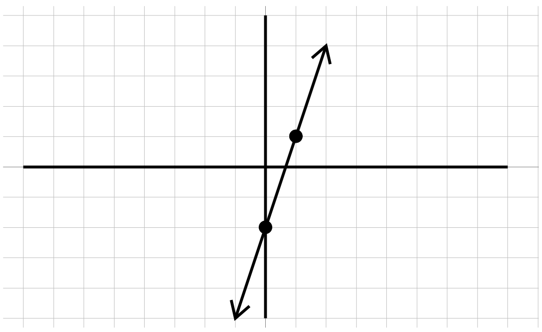
We also could have noticed that both lines had the same slope. Remembering that parallel lines have the same slope we would have known there was no solution even without having to graph the lines.

Example 169.

$$\begin{aligned} 2x - 6y &= 12 \\ 3x - 9y &= 18 \end{aligned} \quad \text{Solve each equation for } y$$

$$\begin{array}{rcl|rcl|l}
2x - 6y &=& 12 & 3x - 9y &=& 18 & \\
\underline{-2x} & & \underline{-2x} & \underline{-3x} & & \underline{-3x} & \text{Subtract } x \text{ terms} \\
-6y &=& -2x + 12 & -9y &=& -3x + 18 & \text{Put } x \text{ terms first} \\
\overline{-6} & & \overline{-6} \quad \overline{-6} & \overline{-9} & & \overline{-9} \quad \overline{-9} & \text{Divide by coefficient of } y \\
y &=& \frac{1}{3}x - 2 & y &=& \frac{1}{3}x - 2 & \text{Identify the slopes and } y - \text{intercepts}
\end{array}$$

$$\begin{aligned} \text{First: } m &= \frac{1}{3}, b = -2 \\ \text{Second: } m &= \frac{1}{3}, b = -2 \end{aligned} \quad \text{Now we can graph both equations together}$$



To graph each equation, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Both equations are the same line! As one line is directly on top of the other line, we can say that the lines "intersect" at all the points! Here we say we have infinite solutions

Once we had both equations in slope-intercept form we could have noticed that the equations were the same. At this point we could have stated that there are infinite solutions without having to go through the work of graphing the equations.

World View Note: The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables and around 300 AD, developed methods for solving systems with any number of unknowns!

4.1 Practice - Graphing

Solve each equation by graphing.

1) $y = -x + 1$
$y = -5x - 3$

2) $y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

3) $y = -3$
$y = -x - 4$

4) $y = -x - 2$
$y = \frac{2}{3}x + 3$

5) $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

6) $y = 2x + 2$
$y = -x - 4$

7) $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

8) $y = 2x - 4$
$y = \frac{1}{2}x + 2$

9) $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

10) $y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

11) $x + 3y = -9$
$5x + 3y = 3$

12) $x + 4y = -12$
$2x + y = 4$

13) $x - y = 4$
$2x + y = -1$

14) $6x + y = -3$
$x + y = 2$

15) $2x + 3y = -6$
$2x + y = 2$

16) $3x + 2y = 2$
$3x + 2y = -6$

17) $2x + y = 2$
$x - y = 4$

18) $x + 2y = 6$
$5x - 4y = 16$

19) $2x + y = -2$
$x + 3y = 9$

20) $x - y = 3$
$5x + 2y = 8$

21) $0 = -6x - 9y + 36$
$12 = 6x - 3y$

22) $-2y + x = 4$
$2 = -x + \frac{1}{2}y$

23) $2x - y = -1$
$0 = -2x - y - 3$

24) $-2y = -4 - x$
$-2y = -5x + 4$

25) $3 + y = -x$
$-4 - 6x = -y$

26) $16 = -x - 4y$
$-2x = -4 - 4y$

27) $-y + 7x = 4$
$-y - 3 + 7x = 0$

28) $-4 + y = x$
$x + 2 = -y$

29) $-12 + x = 4y$
$12 - 5x = 4y$

30) $-5x + 1 = -y$
$-y + x = -3$

4.2

Systems of Equations - Substitution

Objective: Solve systems of equations using substitution.

When solving a system by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or if the answer is a decimal the that graph will not help us find, 3.2134 for example. For these reasons we will rarely use graphing to solve our systems. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several example, then end with a five-step process to solve problems using this method.

Example 170.

$x=5$ $y=2x-3$	We already know $x=5$, substitute this into the other equation
$y=2(\mathbf{5})-3$	Evaluate, multiply first
$y=10-3$	Subtract
$y=7$	We now also have y
$(5,7)$	Our Solution

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parenthesis. The reason for this is shown in the next example.

Example 171.

$2x-3y=7$ $y=3x-7$	We know $y=3x-7$, substitute this into the other equation
$2x-3(\boldsymbol{3x-7})=7$	Solve this equation, distributing -3 first

$$
\begin{array}{rl}
2x - 9x + 21 = 7 & \text{Combine like terms } 2x - 9x \\
-7x + 21 = 7 & \text{Subtract } 21 \\
\underline{\mathbf{-21 \quad -21}} & \\
-7x = -14 & \text{Divide by } -7 \\
\overline{\mathbf{-7}} \quad \overline{\mathbf{-7}} & \\
x = 2 & \text{We now have our } x\text{, plug into the } y = \text{equation to find } y \\
y = 3(\mathbf{2}) - 7 & \text{Evaluate, multiply first} \\
y = 6 - 7 & \text{Subtract} \\
y = -1 & \text{We now also have } y \\
(2, -1) & \text{Our Solution}
\end{array}
$$

By using the entire expression $3x - 7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the lone variable (a variable without a coefficient) is not always alone on one side of the equation. If this happens we can isolate it by solving for the lone variable.

Example 172.

$$
\begin{array}{rl}
3x + 2y = 1 & \\
\boldsymbol{x} - 5y = 6 & \text{Lone variable is } x\text{, isolate by adding } 5y \text{ to both sides.} \\
\underline{\boldsymbol{+5y + 5y}} & \\
x = 6 + 5y & \text{Substitute this into the untouched equation} \\
3(\boldsymbol{6 + 5y}) + 2y = 1 & \text{Solve this equation, distributing 3 first} \\
18 + 15y + 2y = 1 & \text{Combine like terms } 15y + 2y \\
18 + 17y = 1 & \text{Subtract 18 from both sides} \\
\underline{\mathbf{-18 \qquad -18}} & \\
17y = -17 & \text{Divide both sides by 17} \\
\overline{\mathbf{17}} \quad \overline{\mathbf{17}} & \\
y = -1 & \text{We have our } y\text{, plug this into the } x = \text{equation to find } x \\
x = 6 + 5(\mathbf{-1}) & \text{Evaluate, multiply first} \\
x = 6 - 5 & \text{Subtract} \\
x = 1 & \text{We now also have } x \\
(1, -1) & \text{Our Solution}
\end{array}
$$

The process in the previous example is how we will solve problems using substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x-2y=2$ $2x+y=-5$
1. Find the lone variable	Second Equation, y $2x+\boldsymbol{y}=-5$
2. Solve for the lone variable	$\underline{-2x \qquad -2x}$ $\boldsymbol{y=-5-2x}$
3. Substitute into the untouched equation	$4x-2(\boldsymbol{-5-2x})=2$
4. Solve	$4x+10+4x=2$ $8x+10=2$ $\underline{-10-10}$ $\frac{8x}{8}=\frac{-8}{8}$ $\boldsymbol{x=-1}$
5. Plug into lone variable equation and evaluate	$y=-5-2(\boldsymbol{-1})$ $y=-5+2$ $\boldsymbol{y=-3}$
Solution	$(-1,-3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 173.

$$
\begin{array}{rl}
\boldsymbol{x}+y=5 & \text{Find the lone variable: } x \text{ or } y \text{ in first, or } x \text{ in second.} \\
x-y=-1 & \text{We will chose } x \text{ in the first} \\
\boldsymbol{x}+y=5 & \text{Solve for the lone variable, subtract } y \text{ from both sides} \\
\underline{\boldsymbol{-y-y}} & \\
x=5-y & \text{Plug into the untouched equation, the second equation} \\
(\boldsymbol{5-y})-y=-1 & \text{Solve, parenthesis are not needed here, combine like terms} \\
5-2y=-1 & \text{Subtract 5 from both sides} \\
\underline{-5 \qquad\quad -5} & \\
\frac{-2y}{-2}=\frac{-6}{-2} & \text{Divide both sides by } -2 \\
y=3 & \text{We have our } y! \\
x=5-(3) & \text{Plug into lone variable equation, evaluate} \\
x=2 & \text{Now we have our } x
\end{array}
$$

$(2,3)\quad$ Our Solution

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 174.

$$\begin{array}{rl}
\begin{array}{r}\boldsymbol{y}+4=3x\\ 2y-6x=-8\end{array} & \text{Find the lone variable, } y \text{ in the first equation}\\
y+4=3x & \text{Solve for the lone variable, subtract 4 from both sides}\\
\underline{-4\quad -4} & \\
y=3x-4 & \text{Plug into untouched equation}\\
2(\mathbf{3x-4})-6x=-8 & \text{Solve, distribute through parenthesis}\\
6x-8-6x=-8 & \text{Combine like terms } 6x-6x\\
-8=-8 & \text{Variables are gone! } A \text{ true statement.}\\
\text{Infinite solutions} & \text{Our Solution}
\end{array}$$

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 175.

$$\begin{array}{rl}
\begin{array}{r}6x-3y=-9\\ -2x+\boldsymbol{y}=5\end{array} & \text{Find the lone variable, } y \text{ in the second equation}\\
-2x+y=5 & \text{Solve for the lone variable, add } 2x \text{ to both sides}\\
\underline{+2x\quad +2x} & \\
y=5+2x & \text{Plug into untouched equation}\\
6x-3(\mathbf{5+2x})=-9 & \text{Solve, distribute through parenthesis}\\
6x-15-6x=-9 & \text{Combine like terms } 6x-6x\\
-15\neq-9 & \text{Variables are gone! } A \text{ false statement.}\\
\text{No Solution } \varnothing & \text{Our Solution}
\end{array}$$

Because we had a false statement, and no variables, we know that nothing will work in both equations.

World View Note: French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables x, y, and z.

One more question needs to be considered, what if there is no lone variable? If there is no lone variable substitution can still work to solve, we will just have to select one variable to solve for and use fractions as we solve.

Example 176.

$$
\begin{array}{rl}
5\boldsymbol{x} - 6y = -14 & \text{No lone variable,} \\
-2x + 4y = 12 & \text{we will solve for } x \text{ in the first equation} \\
5x - 6y = -14 & \text{Solve for our variable, add } 6y \text{ to both sides} \\
\underline{\boldsymbol{+6y \ \ +6y}} & \\
5x = -14 + 6y & \text{Divide each term by 5} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \\
x = \dfrac{-14}{5} + \dfrac{6y}{5} & \text{Plug into untouched equation} \\
-2\left(\dfrac{-14}{5} + \dfrac{6y}{5}\right) + 4y = 12 & \text{Solve, distribute through parenthesis} \\
\dfrac{28}{5} - \dfrac{12y}{5} + 4y = 12 & \text{Clear fractions by multiplying by 5} \\
\dfrac{28(\mathbf{5})}{5} - \dfrac{12y(\mathbf{5})}{5} + 4y(\mathbf{5}) = 12(\mathbf{5}) & \text{Reduce fractions and multiply} \\
28 - 12y + 20y = 60 & \text{Combine like terms } -12y + 20y \\
28 + 8y = 60 & \text{Subtract 28 from both sides} \\
\underline{\mathbf{-28 \qquad -28}} & \\
8y = 32 & \text{Divide both sides by 8} \\
\overline{\mathbf{8}} \quad \overline{\mathbf{8}} & \\
y = 4 & \text{We have our } y \\
x = \dfrac{-14}{5} + \dfrac{6(4)}{5} & \text{Plug into lone variable equation, multiply} \\
x = \dfrac{-14}{5} + \dfrac{24}{5} & \text{Add fractions} \\
x = \dfrac{10}{5} & \text{Reduce fraction} \\
x = 2 & \text{Now we have our } x \\
(2, 4) & \text{Our Solution}
\end{array}
$$

Using the fractions does make the problem a bit more tricky. This is why we have another method for solving systems of equations that will be discussed in another lesson.

4.2 Practice - Substitution

Solve each system by substitution.

1) $y = -3x$
$y = 6x - 9$

2) $y = x + 5$
$y = -2x - 4$

3) $y = -2x - 9$
$y = 2x - 1$

4) $y = -6x + 3$
$y = 6x + 3$

5) $y = 6x + 4$
$y = -3x - 5$

6) $y = 3x + 13$
$y = -2x - 22$

7) $y = 3x + 2$
$y = -3x + 8$

8) $y = -2x - 9$
$y = -5x - 21$

9) $y = 2x - 3$
$y = -2x + 9$

10) $y = 7x - 24$
$y = -3x + 16$

11) $y = 6x - 6$
$-3x - 3y = -24$

12) $-x + 3y = 12$
$y = 6x + 21$

13) $y = -6$
$3x - 6y = 30$

14) $6x - 4y = -8$
$y = -6x + 2$

15) $y = -5$
$3x + 4y = -17$

16) $7x + 2y = -7$
$y = 5x + 5$

17) $-2x + 2y = 18$
$y = 7x + 15$

18) $y = x + 4$
$3x - 4y = -19$

19) $y = -8x + 19$
$-x + 6y = 16$

20) $y = -2x + 8$
$-7x - 6y = -8$

21) $7x - 2y = -7$
$y = 7$

22) $x - 2y = -13$
$4x + 2y = 18$

23) $x - 5y = 7$
$2x + 7y = -20$

24) $3x - 4y = 15$
$7x + y = 4$

25) $-2x - y = -5$
$x - 8y = -23$

26) $6x + 4y = 16$
$-2x + y = -3$

27) $-6x + y = 20$
$-3x - 3y = -18$

28) $7x + 5y = -13$
$x - 4y = -16$

29) $3x + y = 9$
$2x + 8y = -16$

30) $-5x - 5y = -20$
$-2x + y = 7$

31) $2x + y = 2$
$3x + 7y = 14$

32) $2x + y = -7$
$5x + 3y = -21$

33) $x + 5y = 15$
$-3x + 2y = 6$

34) $2x + 3y = -10$
$7x + y = 3$

35) $-2x + 4y = -16$
$y = -2$

36) $-2x + 2y = -22$
$-5x - 7y = -19$

37) $-6x + 6y = -12$
$8x - 3y = 16$

38) $-8x + 2y = -6$
$-2x + 3y = 11$

39) $2x + 3y = 16$
$-7x - y = 20$

40) $-x - 4y = -14$
$-6x + 8y = 12$

4.3

Systems of Equations - Addition/Elimination

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substituion. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 177.

$$\begin{array}{rl} 3x-4y=8 & \\ \underline{5x+4y=-24} & \text{Notice opposites in front of } y's. \text{ Add columns.} \\ 8x \qquad = -16 & \text{Solve for } x\text{, divide by } 8 \\ \overline{8} \qquad\quad \overline{8} & \\ x=-2 & \text{We have our } x! \\ 5(-2)+4y=-24 & \text{Plug into either original equation, simplify} \\ -10+4y=-24 & \text{Add 10 to both sides} \\ \underline{+10 \qquad\quad +10} & \\ 4y=-14 & \text{Divide by 4} \\ \overline{4} \quad \overline{4} & \\ y=\dfrac{-7}{2} & \text{Now we have our } y! \\ \left(-2,\dfrac{-7}{2}\right) & \text{Our Solution} \end{array}$$

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 178.

$$\begin{array}{rl} -6x+5y=22 & \text{We can get opposites in front of } x\text{, by multiplying the} \\ 2x+3y=2 & \text{second equation by 3, to get } -6x \text{ and } +6x \end{array}$$

$$3(2x+3y)=(2)3 \quad \text{Distribute to get new second equation.}$$

$$\begin{aligned}
6x+9y&=6 && \text{New second equation}\\
\underline{-6x+5y}&\underline{=22} && \text{First equation still the same, add}\\
14y&=28 && \text{Divide both sides by 14}\\
\overline{14}\quad&\ \ \overline{14}\\
y&=2 && \text{We have our } y!\\
2x+3(2)&=2 && \text{Plug into one of the original equations, simplify}\\
2x+6&=2 && \text{Subtract 6 from both sides}\\
\underline{-6}&\underline{-6}\\
2x&=-4 && \text{Divide both sides by 2}\\
\overline{2}\quad&\ \ \overline{2}\\
x&=-2 && \text{We also have our } x!\\
(-2,2)& && \text{Our Solution}
\end{aligned}$$

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$. This illustrates an important point, some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want.

Example 179.

$$\begin{aligned}
3x+6y&=-9 && \text{We can get opposites in front of } x\text{, find LCM of 6 and 9,}\\
2x+9y&=-26 && \text{The LCM is 18. We will multiply to get } 18y \text{ and } -18y
\end{aligned}$$

$$\begin{aligned}
3(3x+6y)&=(-9)3 && \text{Multiply the first equation by 3, both sides!}\\
9x+18y&=-27
\end{aligned}$$

$$\begin{aligned}
-2(2x+9y)&=(-26)(-2) && \text{Multiply the second equation by } -2\text{, both sides!}\\
-4x-18y&=52
\end{aligned}$$

$$\begin{aligned}
9x+18y&=-27 && \text{Add two new equations together}\\
\underline{-4x-18y}&\underline{=52}\\
5x\qquad&=25 && \text{Divide both sides by 5}\\
\overline{5}\qquad&\ \ \overline{5}\\
x&=5 && \text{We have our solution for } x\\
3(5)+6y&=-9 && \text{Plug into either original equation, simplify}\\
15+6y&=-9 && \text{Subtract 15 from both sides}\\
\underline{-15\qquad}&\underline{\ \ -15}
\end{aligned}$$

$$\frac{6y}{6} = \frac{-24}{6} \quad \text{Divide both sides by } 6$$

$$y = -4 \quad \text{Now we have our solution for } y$$

$$(5,-4) \quad \text{Our Solution}$$

It is important for each problem as we get started that all variables and constants are lined up before we start multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x-5y=-13$ $-3y+4=-5x$
1. Line up the variables and constants	Second Equation: $-3y+4=-5x$ $+5x-4 \quad +5x-4$ $5x-3y=-4$
2. Multiply to get opposites (use LCD)	$2x-5y=-13$ $5x-3y=-4$ First Equation: multiply by -5 $-5(2x-5y)=(-13)(-5)$ $-10x+25y=65$ Second Equation: multiply by 2 $2(5x-3y)=(-4)2$ $10x-6y=-8$ $-10x+25y=65$ $10x-6y=-8$
3. Add	$19y=57$
4. Solve	$\frac{19y}{19}=\frac{57}{19}$ $y=3$
5. Plug into either original and solve	$2x-5(3)=-13$ $2x-15=-13$ $+15 \quad +15$ $\frac{2x}{2}=\frac{2}{2}$ $x=1$
Solution	$(1,3)$

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China describes a formula very similar to Gaussian elimination which is very similar to the addition method.

Just as with graphing and substution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables all disappear from our problem, a true statment will indicate infinite solutions and a false statment will indicate no solution.

Example 180.

$$\begin{aligned} 2x-5y&=3 \\ -6x+15y&=-9 \end{aligned} \quad \text{To get opposites in front of } x\text{, multiply first equation by } 3$$

$$\begin{aligned} 3(2x-5y)&=(3)3 && \text{Distribute} \\ 6x-15y&=9 \end{aligned}$$

$$\begin{aligned} 6x-15y&=9 && \text{Add equations together} \\ \underline{-6x+15y}&\underline{=-9} \\ 0&=0 && \text{True statement} \\ \text{Infinite solutions} & && \text{Our Solution} \end{aligned}$$

Example 181.

$$\begin{aligned} 4x-6y&=8 \\ 6x-9y&=15 \end{aligned} \quad \text{LCM for } x's \text{ is } 12.$$

$$\begin{aligned} 3(4x-6y)&=(8)3 && \text{Multiply first equation by } 3 \\ 12x-18y&=24 \end{aligned}$$

$$\begin{aligned} -2(6x-9y)&=(15)(-2) && \text{Multiply second equation by } -2 \\ -12x+18y&=-30 \end{aligned}$$

$$\begin{aligned} 12x-18y&=24 && \text{Add both new equations together} \\ \underline{-12x+18y}&\underline{=-30} \\ 0&=-6 && \text{False statement} \\ \text{No Solution} & && \text{Our Solution} \end{aligned}$$

We have covered three different methods that can be used to solve a system of two equations with two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a lone variable, and addition works great when the other two methods fail. As each method has its own strengths, it is important you are familiar with all three methods.

4.3 Practice - Addition/Elimination

Solve each system by elimination.

1) $4x + 2y = 0$
$-4x - 9y = -28$

2) $-7x + y = -10$
$-9x - y = -22$

3) $-9x + 5y = -22$
$9x - 5y = 13$

4) $-x - 2y = -7$
$x + 2y = 7$

5) $-6x + 9y = 3$
$6x - 9y = -9$

6) $5x - 5y = -15$
$5x - 5y = -15$

7) $4x - 6y = -10$
$4x - 6y = -14$

8) $-3x + 3y = -12$
$-3x + 9y = -24$

9) $-x - 5y = 28$
$-x + 4y = -17$

10) $-10x - 5y = 0$
$-10x - 10y = -30$

11) $2x - y = 5$
$5x + 2y = -28$

12) $-5x + 6y = -17$
$x - 2y = 5$

13) $10x + 6y = 24$
$-6x + y = 4$

14) $x + 3y = -1$
$10x + 6y = -10$

15) $2x + 4y = 24$
$4x - 12y = 8$

16) $-6x + 4y = 12$
$12x + 6y = 18$

17) $-7x + 4y = -4$
$10x - 8y = -8$

18) $-6x + 4y = 4$
$-3x - y = 26$

19) $5x + 10y = 20$
$-6x - 5y = -3$

20) $-9x - 5y = -19$
$3x - 7y = -11$

21) $-7x - 3y = 12$
$-6x - 5y = 20$

22) $-5x + 4y = 4$
$-7x - 10y = -10$

23) $9x - 2y = -18$
$5x - 7y = -10$

24) $3x + 7y = -8$
$4x + 6y = -4$

25) $9x + 6y = -21$
$-10x - 9y = 28$

26) $-4x - 5y = 12$
$-10x + 6y = 30$

27) $-7x + 5y = -8$
$-3x - 3y = 12$

28) $8x + 7y = -24$
$6x + 3y = -18$

29) $-8x - 8y = -8$
$10x + 9y = 1$

30) $-7x + 10y = 13$
$4x + 9y = 22$

31) $9y = 7 - x$
$-18y + 4x = -26$

32) $0 = -9x - 21 + 12y$
$1 + \frac{4}{3}y + \frac{7}{3}x = 0$

33) $0 = 9x + 5y$
$y = \frac{2}{7}x$

34) $-6 - 42y = -12x$
$x - \frac{1}{2} - \frac{7}{2}y = 0$

4.4

Systems of Equations - Three Variables

Objective: Solve systems of equations with three variables using addition/elimination.

Solving systems of equations with 3 variables is very similar to how we solve systems with two variables. When we had two variables we reduced the system down to one with only one variable (by substitution or addition). With three variables we will reduce the system down to one with two variables (usually by addition), which we can then solve by either addition or substitution.

To reduce from three variables down to two it is very important to keep the work organized. We will use addition with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition to eliminate the **same** variable. This second new equation we will call (B). Once we have done this we will have two equations (A) and (B) with the same two variables that we can solve using either method. This is shown in the following examples.

Example 182.

$$\begin{aligned} 3x+2y-z&=-1 \\ -2x-2y+3z&=5 \quad \text{We will eliminate } y \text{ using two different pairs of equations} \\ 5x+2y-z&=3 \end{aligned}$$

$3x+2y-z=-1$	Using the first two equations,
$\underline{-2x-2y+3z=5}$	Add the first two equations
$(A) \quad x \quad +2z=4$	This is equation (A), our first equation
$-2x-2y+3z=5$	Using the second two equations
$\underline{5x+2y-z=3}$	Add the second two equations
$(B) \quad 3x \quad +2z=8$	This is equation (B), our second equation
$(A) \;\; x+2z=4$	Using (A) and (B) we will solve this system.
$(B) \;\; 3x+2z=8$	We will solve by addition
$-1(x+2z)=(4)(-1)$	Multiply (A) by -1
$-x-2z=-4$	
$-x-2z=-4$	Add to the second equation, unchanged
$\underline{3x+2z=8}$	
$2x=4$	Solve, divide by 2
$\frac{2x}{2}=\frac{4}{2}$	
$x=2$	We now have x! Plug this into either (A) or (B)
$(2)+2z=4$	We plug it into (A), solve this equation, subtract 2
$\underline{-2 \qquad -2}$	
$2z=2$	Divide by 2
$\frac{2z}{2}=\frac{2}{2}$	
$z=1$	We now have z! Plug this and x into any original equation
$3(2)+2y-(1)=-1$	We use the first, multiply $3(2)=6$ and combine with -1
$2y+5=-1$	Solve, subtract 5
$\underline{-5 \quad -5}$	
$2y=-6$	Divide by 2
$\frac{2y}{2}=\frac{-6}{2}$	
$y=-3$	We now have y!
$(2,-3,1)$	Our Solution

As we are solving for x, y, and z we will have an ordered triplet (x,y,z) instead of just the ordered pair (x, y). In this above problem, y was easily eliminated using the addition method. However, sometimes we may have to do a bit of work to get a variable to eliminate. Just as with addition of two equations, we may have to multiply equations by something on both sides to get the opposites we want so a variable eliminates. As we do this remmeber it is improtant to eliminate the **same** variable both times using two **different** pairs of equations.

Example 183.

$$\begin{array}{rl} 4x-3y+2z=-29 & \text{No variable will easily eliminate.} \\ 6x+2y-z=-16 & \text{We could choose any variable, so we chose } x \\ -8x-y+3z=23 & \text{We will eliminate } x \text{ twice.} \end{array}$$

$$\begin{array}{rl} 4x-3y+2z=-29 & \text{Start with first two equations. LCM of 4 and 6 is 12.} \\ 6x+2y-z=-16 & \text{Make the first equation have } 12x\text{, the second } -12x \end{array}$$

$$\begin{array}{rl} 3(4x-3y+2z)=(-29)3 & \text{Multiply the first equation by 3} \\ 12x-9y+6z=-87 & \end{array}$$

$$\begin{array}{rl} -2(6x+2y-z)=(-16)(-2) & \text{Multiply the second equation by } -2 \\ -12x-4y+2z=32 & \end{array}$$

$$\begin{array}{lrl} & 12x-9y+6z=-87 & \text{Add these two equations together} \\ & \underline{-12x-4y+2z=32} & \\ (A) & -13y+8z=-55 & \text{This is our } (A) \text{ equation} \end{array}$$

$$\begin{array}{rl} 6x+2y-z=-16 & \text{Now use the second two equations } (a \text{ different pair}) \\ -8x-y+3z=23 & \text{The LCM of 6 and } -8 \text{ is 24.} \end{array}$$

$$\begin{array}{rl} 4(6x+2y-z)=(-16)4 & \text{Multiply the first equation by 4} \\ 24x+8y-4=-64 & \end{array}$$

$$\begin{array}{rl} 3(-8x-y+3z)=(23)3 & \text{Multiply the second equation by 3} \\ -24x-3y+9z=69 & \end{array}$$

$$\begin{array}{lrl} & 24x+8y-4=-64 & \text{Add these two equations together} \\ & \underline{-24x-3y+9z=69} & \\ (B) & 5y+5z=5 & \text{This is our } (B) \text{ equation} \end{array}$$

$$
\begin{array}{rll}
(A) & -13y+8z=-55 & \text{Using } (A) \text{ and } (B) \text{ we will solve this system} \\
(B) & 5y+5z=5 & \text{The second equation is solved for } z \text{ to use substitution}
\end{array}
$$

$$
\begin{array}{rl}
5y+5z=5 & \text{Solving for } z\text{, subtract } 5y \\
\underline{-5y\qquad -5y} & \\
5z=5-5y & \text{Divide each term by } 5 \\
\overline{5}\quad\overline{5}\quad\overline{5} & \\
z=1-y & \text{Plug into untouched equation} \\
-13y+8(1-y)=-55 & \text{Distribute} \\
-13y+8-8y=-55 & \text{Combine like terms } -13y-8y \\
-21y+8=-55 & \text{Subtract } 8 \\
\underline{-8\qquad -8} & \\
-21y=-63 & \text{Divide by } -21 \\
\overline{-21}\quad\overline{-21} & \\
y=3 & \text{We have our } y\text{! Plug this into } z= \text{ equations} \\
z=1-(3) & \text{Evaluate} \\
z=-2 & \text{We have } z\text{, now find } x \text{ from original equation.}
\end{array}
$$

$$
\begin{array}{rl}
4x-3(3)+2(-2)=-29 & \text{Multiply and combine like terms} \\
4x-13=-29 & \text{Add } 13 \\
\underline{+13\quad +13} & \\
4x=-16 & \text{Divide by } 4 \\
\overline{4}\quad\overline{4} & \\
x=-4 & \text{We have our } x\text{!} \\
& \\
(-4,3,-2) & \text{Our Solution!}
\end{array}
$$

World View Note: Around 250, *The Nine Chapters on the Mathematical Art* were published in China. This book had 246 problems, and chapter 8 was about solving systems of equations. One problem had four equations with five variables!

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. The way we interpret the result is identical.

Example 184.

$$
\begin{array}{rl}
5x-4y+3z=-4 & \\
-10x+8y-6z=8 & \text{We will eliminate } x\text{, start with first two equations}
\end{array}
$$

$$15x - 12y + 9z = -12$$

$$\begin{aligned} 5x - 4y + 3z &= -4 && \text{LCM of 5 and } -10 \text{ is } 10. \\ -10x + 8y - 6z &= 8 \end{aligned}$$

$$\begin{aligned} 2(5x - 4y + 3z) &= -4(2) && \text{Multiply the first equation by 2} \\ 10x - 8y + 6z &= -8 \end{aligned}$$

$$\begin{aligned} 10x - 8y + 6z &= -8 && \text{Add this to the second equation, unchanged} \\ \underline{-10x + 8y - 6} &\underline{= 8} \\ 0 &= 0 && A \text{ true statment} \\ \text{Infinite Solutions} & && \text{Our Solution} \end{aligned}$$

Example 185.

$$\begin{aligned} 3x - 4y + z &= 2 \\ -9x + 12y - 3z &= -5 && \text{We will eliminate } z\text{, starting with the first two equations} \\ 4x - 2y - z &= 3 \end{aligned}$$

$$\begin{aligned} 3x - 4y + z &= 2 && \text{The LCM of 1 and } -3 \text{ is } 3 \\ -9x + 12y - 3z &= -5 \end{aligned}$$

$$\begin{aligned} 3(3x - 4y + z) &= (2)3 && \text{Multiply the first equation by 3} \\ 9x - 12y + 3z &= 6 \end{aligned}$$

$$\begin{aligned} 9x - 12y + 3z &= 6 && \text{Add this to the second equation, unchanged} \\ \underline{-9x + 12y - 3z} &\underline{= -5} \\ 0 &= 1 && A \text{ false statement} \\ \text{No Solution } \varnothing & && \text{Our Solution} \end{aligned}$$

Equations with three (or more) variables are not any more difficult than two variables if we are careful to keep our information organized and eliminate the same variable twice using two different pairs of equations. It is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders, but as long as our information is organized and our algebra is correct, we will arrive at the same final solution.

4.4 Practice - Three Variables

Solve each of the following systems of equation.

1) $a-2b+c=5$
$2a+b-c=-1$
$3a+3b-2c=-4$

2) $2x+3y=z-1$
$3x=8z-1$
$5y+7z=-1$

3) $3x+y-z=11$
$x+3y=z+13$
$x+y-3z=11$

4) $x+y+z=2$
$6x-4y+5z=31$
$5x+2y+2z=13$

5) $x+6y+3z=4$
$2x+y+2z=3$
$3x-2y+z=0$

6) $x-y+2z=-3$
$x+2y+3z=4$
$2x+y+z=-3$

7) $x+y+z=6$
$2x-y-z=-3$
$x-2y+3z=6$

8) $x+y-z=0$
$x+2y-4z=0$
$2x+y+z=0$

9) $x+y-z=0$
$x-y-z=0$
$x+y+2z=0$

10) $x+2y-z=4$
$4x-3y+z=8$
$5x-y=12$

11) $-2x+y-3z=1$
$x-4y+z=6$
$4x+16y+4z=24$

12) $4x+12y+16z=4$
$3x+4y+5z=3$
$x+8y+11z=1$

13) $2x+y-3z=0$
$x-4y+z=0$
$4x+16y+4z=0$

14) $4x+12y+16z=0$
$3x+4y+5z=0$
$x+8y+11z=0$

15) $3x+2y+2z=3$
$x+2y-z=5$
$2x-4y+z=0$

16) $p+q+r=1$
$p+2q+3r=4$
$4p+5q+6r=7$

17) $x-2y+3z=4$
$2x-y+z=-1$
$4x+y+z=1$

18) $x+2y-3z=9$
$2x-y+2z=-8$
$3x-y-4z=3$

19) $x-y+2z=0$
$x-2y+3z=-1$
$2x-2y+z=-3$

20) $4x-7y+3z=1$
$3x+y-2z=4$
$4x-7y+3z=6$

21) $4x-3y+2z=40$
$5x+9y-7z=47$
$9x+8y-3z=97$

22) $3x+y-z=10$
$8x-y-6z=-3$
$5x-2y-5z=1$

23) $3x + 3y - 2z = 13$
$6x + 2y - 5z = 13$
$5x - 2y - 5z = -1$

24) $2x - 3y + 5z = 1$
$3x + 2y - z = 4$
$4x + 7y - 7z = 7$

25) $3x - 4y + 2z = 1$
$2x + 3y - 3z = -1$
$x + 10y - 8z = 7$

26) $2x + y = z$
$4x + z = 4y$
$y = x + 1$

27) $m + 6n + 3p = 8$
$3m + 4n = -3$
$5m + 7n = 1$

28) $3x + 2y = z + 2$
$y = 1 - 2x$
$3z = -2y$

29) $-2w + 2x + 2y - 2z = -10$
$w + x + y + z = -5$
$3w + 2x + 2y + 4z = -11$
$w + 3x - 2y + 2z = -6$

30) $-w + 2x - 3y + z = -8$
$-w + x + y - z = -4$
$w + x + y + z = 22$
$-w + x - y - z = -14$

31) $w + x + y + z = 2$
$w + 2x + 2y + 4z = 1$
$-w + x - y - z = -6$
$-w + 3x + y - z = -2$

32) $w + x - y + z = 0$
$-w + 2x + 2y + z = 5$
$-w + 3x + y - z = -4$
$-2w + x + y - 3z = -7$

4.5

Systems of Equations - Value Problems

Objective: Solve value problems by setting up a system of equations.

One application of system of equations are known as value problems. Value problems are ones where each variable has a value attached to it. For example, if our variable is the number of nickles in a person's pocket, those nickles would have a value of five cents each. We will use a table to help us set up and solve value problems. The basic structure of the table is shown below.

	Number	Value	Total
Item 1			
Item 2			
Total			

The first column in the table is used for the number of things we have. Quite often, this will be our variables. The second column is used for the that value each item has. The third column is used for the total value which we calculate by multiplying the number by the value. For example, if we have 7 dimes, each with a value of 10 cents, the total value is $7 \cdot 10 = 70$ cents. The last row of the table is for totals. We only will use the third row (also marked total) for the totals that are given to use. This means sometimes this row may have some blanks in it. Once the table is filled in we can easily make equations by adding each column, setting it equal to the total at the bottom of the column. This is shown in the following example.

Example 186.

In a child's bank are 11 coins that have a value of \$1.85. The coins are either quarters or dimes. How many coins each does child have?

	Number	Value	Total
Quarter	q	25	
Dime	d	10	
Total			

Using value table, use q for quarters, d for dimes
Each quarter$'s$ value is 25 cents, dime$'s$ is 10 cents

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total			

Multiply number by value to get totals

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total	11		185

We have 11 coins total. This is the number total.
We have 1.85 for the final total,
Write final total in cents (185)
Because 25 and 10 are cents

$$q + d = 11$$ First and last columns are our equations by adding

$$25q + 10d = 185$$ Solve by either addition or substitution.

$$-10(q + d) = (11)(-10)$$ Using addition, multiply first equation by -10

$$-10q - 10d = -110$$

$$-10q - 10d = -110$$ Add together equations

$$\underline{25q + 10d = 185}$$

$$\frac{15q}{15} = \frac{75}{15}$$ Divide both sides by 15

$$q = 5$$ We have our q, number of quarters is 5

$$(5) + d = 11$$ Plug into one of original equations

$$\underline{-5 \qquad -5}$$ Subtract 5 from both sides

$$d = 6$$ We have our d, number of dimes is 6

5 quarters and 6 dimes   Our Solution

World View Note: American coins are the only coins that do not state the value of the coin. On the back of the dime it says "one dime" (not 10 cents). On the back of the quarter it says "one quarter" (not 25 cents). On the penny it says "one cent" (not 1 cent). The rest of the world (Euros, Yen, Pesos, etc) all write the value as a number so people who don't speak the language can easily use the coins.

Ticket sales also have a value. Often different types of tickets sell for different prices (values). These problems can be solve in much the same way.

Example 187.

There were 41 tickets sold for an event. Tickets for children cost $1.50 and tickets for adults cost $2.00. Total receipts for the event were $73.50. How many of each type of ticket were sold?

	Number	Value	Total
Child	c	1.5	
Adult	a	2	
Total			

Using our value table, c for child, a for adult
Child tickets have value 1.50, adult value is 2.00
(we can drop the zeros after the decimal point)

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total			

Multiply number by value to get totals

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total	41		73.5

We have 41 tickets sold. This is our number total
The final total was 73.50
Write in dollars as 1.5 and 2 are also dollars

$c + a = 41$ First and last columns are our equations by adding

$1.5c + 2a = 73.5$ We can solve by either addition or substitution

$c + a = 41$ We will solve by substitution.

$\underline{-c \qquad -c}$ Solve for a by subtracting c

$a = 41 - c$

$1.5c + 2(41 - c) = 73.5$ Substitute into untouched equation

$1.5c + 82 - 2c = 73.5$ Distribute

$-0.5c + 82 = 73.5$ Combine like terms

$\underline{-82 - 82}$ Subtract 82 from both sides

$-0.5c = -8.5$ Divide both sides by -0.5

$$\overline{-0.5} \quad \overline{-0.5}$$

$c=17$ We have c, number of child tickets is 17

$a=41-(17)$ Plug into $a=$ equation to find a

$a=24$ We have our a, number of adult tickets is 24

17 child tickets and 24 adult tickets Our Solution

Some problems will not give us the total number of items we have. Instead they will give a relationship between the items. Here we will have statements such as "There are twice as many dimes as nickles". While it is clear that we need to multiply one variable by 2, it may not be clear which variable gets multiplied by 2. Generally the equations are backwards from the English sentence. If there are twice as many dimes, than we multiply the other variable (nickels) by two. So the equation would be $d=2n$. This type of problem is in the next example.

Example 188.

A man has a collection of stamps made up of 5 cent stamps and 8 cent stamps. There are three times as many 8 cent stamps as 5 cent stamps. The total value of all the stamps is $3.48. How many of each stamp does he have?

	Number	Value	Total
Five	f	5	$5f$
Eight	$3f$	8	$24f$
Total			348

Use value table, f for five cent stamp, and e for eight
Also list value of each stamp under value column

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			

Multiply number by value to get total

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			348

The final total was 338 (written in cents)
We do not know the total number, this is left blank.

$e=3f$ **3 times as many 8 cent stamples as 5 cent stamps**

$5f+8e=348$ Total column gives second equation

$5f+8(3f)=348$ Substitution, substitute first equation in second

$5f+24f=348$ Multiply first

$29f=348$ Combine like terms

$\overline{29} \quad \overline{29}$ Divide both sides by 39

$f=12$ We have f. There are 12 five cent stamps

$e=3(12)$ Plug into first equation

$e = 36$ We have e, There are 36 eight cent stamps

12 five cent, 36 eight cent stamps Our Solution

The same process for solving value problems can be applied to solving interest problems. Our table titles will be adjusted slightly as we do so.

	Invest	Rate	Interest
Account 1			
Account 2			
Total			

Our first column is for the amount invested in each account. The second column is the interest rate earned (written as a decimal - move decimal point twice left), and the last column is for the amount of interset earned. Just as before, we multiply the investment amount by the rate to find the final column, the interest earned. This is shown in the following example.

Example 189.

A woman invests \$4000 in two accounts, one at 6% interest, the other at 9% interest for one year. At the end of the year she had earned \$270 in interest. How much did she have invested in each account?

	Invest	Rate	Interest
Account 1	x	0.06	
Account 2	y	0.09	
Total			

Use our investment table, x and y for accounts
Fill in interest rates as decimals

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total			

Multiply across to find interest earned.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total	4000		270

Total investment is 4000,
Total interest was 276

$x + y = 4000$ First and last column give our two equations

$0.06x + 0.09y = 270$ Solve by either substitution or addition

$-0.06(x + y) = (4000)(-0.06)$ Use Addition, multiply first equation by -0.06

$-0.06x - 0.06y = -240$

$$
\begin{array}{rl}
-0.06x-0.06y=-240 & \text{Add equations together} \\
\underline{0.06x+0.09y=270} & \\
0.03y=30 & \text{Divide both sides by } 0.03 \\
\overline{0.03}\quad\overline{0.03} & \\
y=1000 & \text{We have } y\text{, \$1000 invested at } 9\% \\
x+1000=4000 & \text{Plug into original equation} \\
-1000\ -1000 & \text{Subtract 1000 from both sides} \\
x=3000 & \text{We have } x\text{, \$3000 invested at } 6\% \\
\text{\$1000 at 9\% and \$3000 at 6\%} & \text{Our Solution}
\end{array}
$$

The same process can be used to find an unknown interest rate.

Example 190.

John invests \$5000 in one account and \$8000 in an account paying 4% more in interest. He earned \$1230 in interest after one year. At what rates did he invest?

	Invest	Rate	Interest
Account 1	5000	x	
Account 2	8000	$x+0.04$	
Total			

Our investment table. Use x for first rate
The second rate is 4% higher, or $x+0.04$
Be sure to write this rate as *a* decimal!

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000x+320$
Total			

Multiply to fill in interest column.
Be sure to distribute $8000(x+0.04)$

	Invest	Rate	Interest
Account 2	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000x+320$
Total			1230

Total interest was 1230.

$$
\begin{array}{rl}
5000x+8000x+320=1230 & \text{Last column gives our equation} \\
13000x+320=1230 & \text{Combine like terms} \\
\underline{-320\ \ -320} & \text{Subtract 320 from both sides} \\
13000x=910 & \text{Divide both sides by 13000} \\
\overline{13000}\ \overline{13000} & \\
x=0.07 & \text{We have our } x\text{, } 7\% \text{ interest} \\
(0.07)+0.04 & \text{Second account is } 4\% \text{ higher} \\
0.11 & \text{The account with \$8000 is at } 11\% \\
\text{\$5000 at 7\% and \$8000 at 11\%} & \text{Our Solution}
\end{array}
$$

4.5 Practice - Value Problems

Solve.

1) A collection of dimes and quaters is worth \$15.25. There are 103 coins in all. How many of each is there?

2) A collection of half dollars and nickels is worth \$13.40. There are 34 coins in all. How many are there?

3) The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

4) A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters were there?

5) A boy has \$2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind has he?

6) \$3.75 is made up of quarters and half dollars. If the number of quarters exceeds the number of half dollars by 3, how many coins of each denomination are there?

7) A collection of 27 coins consisting of nickels and dimes amounts to \$2.25. How many coins of each kind are there?

8) \$3.25 in dimes and nickels, were distributed among 45 boys. If each received one coin, how many received dimes and how many received nickels?

9) There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

10) There were 200 tickets sold for a women's basketball game. Tickets for students were 50 cents each and for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for noncard holders the price was \$2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

12) At a local ball game the hotdogs sold for \$2.50 each and the hamburgers sold for \$2.75 each. There were 131 total sandwiches sold for a total value of \$342. How many of each sandwich was sold?

13) At a recent Vikings game \$445 in admission tickets was taken in. The cost of a student ticket was \$1.50 and the cost of a non-student ticket was \$2.50. A total of 232 tickets were sold. How many students and how many non-students attented the game?

14) A bank contains 27 coins in dimes and quarters. The coins have a total value of \$4.95. Find the number of dimes and quarters in the bank.

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of \$1.15. Find the number of nickels and dimes in the coin purse.

16) A business executive bought 40 stamps for \$9.60. The purchase included 25¢ stamps and 20¢ stamps. How many of each type of stamp were bought?

17) A postal clerk sold some 15¢ stamps and some 25¢ stamps. Altogether, 15 stamps were sold for a total cost of \$3.15. How many of each type of stamps were sold?

18) A drawer contains 15¢ stamps and 18¢ stamps. The number of 15¢ stamps is four less than three times the number of 18¢ stamps. The total value of all the stamps is \$1.29. How many 15¢ stamps are in the drawer?

19) The total value of dimes and quarters in a bank is \$6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

20) A child's piggy bank contains 44 coins in quarters and dimes. The coins have a total value of \$8.60. Find the number of quaters in the bank.

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

22) A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is \$50. Find the number of each type of bill in the cash box.

23) A bank teller cashed a check for \$200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

24) A collection of stamps consists of 22¢ stamps and 40¢ stamps. The number of 22¢ stamps is three more than four times the number of 40¢ stamps. The total value of the stamps is \$8.34. Find the number of 22¢ stamps in the collection.

25) A total of \$27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3385. How much was invested at each rate?

26) A total of \$50000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is \$3250. How much was invested at each rate?

27) A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

28) A total of \$18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1248. How much was invested at each rate?

29) An inheritance of \$10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was \$1038.50. How much was invested at each rate?

30) Kerry earned a total of \$900 last year on his investments. If \$7000 was invested at a certain rate of return and \$9000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

31) Jason earned \$256 interest last year on his investments. If \$1600 was invested at a certain rate of return and \$2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

32) Millicent earned \$435 last year in interest. If \$3000 was invested at a certain rate of return and \$4500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

33) A total of \$8500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is \$385. How much was invested at each rate?

34) A total of \$12000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is \$1005. How much was invested at each rate?

35) A total of \$15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is \$1455. How much was invested at each rate?

36) A total of \$17500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is \$1227.50. How much was invested at each rate?

37) A total of \$6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$300. How much was invested at each rate?

38) A total of \$14000 is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is \$910. How much was invested at each rate?

39) A total of \$11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

40) An investment portfolio earned \$2010 in interest last year. If \$3000 was invested at a certain rate of return and \$24000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

41) Samantha earned \$1480 in interest last year on her investments. If \$5000 was invested at a certain rate of return and \$11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

42) A man has \$5.10 in nickels, dimes, and quarters. There are twice as many nickels as dimes and 3 more dimes than quarters. How many coins of each kind were there?

43) 30 coins having a value of \$3.30 consists of nickels, dimes and quarters. If there are twice as many quarters as dimes, how many coins of each kind were there?

44) A bag contains nickels, dimes and quarters having a value of \$3.75. If there are 40 coins in all and 3 times as many dimes as quarters, how many coins of each kind were there?

4.6

Systems of Equations - Mixture Problems

Objective: Solve mixture problems by setting up a system of equations.

One application of systems of equations are mixture problems. Mixture problems are ones where two different solutions are mixed together resulting in a new final solution. We will use the following table to help us solve mixture problems:

	Amount	Part	Total
Item 1			
Item 2			
Final			

The first column is for the amount of each item we have. The second column is labeled "part". If we mix percentages we will put the rate (written as a decimal) in this column. If we mix prices we will put prices in this column. Then we can multiply the amount by the part to find the total. Then we can get an equation by adding the amount and/or total columns that will help us solve the problem and answer the questions.

These problems can have either one or two variables. We will start with one variable problems.

Example 191.

A chemist has 70 mL of a 50% methane solution. How much of a 80% solution must she add so the final solution is 60% methane?

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final			

Set up the mixture table. We start with 70, but don't know how much we add, that is x. The part is the percentages, 0.5 for start, 0.8 for add.

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final	$70+x$	0.6	

Add amount column to get final amount. The part for this amount is 0.6 because we want the final solution to be 60% methane.

	Amount	Part	Total
Start	70	0.5	35
Add	x	0.8	$0.8x$
Final	$70+x$	0.6	$42+0.6x$

Multiply amount by part to get total.
be sure to distribute on the last row: $(70+x)0.6$

$$35+0.8x=42+0.6x$$
The last column is our equation by adding

$$\underline{-0.6x \qquad -0.6x}$$
Move variables to one side, subtract $0.6x$

$$35+0.2x=42$$
Subtract 35 from both sides

$$\underline{-35 \qquad -35}$$

$$0.2x=7$$
Divide both sides by 0.2

$$\overline{0.2} \quad \overline{0.2}$$

$$x=35$$
We have our x!

35 mL must be added — Our Solution

The same process can be used if the starting and final amount have a price attached to them, rather than a percentage.

Example 192.

A coffee mix is to be made that sells for \$2.50 by mixing two types of coffee. The cafe has 40 mL of coffee that costs \$3.00. How much of another coffee that costs \$1.50 should the cafe mix with the first?

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final			

Set up mixture table. We know the starting amount and its cost, \$3. The added amount we do not know but we do know its cost is \$1.50.

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final	$40+x$	2.5	

Add the amounts to get the final amount.
We want this final amount to sell for \$2.50.

	Amount	Part	Total
Start	40	3	120
Add	x	1.5	$1.5x$
Final	$40+x$	2.5	$100+2.5x$

Multiply amount by part to get the total.
Be sure to distribute on the last row $(40+x)2.5$

$$120+1.5x=100+2.5x$$ Adding down the total column gives our equation

$$\underline{-1.5x \qquad -1.5x}$$ Move variables to one side by subtracting $1.5x$

$$120=100+x$$ Subtract 100 from both sides

$$\underline{-100-100}$$

$$20=x$$ We have our x.

20mL must be added. Our Solution

World View Note: Brazil is the world's largest coffee producer, producing 2.59 million metric tons of coffee a year! That is over three times as much coffee as second place Vietnam!

The above problems illustrate how we can put the mixture table together and get an equation to solve. However, here we are interested in systems of equations, with two unknown values. The following example is one such problem.

Example 193.

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

	Amount	Part	Total
Milk 1	x	0.24	
Milk 2	y	0.18	
Final	42	0.2	

We don't know either start value, but we do know final is 42. Also fill in part column with percentage of each type of milk including the final solution

	Amount	Part	Total
Milk 1	x	0.24	$0.24x$
Milk 2	y	0.18	$0.18y$
Final	42	0.2	8.4

Multiply amount by part to get totals.

$$x+y=42$$ The amount column gives one equation

$$0.24x+0.18y=8.4$$ The total column gives *a* second equation.

$$-0.18(x+y)=(42)(-0.18) \quad \text{Use addition. Multiply first equation by } -0.18$$
$$-0.18x-0.18y=-7.56$$

$$-0.18x-0.18y=-7.56 \quad \text{Add the equations together}$$
$$\underline{0.24x+0.18y=8.4}$$
$$0.06x \quad =0.84 \quad \text{Divide both sides by 0.06}$$
$$\overline{0.06} \quad \overline{0.06}$$
$$x=14 \quad \text{We have our } x\text{, 14 gal of 24\% butterfat}$$
$$(14)+y=42 \quad \text{Plug into original equation to find } y$$
$$\underline{-14 \quad -14} \quad \text{Subtract 14 from both sides}$$
$$y=28 \quad \text{We have our } y\text{, 28 gal of 18\% butterfat}$$
$$\text{14 gal of 24\% and 28 gal of 18\%} \quad \text{Our Solution}$$

The same process can be used to solve mixtures of prices with two unknowns.

Example 194.

In a candy shop, chocolate which sells for \$4 a pound is mixed with nuts which are sold for \$2.50 a pound are mixed to form a chocolate-nut candy which sells for \$3.50 a pound. How much of each are used to make 30 pounds of the mixture?

	Amount	Part	Total
Chocolate	c	4	
Nut	n	2.5	
Final	30	3.5	

Using our mixture table, use c and n for variables
We do know the final amount (30) and price, include this in the table

	Amount	Part	Total
Chocolate	c	4	$4c$
Nut	n	2.5	$2.5n$
Final	30	3.5	105

Multiply amount by part to get totals

$$c+n=30 \quad \text{First equation comes from the first column}$$
$$4c+2.5n=105 \quad \text{Second equation comes from the total column}$$

$$c+n=30 \quad \text{We will solve this problem with substitution}$$
$$-n-n \quad \text{Solve for } c \text{ by subtracting } n \text{ from the first equation}$$
$$c=30-n$$

$4(30-n)+2.5n=105$	Substitute into untouched equation
$120-4n+2.5n=105$	Distribute
$120-1.5n=105$	Combine like terms
$\underline{-120 \qquad -120}$	Subtract 120 from both sides
$-1.5n=-15$	Divide both sides by -1.5
$\overline{-1.5} \quad \overline{-1.5}$	
$n=10$	We have our n, 10 lbs of nuts
$c=30-(10)$	Plug into $c=$ equation to find c
$c=20$	We have our c, 20 lbs of chocolate
10 lbs of nuts and 20 lbs of chocolate	Our Solution

With mixture problems we often are mixing with a pure solution or using water which contains none of our chemical we are interested in. For pure solutions, the percentage is 100% (or 1 in the table). For water, the percentage is 0%. This is shown in the following example.

Example 195.

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 L?

	Amount	Part	Final
Antifreeze	a	1	
Water	w	0	
Final	70	0.65	

We use a and w for our variables. Antifreeze is pure, 100% or 1 in our table, written as a decimal. Water has no antifreeze, its percentage is 0. We also fill in the final percent

	Amount	Part	Final
Antifreeze	a	1	a
Water	w	0	0
Final	70	0.65	45.5

Multiply to find final amounts

$a+w=70$	First equation comes from first column
$a=45.5$	Second equation comes from second column
$(45.5)+w=70$	We have a, plug into to other equation
$-45.5 \qquad -45.5$	Subtract 45.5 from both sides
$w=24.5$	We have our w
$45.5\,L$ of antifreeze and $24.5\,L$ of water	Our Solution

4.6 Practice - Mixture Problems

Solve.

1) A tank contains 8000 liters of a solution that is 40% acid. How much water should be added to make a solution that is 30% acid?

2) How much antifreeze should be added to 5 quarts of a 30% mixture of antifreeze to make a solution that is 50% antifreeze?

3) Of 12 pounds of salt water 10% is salt; of another mixture 3% is salt. How many pounds of the second should be added to the first in order to get a mixture of 5% salt?

4) How much alcohol must be added to 24 gallons of a 14% solution of alcohol in order to produce a 20% solution?

5) How many pounds of a 4% solution of borax must be added to 24 pounds of a 12% solution of borax to obtain a 10% solution of borax?

6) How many grams of pure acid must be added to 40 grams of a 20% acid solution to make a solution which is 36% acid?

7) A 100 LB bag of animal feed is 40% oats. How many pounds of oats must be added to this feed to produce a mixture which is 50% oats?

8) A 20 oz alloy of platinum that costs $220 per ounce is mixed with an alloy that costs $400 per ounce. How many ounces of the $400 alloy should be used to make an alloy that costs $300 per ounce?

9) How many pounds of tea that cost $4.20 per pound must be mixed with 12 lb of tea that cost $2.25 per pound to make a mixture that costs $3.40 per pound?

10) How many liters of a solvent that costs $80 per liter must be mixed with 6 L of a solvent that costs $25 per liter to make a solvent that costs $36 per liter?

11) How many kilograms of hard candy that cost $7.50 per kilogram must be mixed with 24 kg of jelly beans that cost $3.25 per kilogram to make a mixture that sells for $4.50 per kilogram?

12) How many kilograms of soil supplement that costs $7.00 per kilogram must be mixed with 20 kg of aluminum nitrate that costs $3.50 per kilogram to make a fertilizer that costs $4.50 per kilogram?

13) How many pounds of lima beans that cost 90¢ per pound must be mixed with 16 lb of corn that cost 50¢ per pound to make a mixture of vegetables that costs 65¢ per pound?

14) How many liters of a blue dye that costs $1.60 per liter must be mixed with 18 L of anil that costs $2.50 per liter to make a mixture that costs $1.90 per liter?

15) Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100cc. of a solution that is 68% acid?

16) A certain grade of milk contains 10% butter fat and a certain grade of cream 60% butter fat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butter fat?

17) A farmer has some cream which is 21% butterfat and some which is 15% butter fat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

18) A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150L which is 96% maple syrup?

19) A chemist wants to make 50ml of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

20) A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many mililiters of each are used to make a 300 ml solution that is 5% hydrogen peroxide?

21) A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gal of paint that is 19% green dye?

22) A candy mix sells for \$2.20 per kilogram. It contains chocolates worth \$1.80 per kilogram and other candy worth \$3.00 per kilogram. How much of each are in 15 kilograms of the mixture?

23) To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth \$4.00/lb. with a weed killer worth \$8.00/lb. The mixture will cost \$6.00/lb. How much of each should be used to prepare 500 lb. of the mixture?

24) A grocer is mixing 40 cent per lb. coffee with 60 cent per lb. coffee to make a mixture worth 54¢ per lb. How much of each kind of coffee should be used to make 70 lb. of the mixture?

25) A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

26) A high-protein diet supplement that costs \$6.75 per pound is mixed with a vitamin supplement that costs \$3.25 per pound. How many pounds of each should be used to make 5 lb of a mixture that costs \$4.65 per pound?

27) A goldsmith combined an alloy that costs \$4.30 per ounce with an alloy that costs \$1.80 per ounce. How many ounces of each were used to make a mixture of 200 oz costing \$2.50 per ounce?

28) A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs \$8 per kilogram with kiwis that cost \$3 per kilogram. How many kilograms of each were used to make a 5 kg mixture that costs \$4.50 per kilogram?

29) The manager of a garden shop mixes grass seed that is 60% rye grass with 70 lb of grass seed that is 80% rye grass to make a mixture that is 74% rye grass. How much of the 60% mixture is used?

30) How many ounces of water evaporated from 50 oz of a 12% salt solution to produce a 15% salt solution?

31) A caterer made an ice cream punch by combining fruit juice that cost $2.25 per gallon with ice cream that costs $3.25 per gallon. How many gallons of each were used to make 100 gal of punch costing $2.50 per pound?

32) A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kg of cloth that is 96% silk?

33) A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 lb of a fabric that is 28% wool?

34) How many pounds of coffee that is 40% java beans must be mixed with 80 lb of coffee that is 30% java beans to make a coffee blend that is 32% java beans?

35) The manager of a specialty food store combined almonds that cost $4.50 per pound with walnuts that cost $2.50 per pound. How many pounds of each were used to make a 100 lb mixture that cost $3.24 per pound?

36) A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 lb of tea that is 18% jasmine?

37) How many ounces of dried apricots must be added to 18 oz of a snack mix that contains 20% dried apricots to make a mixture that is 25% dried apricots?

38) How many mililiters of pure chocolate must be added to 150 ml of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

39) How many ounces of pure bran flakes must be added to 50 oz of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

40) A ground meat mixture is formed by combining meat that costs $2.20 per pound with meat that costs $4.20 per pound. How many pounds of each were used to make a 50 lb mixture tha costs $3.00 per pound?

41) How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

42) A lumber company combined oak wood chips that cost $3.10 per pound with pine wood chips that cost $2.50 per pound. How many pounds of each were used to make an 80 lb mixture costing $2.65 per pound?

43) How many ounces of pure water must be added to 50 oz of a 15% saline solution to make a saline solution that is 10% salt?

Chapter 5 : Polynomials

5.1

Polynomials - Exponent Properties

Objective: Simplify expressions using the properties of exponents.

Problems with expoenents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication. We will use this fact to discover the important properties.

World View Note: The word exponent comes from the Latin "expo" meaning out of and "ponere" meaning place. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents (dating back to the 23rd century BC or earlier!)

Example 196.

$$\begin{array}{rl} a^3a^2 & \text{Expand exponents to multiplication problem} \\ (aaa)(aa) & \text{Now we have } 5\, a's \text{ being multiplied together} \\ a^5 & \text{Our Solution} \end{array}$$

A quicker method to arrive at our answer would have been to just add the exponents: $a^3a^2 = a^{3+2} = a^5$ This is known as the **product rule of exponents**

$$\textbf{Product Rule of Exponents: } \boldsymbol{a^m a^n = a^{m+n}}$$

The product rule of exponents can be used to simplify many problems. We will add the exponent on like variables. This is shown in the following examples

Example 197.

$$\begin{array}{rl} 3^2\cdot 3^6\cdot 3 & \text{Same base, add the exponents } 2+6+1 \\ 3^9 & \text{Our Solution} \end{array}$$

Example 198.

$$\begin{array}{rl} 2x^3y^5z\cdot 5xy^2z^3 & \text{Multiply } 2\cdot 5\text{, add exponents on } x, y \text{ and } z \\ 10x^4y^7z^4 & \text{Our Solution} \end{array}$$

Rather than multiplying, we will now try to divide with exponents

Example 199.

$$\begin{array}{rl} \dfrac{a^5}{a^2} & \text{Expand exponents} \\ \dfrac{aaaaa}{aa} & \text{Divide out two of the } a's \\ aaa & \text{Convert to exponents} \\ a^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to just subtract the exponents, $\frac{a^5}{a^2} = a^{5-2} = a^3$. This is known as the quotient rule of exponents.

$$\textbf{Quotient Rule of Exponents: } \frac{a^m}{a^n} = a^{m-n}$$

The quotient rule of exponents can similarly be used to simplify exponent problems by subtracting exponents on like variables. This is shown in the following examples.

Example 200.

$$\begin{array}{rl} \dfrac{7^{13}}{7^5} & \text{Same base, subtract the exponents} \\ 7^8 & \text{Our Solution} \end{array}$$

Example 201.

$$\begin{array}{rl} \dfrac{5a^3b^5c^2}{2\text{ab}^3c} & \text{Subtract exponents on } a, b \text{ and } c \\ \\ \dfrac{5}{2}a^2b^2c & \text{Our Solution} \end{array}$$

A third property we will look at will have an exponent problem raised to a second exponent. This is investigated in the following example.

Example 202.

$$\begin{array}{rl} \left(a^2\right)^3 & \text{This means we have } a^2 \text{ three times} \\ a^2 \cdot a^2 \cdot a^2 & \text{Add exponents} \\ a^6 & \text{Our solution} \end{array}$$

A quicker method to arrive at the solution would have been to just multiply the exponents, $(a^2)^3 = a^{2 \cdot 3} = a^6$. This is known as the power of a power rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Power Rule of Exponents: } (a^m)^n = a^{mn}$$

This property is often combined with two other properties which we will investigate now.

Example 203.

$$\begin{array}{rl} (ab)^3 & \text{This means we have } (ab) \text{ three times} \\ (ab)(ab)(ab) & \text{Three } a's \text{ and three } b's \text{ can be written with exponents} \\ a^3b^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to take the exponent of three and put it on each factor in parenthesis, $(ab)^3 = a^3b^3$. This is known as the power of a product rule or exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Product Rule of Exponents: } (\mathbf{ab})^{\boldsymbol{m}} = \boldsymbol{a^m b^m}$$

It is important to be careful to only use the power of a product rule with multiplication inside parenthesis. This property does NOT work if there is addition or subtraction.

Warning 204.

$$(a+b)^m \neq a^m + b^m \qquad \text{These are } \textbf{NOT} \text{ equal, beware of this error!}$$

Another property that is very similar to the power of a product rule is considered next.

Example 205.

$$\left(\frac{a}{b}\right)^3 \qquad \text{This means we have the fraction three timse}$$

$$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \qquad \text{Multiply fractions across the top and bottom, using exponents}$$

$$\frac{a^3}{b^3} \qquad \text{Our Solution}$$

A quicker method to arrive at the solution would have been to put the exponent on every factor in both the numerator and denominator, $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. This is known as the power of a quotient rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Quotient Rule of Exponents: } \left(\boldsymbol{\frac{a}{b}}\right)^{\boldsymbol{m}} = \boldsymbol{\frac{a^m}{b^m}}$$

The power of a power, product and quotient rules are often used together to simplify expressions. This is shown in the following examples.

Example 206.

$$(x^3yz^2)^4 \qquad \text{Put the exponent of 4 on each factor, multiplying powers}$$
$$x^{12}y^4z^8 \qquad \text{Our solution}$$

Example 207.

$\left(\dfrac{a^3b}{c^8d^5}\right)^2$	Put the exponent of 2 on each factor, multiplying powers
$\dfrac{a^6b^2}{c^8d^{10}}$	Our Solution

As we multiply exponents its important to remember these properties apply to exponents, not bases. An expressions such as 5^3 does not mean we multipy 5 by 3, rather we multiply 5 three times, $5 \times 5 \times 5 = 125$. This is shown in the next example.

Example 208.

$(4x^2y^5)^3$	Put the exponent of 3 on each factor, multiplying powers
$4^3x^6y^{15}$	Evaluate 4^3
$64x^6y^{15}$	Our Solution

In the previous example we did not put the 3 on the 4 and multipy to get 12, this would have been incorrect. Never multipy a base by the exponent. These properties pertain to exponents only, not bases.

In this lesson we have discussed 5 different exponent properties. These rules are summarized in the following table.

Rules of Exponents

Product Rule of Exponents	$\boldsymbol{a^m a^n = a^{m+n}}$
Quotient Rule of Exponents	$\boldsymbol{\dfrac{a^m}{a^n} = a^{m-n}}$
Power of a Power Rule of Exponents	$\boldsymbol{(a^m)^n = a^{mn}}$
Power of a Product Rule of Exponents	$\boldsymbol{(ab)^m = a^m b^m}$
Power of a Quotient Rule of Exponents	$\boldsymbol{\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}}$

These five properties are often mixed up in the same problem. Often there is a bit of flexibility as to which property is used first. However, order of operations still applies to a problem. For this reason it is the suggestion of the auther to simplify inside any parenthesis first, then simplify any exponents (using power rules), and finally simplify any multiplication or division (using product and quotient rules). This is illustrated in the next few examples.

Example 209.

$(4x^3y \cdot 5x^4y^2)^3$	In parenthesis simplify using product rule, adding exponents
$(20x^7y^3)^3$	With power rules, put three on each factor, multiplying exponents
$20^3x^{21}y^9$	Evaluate 20^3
$8000x^{21}y^9$	Our Solution

Example 210.

$$\begin{array}{rl} 7a^3(2a^4)^3 & \text{Parenthesis are already simplified, next use power rules} \\ 7a^3(8a^{12}) & \text{Using product rule, add exponents and multiply numbers} \\ 56a^{15} & \text{Our Solution} \end{array}$$

Example 211.

$$\begin{array}{rl} \dfrac{3a^3b\cdot 10a^4b^3}{2a^4b^2} & \text{Simplify numerator with product rule, adding exponents} \\ \dfrac{30a^7b^4}{2a^4b^2} & \text{Now use the quotient rule to subtract exponents} \\ 15a^3b^2 & \text{Our Solution} \end{array}$$

Example 212.

$$\begin{array}{rl} \dfrac{3m^8n^{12}}{(m^2n^3)^3} & \text{Use power rule in denominator} \\ \dfrac{3m^8n^{12}}{m^6n^9} & \text{Use quotient rule} \\ 3m^2n^3 & \text{Our solution} \end{array}$$

Example 213.

$$\begin{array}{rl} \left(\dfrac{3\mathrm{ab}^2(2a^4b^2)^3}{6a^5b^7}\right)^2 & \text{Simplify inside parenthesis first, using power rule in numerator} \\ \left(\dfrac{3\mathrm{ab}^2(8a^{12}b^6)}{6a^5b^7}\right)^2 & \text{Simplify numerator using product rule} \\ \left(\dfrac{24a^{13}b^8}{6a^5b^7}\right)^2 & \text{Simplify using the quotient rule} \\ \left(4a^8b\right)^2 & \text{Now that the parenthesis are simplified, use the power rules} \\ 16a^{16}b^2 & \text{Our Solution} \end{array}$$

Clearly these problems can quickly become quite involved. Remember to follow order of operations as a guide, simplify inside parenthesis first, then power rules, then product and quotient rules.

5.1 Practice - Exponent Properties

Simplify.

1) $4\cdot4^4\cdot4^4$

2) $4\cdot4^4\cdot4^2$

3) $4\cdot2^2$

4) $3\cdot3^3\cdot3^2$

5) $3m\cdot4\mathrm{mn}$

6) $3x\cdot4x^2$

7) $2m^4n^2\cdot4nm^2$

8) $x^2y^4\cdot xy^2$

9) $(3^3)^4$

10) $(4^3)^4$

11) $(4^4)^2$

12) $(3^2)^3$

13) $(2u^3v^2)^2$

14) $(xy)^3$

15) $(2a^4)^4$

16) $(2xy)^4$

17) $\frac{4^5}{4^3}$

18) $\frac{3^7}{3^3}$

19) $\frac{3^2}{3}$

20) $\frac{3^4}{3}$

21) $\frac{3nm^2}{3n}$

22) $\frac{x^2y^4}{4xy}$

23) $\frac{4x^3y^4}{3xy^3}$

24) $\frac{xy^3}{4xy}$

25) $(x^3y^4\cdot2x^2y^3)^2$

26) $(u^2v^2\cdot2u^4)^3$

27) $2x(x^4y^4)^4$

28) $\frac{3vu^5\cdot2v^3}{uv^2\cdot2u^3v}$

29) $\frac{2x^7y^5}{3x^3y\cdot4x^2y^3}$

30) $\frac{2ba^7\cdot2b^4}{ba^2\cdot3a^3b^4}$

31) $\left(\frac{(2x)^3}{x^3}\right)^2$

32) $\frac{2a^2b^2a^7}{(ba^4)^2}$

33) $\left(\frac{2y^{17}}{(2x^2y^4)^4}\right)^3$

34) $\frac{yx^2\cdot(y^4)^2}{2y^4}$

35) $\left(\frac{2mn^4\cdot2m^4n^4}{mn^4}\right)^3$

36) $\frac{n^3(n^4)^2}{2mn}$

37) $\frac{2xy^5\cdot2x^2y^3}{2xy^4\cdot y^3}$

38) $\frac{(2y^3x^2)^2}{2x^2y^4\cdot x^2}$

39) $\frac{q^3r^2\cdot(2p^2q^2r^3)^2}{2p^3}$

40) $\frac{2x^4y^5\cdot2z^{10}x^2y^7}{(xy^2z^2)^4}$

41) $\left(\frac{zy^3\cdot z^3x^4y^4}{x^3y^3z^3}\right)^4$

42) $\left(\frac{2q^3p^3r^4\cdot2p^3}{(qrp^3)^2}\right)^4$

43) $\frac{2x^2y^2z^6\cdot2zx^2y^2}{(x^2z^3)^2}$

5.2

Polynomials - Negative Exponents

Objective: Simplify expressions with negative exponents using the properties of exponents.

There are a few special exponent properties that deal with exponents that are not positive. The first is considered in the following example, which is worded out 2 different ways:

Example 214.

$$\frac{a^3}{a^3} \quad \text{Use the quotient rule to subtract exponents}$$

$$a^0 \quad \text{Our Solution, but now we consider the problem } a \text{ the second way:}$$

$$\frac{a^3}{a^3} \quad \text{Rewrite exponents as repeated multiplication}$$

$$\frac{aaa}{aaa} \quad \text{Reduce out all the } a's$$

$$\frac{1}{1} = 1 \quad \text{Our Solution, when we combine the two solutions we get:}$$

$$a^0 = 1 \quad \text{Our final result.}$$

This final result is an imporant property known as the zero power rule of exponents

Zero Power Rule of Exponents: $a^0 = 1$

Any number or expression raised to the zero power will always be 1. This is illustrated in the following example.

Example 215.

$$(3x^2)^0 \quad \text{Zero power rule}$$

$$1 \quad \text{Our Solution}$$

Another property we will consider here deals with negative exponents. Again we will solve the following example two ways.

Example 216.

$$\frac{a^3}{a^5} \quad \text{Using the quotient rule, subtract exponents}$$

$$a^{-2} \quad \text{Our Solution, but we will also solve this problem another way.}$$

$$\frac{a^3}{a^5} \quad \text{Rewrite exponents as repeated multiplication}$$

$$\frac{aaa}{aaaaa} \quad \text{Reduce three } a's \text{ out of top and bottom}$$

$$\frac{1}{aa} \quad \text{Simplify to exponents}$$

$$\frac{1}{a^2} \quad \text{Our Solution, putting these solutions together gives:}$$

$$a^{-2} = \frac{1}{a^2} \quad \text{Our Final Solution}$$

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprical the exponent is now positive. Also, it is important to note a negative exponent does not mean the expression is negative, only that we need the reciprocal of the base. Following are the rules of negative exponents

$$\boldsymbol{a^{-m} = \frac{1}{m}}$$

Rules of Negative Exponets: $\boldsymbol{\frac{1}{a^{-m}} = a^m}$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator, likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 217.

$$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2} \quad \text{Negative exponents on } b, d, \text{ and } e \text{ need to flip}$$

$$\frac{a^3cde^4}{2b^2f^2} \quad \text{Our Solution}$$

As we simplified our fraction we took special care to move the bases that had a negative exponent, but the expression itself did not become negative because of those exponents. Also, it is important to remember that exponents only effect what they are attached to. The 2 in the denominator of the above example does not have an exponent on it, so it does not move with the d.

We now have the following nine properties of exponents. It is important that we are very familiar with all of them.

Properties of Exponents

$$\boldsymbol{a^m a^n = a^{m+n}} \qquad \boldsymbol{(ab)^m = a^m b^m} \qquad \boldsymbol{a^{-m} = \frac{1}{a^m}}$$

$$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}} \qquad \boldsymbol{\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}} \qquad \boldsymbol{\frac{1}{a^{-m}} = a^m}$$

$$\boldsymbol{(a^m)^n = a^{mn}} \qquad \boldsymbol{a^0 = 1} \qquad \boldsymbol{\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}}$$

World View Note: Nicolas Chuquet, the French mathematician of the 15th century wrote $12^{1\bar{m}}$ to indicate $12x^{-1}$. This was the first known use of the negative exponent.

Simplifying with negative exponents is much the same as simplifying with positive exponents. It is the advice of the author to keep the negative exponents until the end of the problem and then move them around to their correct location (numerator or denominator). As we do this it is important to be very careful of rules for adding, subtracting, and multiplying with negatives. This is illustrated in the following examples

Example 218.

$$\frac{4x^{-5}y^{-3}\cdot 3x^3y^{-2}}{6x^{-5}y^3}$$ Simplify numerator with product rule, adding exponents

$$\frac{12x^{-2}y^{-5}}{6x^{-5}y^3}$$ Quotient rule to subtract exponets, be careful with negatives!

$(-2)-(-5)=(-2)+5=3$

$(-5)-3=(-5)+(-3)=-8$

$2x^3y^{-8}$ Negative exponent needs to move down to denominator

$$\frac{2x^3}{y^8}$$ Our Solution

Example 219.

$$\frac{(3\mathrm{ab}^3)^{-2}\mathrm{ab}^{-3}}{2a^{-4}b^0}$$ In numerator, use power rule with -2, multiplying exponents
In denominator, $b^0=1$

$$\frac{3^{-2}a^{-2}b^{-6}\mathrm{ab}^{-3}}{2a^{-4}}$$ In numerator, use product rule to add exponents

$$\frac{3^{-2}a^{-1}b^{-9}}{2a^{-4}}$$ Use quotient rule to subtract exponents, be careful with negatives

$(-1)-(-4)=(-1)+4=3$

$$\frac{3^{-2}a^3b^{-9}}{2}$$ Move 3 and b to denominator because of negative exponents

$$\frac{a^3}{3^2 2b^9}$$ Evaluate $3^2 2$

$$\frac{a^3}{18b^9}$$ Our Solution

In the previous example it is important to point out that when we simplified 3^{-2} we moved the three to the denominator and the exponent became positive. We did not make the number negative! Negative exponents never make the bases negative, they simply mean we have to take the reciprocal of the base. One final example with negative exponents is given here.

Example 220.

$$\left(\frac{3x^{-2}y^5z^3\cdot 6x^{-6}y^{-2}z^{-3}}{9(x^2y^{-2})^{-3}}\right)^{-3}$$ In numerator, use product rule, adding exponents
In denominator, use power rule, multiplying exponets

$$\left(\frac{18x^{-8}y^3z^0}{9x^{-6}y^6}\right)^{-3}$$ Use quotient rule to subtract exponents,

be careful with negatives:

$(-8)-(-6)=(-8)+6=-2$

$3-6=3+(-6)=-3$

$(2x^{-2}y^{-3}z^0)^{-3}$ Parenthesis are done, use power rule with -3

$2^{-3}x^6y^9z^0$ Move 2 with negative exponent down and $z^0=1$

$$\frac{x^6y^9}{2^3}$$ Evaluate 2^3

$$\frac{x^6y^9}{8}$$ Our Solution

5.2 Practice - Negative Exponents

Simplify. Your answer should contain only positive expontents.

1) $2x^4y^{-2}\cdot(2xy^3)^4$

2) $2a^{-2}b^{-3}\cdot(2a^0b^4)^4$

3) $(a^4b^{-3})^3\cdot 2a^3b^{-2}$

4) $2x^3y^2\cdot(2x^3)^0$

5) $(2x^2y^2)^4x^{-4}$

6) $(m^0n^3\cdot 2m^{-3}n^{-3})^0$

7) $(x^3y^4)^3\cdot x^{-4}y^4$

8) $2m^{-1}n^{-3}\cdot(2m^{-1}n^{-3})^4$

9) $\frac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0}$

10) $\frac{3y^3}{3yx^3\cdot 2x^4y^{-3}}$

11) $\frac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}}$

12) $\frac{3x^3y^2}{4y^{-2}\cdot 3x^{-2}y^{-4}}$

13) $\frac{u^2v^{-1}}{2u^0v^4\cdot 2uv}$

14) $\frac{2xy^2\cdot 4x^3y^{-4}}{4x^{-4}y^{-4}\cdot 4x}$

15) $\frac{u^2}{4u^0v^3\cdot 3v^2}$

16) $\frac{2x^{-2}y^2}{4yx^2}$

17) $\frac{2y}{(x^0y^2)^4}$

18) $\frac{(a^4)^4}{2b}$

19) $(\frac{2a^2b^3}{a^{-1}})^4$

20) $(\frac{2y^{-4}}{x^2})^{-2}$

21) $\frac{2nm^4}{(2m^2n^2)^4}$

22) $\frac{2y^2}{(x^4y^0)^{-4}}$

23) $\frac{(2mn)^4}{m^0n^{-2}}$

24) $\frac{2x^{-3}}{(x^4y^{-3})^{-1}}$

25) $\frac{y^3\cdot x^{-3}y^2}{(x^4y^2)^3}$

26) $\frac{2x^{-2}y^0\cdot 2xy^4}{(xy^0)^{-1}}$

27) $\frac{2u^{-2}v^3\cdot(2uv^4)^{-1}}{2u^{-4}v^0}$

28) $\frac{2yx^2\cdot x^{-2}}{(2x^0y^4)^{-1}}$

29) $(\frac{2x^0\cdot y^4}{y^4})^3$

30) $\frac{u^{-3}v^{-4}}{2v(2u^{-3}v^4)^0}$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0}$

32) $\frac{b^{-1}}{(2a^4b^0)^0\cdot 2a^{-3}b^2}$

33) $\frac{2yzx^2}{2x^4y^4z^{-2}\cdot(zy^2)^4}$

34) $\frac{2b^4c^{-2}\cdot(2b^3c^2)^{-4}}{a^{-2}b^4}$

35) $\frac{2kh^0\cdot 2h^{-3}k^0}{(2kj^3)^2}$

36) $(\frac{(2x^{-3}y^0z^{-1})^3\cdot x^{-3}y^2}{2x^3})^{-2}$

37) $\frac{(cb^3)^2\cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3}$

38) $\frac{2q^4\cdot m^2p^2q^4}{(2m^{-4}p^2)^3}$

39) $\frac{(yx^{-4}z^2)^{-1}}{z^3\cdot x^2y^3z^{-1}}$

40) $\frac{2mpn^{-3}}{(m^0n^{-4}p^2)^3\cdot 2n^2p^0}$

5.3

Polynomials - Scientific Notation

Objective: Multiply and divide expressions using scientific notation and exponent properties.

One application of exponent properties comes from scientific notation. Scientific notation is used to represent really large or really small numbers. An example of really large numbers would be the distance that light travels in a year in miles. An example of really small numbers would be the mass of a single hydrogen atom in grams. Doing basic operations such as multiplication and division with these numbers would normally be very combersome. However, our exponent properties make this process much simpler.

First we will take a look at what scientific notation is. Scientific notation has two parts, a number between one and ten (it can be equal to one, but not ten), and that number multiplied by ten to some exponent.

$$\textbf{Scientific Notation: } \boldsymbol{a \times 10^b} \textbf{ where } \boldsymbol{1 \leqslant a < 10}$$

The exponent, b, is very important to how we convert between scientific notation and normal numbers, or standard notation. The exponent tells us how many times we will multiply by 10. Multiplying by 10 in affect moves the decimal point one place. So the exponent will tell us how many times the exponent moves between scientific notation and standard notation. To decide which direction to move the decimal (left or right) we simply need to remember that positive exponents mean in standard notation we have a big number (bigger than ten) and negative exponents mean in standard notation we have a small number (less than one).

Keeping this in mind, we can easily make conversions between standard notation and scientific notation.

Example 221.

Convert $14,200$ to scientific notation	Put decimal after first nonzero number
1.42	Exponent is how many times decimal moved, 4
$\times 10^4$	Positive exponent, standard notation is big
1.42×10^4	Our Solution

Example 222.

Convert 0.0042 to scientific notation	Put decimal after first nonzero number
4.2	Exponent is how many times decimal moved, 3
$\times 10^{-3}$	Negative exponent, standard notation is small
4.2×10^{-3}	Our Solution

Example 223.

Convert 3.21×10^5 to standard notation	Positive exponent means standard notation big number. Move decimal right 5 places
$321,000$	Our Solution

Example 224.

Conver 7.4×10^{-3} to standard notation	Negative exponent means standard notation is *a* small number. Move decimal left 3 places
0.0074	Our Solution

Converting between standard notation and scientific notation is important to understand how scientific notation works and what it does. Here our main interest is to be able to multiply and divide numbers in scientific notation using exponent properties. The way we do this is first do the operation with the front number (multiply or divide) then use exponent properties to simplify the 10's. Scientific notation is the only time where it will be allowed to have negative exponents in our final solution. The negative exponent simply informs us that we are dealing with small numbers. Consider the following examples.

Example 225.

$(2.1 \times 10^{-7})(3.7 \times 10^5)$	Deal with numbers and $10's$ separately
$(2.1)(3.7) = 7.77$	Multiply numbers
$10^{-7}10^5 = 10^{-2}$	Use product rule on $10's$ and add exponents
7.77×10^{-2}	Our Solution

Example 226.

$\dfrac{4.96 \times 10^4}{3.1 \times 10^{-3}}$	Deal with numbers and $10's$ separately
$\dfrac{4.96}{3.1} = 1.6$	Divide Numbers
$\dfrac{10^4}{10^{-3}} = 10^7$	Use quotient rule to subtract exponents, be careful with negatives!
	Be careful with negatives, $4 - (-3) = 4 + 3 = 7$
1.6×10^7	Our Solution

Example 227.

$$
\begin{array}{rl}
(1.8\times10^{-4})^3 & \text{Use power rule to deal with numbers and } 10's \text{ separately}\\
1.8^3 = 5.832 & \text{Evaluate } 1.8^3\\
(10^{-4})^3 = 10^{-12} & \text{Multiply exponents}\\
5.832\times10^{-12} & \text{Our Solution}
\end{array}
$$

Often when we multiply or divide in scientific notation the end result is not in scientific notation. We will then have to convert the front number into scientific notation and then combine the 10's using the product property of exponents and adding the exponents. This is shown in the following examples.

Example 228.

$$
\begin{array}{rl}
(4.7\times10^{-3})(6.1\times10^{9}) & \text{Deal with numbers and } 10's \text{ separately}\\
(4.7)(6.1) = 28.67 & \text{Multiply numbers}\\
2.867\times10^{1} & \text{Convert this number into scientific notation}\\
\mathbf{10^{1}}10^{-3}10^{9} = 10^{7} & \text{Use product rule, add exponents, using } 10^1 \text{ from conversion}\\
2.867\times10^{7} & \text{Our Solution}
\end{array}
$$

World View Note: Archimedes (287 BC - 212 BC), the Greek mathematician, developed a system for representing large numbers using a system very similar to scientific notation. He used his system to calculate the number of grains of sand it would take to fill the universe. His conclusion was 10^{63} grains of sand because he figured the universe to have a diameter of 10^{14} stadia or about 2 light years.

Example 229.

$$
\begin{array}{rl}
\dfrac{2.014\times10^{-3}}{3.8\times10^{-7}} & \text{Deal with numbers and } 10's \text{ separately}\\[2ex]
\dfrac{2.014}{3.8} = 0.53 & \text{Divide numbers}\\[2ex]
0.53 = 5.3\times10^{-1} & \text{Change this number into scientific notation}\\
\dfrac{\mathbf{10^{-1}}10^{-3}}{10^{-7}} = 10^{3} & \text{Use product and quotient rule, using } 10^{-1} \text{ from the conversion}\\
 & \text{Be careful with signs:}\\
 & (-1)+(-3)-(-7) = (-1)+(-3)+7 = 3\\
5.3\times10^{3} & \text{Our Solution}
\end{array}
$$

5.3 Practice - Scientific Notation

Write each number in scientific notiation

1) 885

2) 0.000744

3) 0.081

4) 1.09

5) 0.039

6) 15000

Write each number in standard notation

7) $8.7 \text{ x } 10^5$

8) $2.56 \text{ x } 10^2$

9) $9 \text{ x } 10^{-4}$

10) $5 \text{ x } 10^4$

11) $2 \text{ x } 10^0$

12) $6 \text{ x } 10^{-5}$

Simplify. Write each answer in scientific notation.

13) $(7 \text{ x } 10^{-1})(2 \text{ x } 10^{-3})$

14) $(2 \times 10^{-6})(8.8 \times 10^{-5})$

15) $(5.26 \text{ x } 10^{-5})(3.16 \text{ x } 10^{-2})$

16) $(5.1 \times 10^6)(9.84 \times 10^{-1})$

17) $(2.6 \text{ x } 10^{-2})(6 \text{ x } 10^{-2})$

18) $\frac{7.4 \times 10^4}{1.7 \times 10^{-4}}$

19) $\frac{4.9 \times 10^1}{2.7 \times 10^{-3}}$

20) $\frac{7.2 \times 10^{-1}}{7.32 \times 10^{-1}}$

21) $\frac{5.33 \times 10^{-6}}{9.62 \times 10^{-2}}$

22) $\frac{3.2 \times 10^{-3}}{5.02 \times 10^0}$

23) $(5.5 \times 10^{-5})^2$

24) $(9.6 \times 10^3)^{-4}$

25) $(7.8 \times 10^{-2})^5$

26) $(5.4 \times 10^6)^{-3}$

27) $(8.03 \times 10^4)^{-4}$

28) $(6.88 \times 10^{-4})(4.23 \times 10^1)$

29) $\frac{6.1 \times 10^{-6}}{5.1 \times 10^{-4}}$

30) $\frac{8.4 \times 10^5}{7 \times 10^{-2}}$

31) $(3.6 \times 10^0)(6.1 \times 10^{-3})$

32) $(3.15 \times 10^3)(8 \times 10^{-1})$

33) $(1.8 \times 10^{-5})^{-3}$

34) $\frac{9.58 \times 10^{-2}}{1.14 \times 10^{-3}}$

35) $\frac{9 \times 10^4}{7.83 \times 10^{-2}}$

36) $(8.3 \times 10^1)^5$

37) $\frac{3.22 \times 10^{-3}}{7 \times 10^{-6}}$

38) $\frac{5 \times 10^6}{6.69 \times 10^2}$

39) $\frac{2.4 \times 10^{-6}}{6.5 \times 10^0}$

40) $(9 \times 10^{-2})^{-3}$

41) $\frac{6 \times 10^3}{5.8 \times 10^{-3}}$

42) $(2 \times 10^4)(6 \times 10^1)$

5.4

Polynomials - Introduction to Polynomials

Objective: Evaluate, add, and subtract polynomials.

Many applications in mathematics have to do with what are called polynomials. Polynomials are made up of terms. **Terms** are a product of numbers and/or variables. For example, $5x$, $2y^2$, -5, ab^3c, and x are all terms. Terms are connected to each other by addition or subtraction. Expressions are often named based on the number of terms in them. A **monomial** has one term, such as $3x^2$. A **binomial** has two terms, such as $a^2 - b^2$. A Trinomial has three terms, such as $ax^2 + bx + c$. The term **polynomial** means many terms. Monomials, binomials, trinomials, and expressions with more terms all fall under the umbrella of "polynomials".

If we know what the variable in a polynomial represents we can replace the variable with the number and evaluate the polynomial as shown in the following example.

Example 230.

$$\begin{array}{rl} 2x^2-4x+6 \text{ when } x=-4 & \text{Replace variable } x \text{ with } -4 \\ 2(-4)^2-4(-4)+6 & \text{Exponents first} \\ 2(16)-4(-4)+6 & \text{Multiplication (we can do all terms at once)} \\ 32+16+6 & \text{Add} \\ 54 & \text{Our Solution} \end{array}$$

It is important to be careful with negative variables and exponents. Remember the exponent only effects the number it is physically attached to. This means $-3^2=-9$ because the exponent is only attached to the 3. Also, $(-3)^2=9$ because the exponent is attached to the parenthesis and effects everything inside. When we replace a variable with parenthesis like in the previous example, the substituted value is in parenthesis. So the $(-4)^2=16$ in the example. However, consider the next example.

Example 231.

$$\begin{array}{rl} -x^2+2x+6 \text{ when } x=3 & \text{Replace variable } x \text{ with } 3 \\ -(3)^2+2(3)+6 & \text{Exponent only on the 3, not negative} \\ -9+2(3)+6 & \text{Multiply} \\ -9+6+6 & \text{Add} \\ 3 & \text{Our Solution} \end{array}$$

World View Note: Ada Lovelace in 1842 described a Difference Engine that would be used to caluclate values of polynomials. Her work became the foundation for what would become the modern computer (the programming language Ada was named in her honor), more than 100 years after her death from cancer.

Generally when working with polynomials we do not know the value of the variable, so we will try and simplify instead. The simplest operation with polynomials is addition. When adding polynomials we are mearly combining like terms. Consider the following example

Example 232.

$$\begin{array}{rl} (4x^3-2x+8)+(3x^3-9x^2-11) & \text{Combine like terms } 4x^3+3x^3 \text{ and } 8-11 \\ 7x^3-9x^2-2x-3 & \text{Our Solution} \end{array}$$

Generally final answers for polynomials are written so the exponent on the variable counts down. Example 3 demonstrates this with the exponent counting down 3, 2, 1, 0 (recall $x^0=1$). Subtracting polynomials is almost as fast. One extra step comes from the minus in front of the parenthesis. When we have a negative in front of parenthesis we distribute it through, changing the signs of everything inside. The same is done for the subtraction sign.

Example 233.

$$\begin{array}{rl} (5x^2-2x+7)-(3x^2+6x-4) & \text{Distribute negative through second part} \\ 5x^2-2x+7-3x^2-6x+4 & \text{Combine like terms } 5x^2-3x^3, -2x-6x, \text{ and } 7+4 \\ 2x^2-8x+11 & \text{Our Solution} \end{array}$$

Addition and subtraction can also be combined into the same problem as shown in this final example.

Example 234.

$$\begin{array}{rl} (2x^2-4x+3)+(5x^2-6x+1)-(x^2-9x+8) & \text{Distribute negative through} \\ 2x^2-4x+3+5x^2-6x+1-x^2+9x-8 & \text{Combine like terms} \\ 6x^2-x-4 & \text{Our Solution} \end{array}$$

5.4 Practice - Introduction to Polynomials

Simplify each expression.

1) $-a^3-a^2+6a-21$ when $a=-4$

2) $n^2+3n-11$ when $n=-6$

3) $n^3-7n^2+15n-20$ when $n=2$

4) $n^3-9n^2+23n-21$ when $n=5$

5) $-5n^4-11n^3-9n^2-n-5$ when $n=-1$

6) x^4-5x^3-x+13 when $x=5$

7) $x^2+9x+23$ when $x=-3$

8) $-6x^3+41x^2-32x+11$ when $x=6$

9) $x^4-6x^3+x^2-24$ when $x=6$

10) $m^4+8m^3+14m^2+13m+5$ when $m=-6$

11) $(5p-5p^4)-(8p-8p^4)$

12) $(7m^2+5m^3)-(6m^3-5m^2)$

13) $(3n^2+n^3)-(2n^3-7n^2)$

14) $(x^2+5x^3)+(7x^2+3x^3)$

15) $(8n+n^4)-(3n-4n^4)$

16) $(3v^4+1)+(5-v^4)$

17) $(1+5p^3)-(1-8p^3)$

18) $(6x^3+5x)-(8x+6x^3)$

19) $(5n^4+6n^3)+(8-3n^3-5n^4)$

20) $(8x^2+1)-(6-x^2-x^4)$

21) $(3+b^4)+(7+2b+b^4)$

22) $(1+6r^2)+(6r^2-2-3r^4)$

23) $(8x^3+1)-(5x^4-6x^3+2)$

24) $(4n^4+6)-(4n-1-n^4)$

25) $(2a+2a^4)-(3a^2-5a^4+4a)$

26) $(6v+8v^3)+(3+4v^3-3v)$

27) $(4p^2-3-2p)-(3p^2-6p+3)$

28) $(7+4m+8m^4)-(5m^4+1+6m)$

29) $(4b^3+7b^2-3)+(8+5b^2+b^3)$

30) $(7n+1-8n^4)-(3n+7n^4+7)$

31) $(3+2n^2+4n^4)+(n^3-7n^2-4n^4)$

32) $(7x^2+2x^4+7x^3)+(6x^3-8x^4-7x^2)$

33) $(n-5n^4+7)+(n^2-7n^4-n)$

34) $(8x^2+2x^4+7x^3)+(7x^4-7x^3+2x^2)$

35) $(8r^4-5r^3+5r^2)+(2r^2+2r^3-7r^4+1)$

36) $(4x^3+x-7x^2)+(x^2-8+2x+6x^3)$

37) $(2n^2+7n^4-2)+(2+2n^3+4n^2+2n^4)$

38) $(7b^3-4b+4b^4)-(8b^3-4b^2+2b^4-8b)$

39) $(8-b+7b^3)-(3b^4+7b-8+7b^2)+(3-3b+6b^3)$

40) $(1-3n^4-8n^3)+(7n^4+2-6n^2+3n^3)+(4n^3+8n^4+7)$

41) $(8x^4+2x^3+2x)+(2x+2-2x^3-x^4)-(x^3+5x^4+8x)$

42) $(6x-5x^4-4x^2)-(2x-7x^2-4x^4-8)-(8-6x^2-4x^4)$

5.5

Polynomials - Multiplying Polynomials

Objective: Multiply polynomials.

Multiplying polynomials can take several different forms based on what we are multiplying. We will first look at multiplying monomials, then monomials by polynomials and finish with polynomials by polynomials.

Multiplying monomials is done by multiplying the numbers or coefficients and then adding the exponents on like factors. This is shown in the next example.

Example 235.

$(4x^3y^4z)(2x^2y^6z^3)$	Multiply numbers and add exponents for x, y, and z
$8x^5y^{10}z^4$	Our Solution

In the previous example it is important to remember that the z has an exponent of 1 when no exponent is written. Thus for our answer the z has an exponent of $1+3=4$. Be very careful with exponents in polynomials. If we are adding or subtracting the exponnets will stay the same, but when we multiply (or divide) the exponents will be changing.

Next we consider multiplying a monomial by a polynomial. We have seen this operation before with distributing through parenthesis. Here we will see the exact same process.

Example 236.

$4x^3(5x^2-2x+5)$	Distribute the $4x^3$, multiplying numbers, adding exponents
$20x^5-8x^4+20x^3$	Our Solution

Following is another example with more variables. When distributing the exponents on a are added and the exponents on b are added.

Example 237.

$2a^3b(3\mathrm{ab}^2-4a)$	Distribute, multiplying numbers and adding exponents
$6a^4b^3-8a^4b$	Our Solution

There are several different methods for multiplying polynomials. All of which work, often students prefer the method they are first taught. Here three methods will be discussed. All three methods will be used to solve the same two multiplication problems.

Multiply by Distributing

Just as we distribute a monomial through parenthesis we can distribute an entire polynomial. As we do this we take each term of the second polynomial and put it in front of the first polynomial.

Example 238.

$$\begin{array}{rl} (4x+7y)(3x-2y) & \text{Distribute } (4x+7y) \text{ through parenthesis} \\ 3x\boldsymbol{(4x+7y)}-2y\boldsymbol{(4x+7y)} & \text{Distribute the } 3x \text{ and } -2y \\ 12x^2+21xy-8xy-14y^2 & \text{Combine like terms } 21xy-8xy \\ 12x^2+13xy-14y^2 & \text{Our Solution} \end{array}$$

This example illustrates an important point, the negative/subtraction sign stays with the $2y$. Which means on the second step the negative is also distributed through the last set of parenthesis.

Multiplying by distributing can easily be extended to problems with more terms. First distribute the front parenthesis onto each term, then distribute again!

Example 239.

$$\begin{array}{rl} (2x-5)(4x^2-7x+3) & \text{Distribute } (2x-5) \text{ through parenthesis} \\ 4x^2\boldsymbol{(2x-5)}-7x\boldsymbol{(2x-5)}+3\boldsymbol{(2x-5)} & \text{Distribute again through each parenthesis} \\ 8x^3-20x^2-14x^2+35x+6x-15 & \text{Combine like terms} \\ 8x^3-34x^2+41x-15 & \text{Our Solution} \end{array}$$

This process of multiplying by distributing can easily be reversed to do an important procedure known as factoring. Factoring will be addressed in a future lesson.

Multiply by FOIL

Another form of multiplying is known as FOIL. Using the FOIL method we multiply each term in the first binomial by each term in the second binomial. The letters of FOIL help us remember every combination. F stands for First, we multiply the first term of each binomial. O stand for Outside, we multiply the outside two terms. I stands for Inside, we multiply the inside two terms. L stands for Last, we multiply the last term of each binomial. This is shown in the next example:

Example 240.

$$\begin{array}{rl} (4x+7y)(3x-2y) & \text{Use FOIL to multiply} \\ (4x)(3x)=12x^2 & F-\text{First terms } (4x)(3x) \\ (4x)(-2y)=-8xy & O-\text{Outside terms } (4x)(-2y) \\ (7y)(3x)=21xy & I-\text{Inside terms } (7y)(3x) \\ (7y)(-2y)=-14y^2 & L-\text{Last terms } (7y)(-2y) \\ 12x^2-8xy+21xy-14y^2 & \text{Combine like terms } -8xy+21xy \\ 12x^2+13xy-14y^2 & \text{Our Solution} \end{array}$$

Some students like to think of the FOIL method as distributing the first term $4x$ through the $(3x-2y)$ and distributing the second term $7y$ through the $(3x-2y)$. Thinking about FOIL in this way makes it possible to extend this method to problems with more terms.

Example 241.

$$\begin{array}{rl} (2x-5)(4x^2-7x+3) & \text{Distribute } 2x \text{ and } -5 \\ (2x)(4x^2)+(2x)(-7x)+(2x)(3)-5(4x^2)-5(-7x)-5(3) & \text{Multiply out each term} \\ 8x^3-14x^2+6x-20x^2+35x-15 & \text{Combine like terms} \\ 8x^3-34x^2+41x-15 & \text{Our Solution} \end{array}$$

The second step of the FOIL method is often not written, for example, consider the previous example, a student will often go from the problem $(4x+7y)(3x-2y)$ and do the multiplication mentally to come up with $12x^2-8xy+21xy-14y^2$ and then combine like terms to come up with the final solution.

Multiplying in rows

A third method for multiplying polynomials looks very similar to multiplying numbers. Consider the problem:

$$\begin{array}{rl} 35 & \\ \underline{\times 27} & \\ 245 & \text{Multiply 7 by 5 then 3} \\ \underline{700} & \text{Use 0 for placeholder, multiply 2 by 5 then 3} \\ 945 & \text{Add to get Our Solution} \end{array}$$

World View Note: The first known system that used place values comes from Chinese mathematics, dating back to 190 AD or earlier.

The same process can be done with polynomials. Multiply each term on the bottom with each term on the top.

Example 242.

$$\begin{array}{rl} (4x+7y)(3x-2y) & \text{Rewrite as vertical problem} \\ 4x+7y & \\ \underline{\times\, 3x-2y} & \\ -8xy-14y^2 & \text{Multiply } -2y \text{ by } 7y \text{ then } 4x \\ \underline{12x^2+21xy \qquad\quad} & \text{Multiply } 3x \text{ by } 7y \text{ then } 4x\text{. Line up like terms} \\ 12x^2+13xy-14y^2 & \text{Add like terms to get Our Solution} \end{array}$$

This same process is easily expanded to a problem with more terms.

Example 243.

$(2x-5)(4x^2-7x+3)$	Rewrite as vertical problem
$4x^3-7x+3$	Put polynomial with most terms on top
$\underline{\times\, 2x-5}$	
$-20x^2+35x-15$	Multiply -5 by each term
$\underline{8x^3-14x^2+6x}$	Multiply $2x$ by each term. Line up like terms
$8x^3-34x^2+41x-15$	Add like terms to get our solution

This method of multiplying in rows also works with multiplying a monomial by a polynomial!

Any of the three described methods work to multiply polynomials. It is suggested that you are very comfortable with at least one of these methods as you work through the practice problems. All three methods are shown side by side in the example.

Example 244.

$$(2x-y)(4x-5y)$$

Distribute	**FOIL**	**Rows**
$4x(2x-y)-5y(2x-y)$	$2x(4x)+2x(-5y)-y(4x)-y(-5y)$	$2x-y$
$8x^2-4xy-10xy-5y^2$	$8x^2-10xy-4xy+5y^2$	$\underline{\times\, 4x-5y}$
$8x^2-14xy-5y^2$	$8x^2-14xy+5y^2$	$-10xy+5y^2$
		$\underline{8x^2-4xy}$
		$8x^2-14xy+5y^2$

When we are multiplying a monomial by a polynomial by a polynomial we can solve by first multiplying the polynomials then distributing the coefficient last. This is shown in the last example.

Example 245.

$3(2x-4)(x+5)$	Multiply the binomials, we will use FOIL
$3(2x^2+10x-4x-20)$	Combine like terms
$3(2x^2+6x-20)$	Distribute the 3
$6x^2+18x-60$	Our Solution

A common error students do is distribute the three at the start into both parenthesis. While we can distribute the 3 into the $(2x-4)$ factor, distributing into both would be wrong. Be careful of this error. This is why it is suggested to multiply the binomials first, then distribute the coeffienct last.

5.5 Practice - Multiply Polynomials

Find each product.

1) $6(p-7)$

2) $4k(8k+4)$

3) $2(6x+3)$

4) $3n^2(6n+7)$

5) $5m^4(4m+4)$

6) $3(4r-7)$

7) $(4n+6)(8n+8)$

8) $(2x+1)(x-4)$

9) $(8b+3)(7b-5)$

10) $(r+8)(4r+8)$

11) $(4x+5)(2x+3)$

12) $(7n-6)(n+7)$

13) $(3v-4)(5v-2)$

14) $(6a+4)(a-8)$

15) $(6x-7)(4x+1)$

16) $(5x-6)(4x-1)$

17) $(5x+y)(6x-4y)$

18) $(2u+3v)(8u-7v)$

19) $(x+3y)(3x+4y)$

20) $(8u+6v)(5u-8v)$

21) $(7x+5y)(8x+3y)$

22) $(5a+8b)(a-3b)$

23) $(r-7)(6r^2-r+5)$

24) $(4x+8)(4x^2+3x+5)$

25) $(6n-4)(2n^2-2n+5)$

26) $(2b-3)(4b^2+4b+4)$

27) $(6x+3y)(6x^2-7xy+4y^2)$

28) $(3m-2n)(7m^2+6mn+4n^2)$

29) $(8n^2+4n+6)(6n^2-5n+6)$

30) $(2a^2+6a+3)(7a^2-6a+1)$

31) $(5k^2+3k+3)(3k^2+3k+6)$

32) $(7u^2+8uv-6v^2)(6u^2+4uv+3v^2)$

33) $3(3x-4)(2x+1)$

34) $5(x-4)(2x-3)$

35) $3(2x+1)(4x-5)$

36) $2(4x+1)(2x-6)$

37) $7(x-5)(x-2)$

38) $5(2x-1)(4x+1)$

39) $6(4x-1)(4x+1)$

40) $3(2x+3)(6x+9)$

5.6

Polynomials - Multiply Special Products

Objective: Recognize and use special product rules of a sum and difference and perfect squares to multiply polynomials.

There are a few shortcuts that we can take when multiplying polynomials. If we can recognize them the shortcuts can help us arrive at the solution much quicker. These shortcuts will also be useful to us as our study of algebra continues.

The first shortcut is often called a **sum and a difference**. A sum and a difference is easily recognized as the numbers and variables are exactly the same, but the sign in the middle is different (one sum, one difference). To illustrate the shortcut consider the following example, multiplied by the distributing method.

Example 246.

$$\begin{array}{rl} (a+b)(a-b) & \text{Distribute } (a+b) \\ a(a+b)-b(a+b) & \text{Distribute } a \text{ and } -b \\ a^2+ab-ab-b^2 & \text{Combine like terms } ab-ab \\ a^2-b^2 & \text{Our Solution} \end{array}$$

The important part of this example is the middle terms subtracted to zero. Rather than going through all this work, when we have a sum and a difference we will jump right to our solution by squaring the first term and squaring the last term, putting a subtraction between them. This is illustrated in the following example

Example 247.

$$\begin{array}{rl} (x-5)(x+5) & \text{Recognize sum and difference} \\ x^2-25 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

This is much quicker than going through the work of multiplying and combining like terms. Often students ask if they can just multiply out using another method and not learn the shortcut. These shortcuts are going to be very useful when we get to factoring polynomials, or reversing the multiplication process. For this reason it is very important to be able to recognize these shortcuts. More examples are shown here.

Example 248.

$$\begin{array}{rl} (3x+7)(3x-7) & \text{Recognize sum and difference} \\ 9x^2-49 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

Example 249.

$$\begin{array}{rl} (2x-6y)(2x+6y) & \text{Recognize sum and difference} \\ 4x^2-36y^2 & \text{Square both, put subtraction between. Our Solution} \end{array}$$

It is interesting to note that while we can multiply and get an answer like $a^2 - b^2$ (with subtraction), it is impossible to multiply real numbers and end up with a product such as $a^2 + b^2$ (with addition).

Another shortcut used to multiply is known as a **perfect square**. These are easy to recognize as we will have a binomial with a 2 in the exponent. The following example illustrates multiplying a perfect square

Example 250.

$$\begin{array}{rl} (a+b)^2 & \text{Squared is same as multiplying by itself} \\ (a+b)(a+b) & \text{Distribute } (a+b) \\ a(a+b)+b(a+b) & \text{Distribute again through final parenthesis} \\ a^2+ab+ab+b^2 & \text{Combine like terms } ab+ab \\ a^2+2\text{ab}+b^2 & \text{Our Solution} \end{array}$$

This problem also helps us find our shortcut for multiplying. The first term in the answer is the square of the first term in the problem. The middle term is 2 times the first term times the second term. The last term is the square of the last term. This can be shortened to square the first, twice the product, square the last. If we can remember this shortcut we can square any binomial. This is illustrated in the following example

Example 251.

$$\begin{array}{rl} (x-5)^2 & \text{Recognize perfect square} \\ x^2 & \text{Square the first} \\ 2(x)(-5)=-10x & \text{Twice the product} \\ (-5)^2=25 & \text{Square the last} \\ x^2-10x+25 & \text{Our Solution} \end{array}$$

Be very careful when we are squaring a binomial to **NOT** distribute the square through the parenthesis. A common error is to do the following: $(x-5)^2 = x^2 - 25$ (or $x^2 + 25$). Notice both of these are missing the middle term, $-10x$. This is why it is important to use the shortcut to help us find the correct solution. Another important observation is that the middle term in the solution always has the same sign as the middle term in the problem. This is illustrated in the next examples.

Example 252.

$$\begin{array}{rl} (2x+5)^2 & \text{Recognize perfect square} \\ (2x)^2 = 4x^2 & \text{Square the first} \\ 2(2x)(5) = 20x & \text{Twice the product} \\ 5^2 = 25 & \text{Square the last} \\ 4x^2 + 20x + 25 & \text{Our Solution} \end{array}$$

Example 253.

$$\begin{array}{rl} (3x-7y)^2 & \text{Recognize perfect square} \\ 9x^2 - 42\text{xy} + 49y^2 & \text{Square the first, twice the product, square the last. Our Solution} \end{array}$$

Example 254.

$$\begin{array}{rl} (5a+9b)^2 & \text{Recognize perfect square} \\ 25a^2 + 90\text{ab} + 81b^2 & \text{Square the first, twice the product, square the last. Our Solution} \end{array}$$

These two formulas will be important to commit to memory. The more familiar we are with them, the easier factoring, or multiplying in reverse, will be. The final example covers both types of problems (two perfect squares, one positive, one negative), be sure to notice the difference between the examples and how each formula is used

Example 255.

$$\begin{array}{ccc} (4x-7)(4x+7) & (4x+7)^2 & (4x-7)^2 \\ 16x^2 - 49 & 16x^2 + 56x + 49 & 16x^2 - 56x + 49 \end{array}$$

World View Note: There are also formulas for higher powers of binomials as well, such as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. While French mathematician Blaise Pascal often gets credit for working with these expansions of binomials in the 17th century, Chinese mathematicians had been working with them almost 400 years earlier!

5.6 Practice - Multiply Special Products

Find each product.

1) $(x+8)(x-8)$

2) $(a-4)(a+4)$

3) $(1+3p)(1-3p)$

4) $(x-3)(x+3)$

5) $(1-7n)(1+7n)$

6) $(8m+5)(8m-5)$

7) $(5n-8)(5n+8)$

8) $(2r+3)(2r-3)$

9) $(4x+8)(4x-8)$

10) $(b-7)(b+7)$

11) $(4y-x)(4y+x)$

12) $(7a+7b)(7a-7b)$

13) $(4m-8n)(4m+8n)$

14) $(3y-3x)(3y+3x)$

15) $(6x-2y)(6x+2y)$

16) $(1+5n)^2$

17) $(a+5)^2$

18) $(v+4)^2$

19) $(x-8)^2$

20) $(1-6n)^2$

21) $(p+7)^2$

22) $(7k-7)^2$

23) $(7-5n)^2$

24) $(4x-5)^2$

25) $(5m-8)^2$

26) $(3a+3b)^2$

27) $(5x+7y)^2$

28) $(4m-n)^2$

29) $(2x+2y)^2$

30) $(8x+5y)^2$

31) $(5+2r)^2$

32) $(m-7)^2$

33) $(2+5x)^2$

34) $(8n+7)(8n-7)$

35) $(4v-7)(4v+7)$

36) $(b+4)(b-4)$

37) $(n-5)(n+5)$

38) $(7x+7)^2$

39) $(4k+2)^2$

40) $(3a-8)(3a+8)$

5.7

Polynomials - Divide Polynomials

Objective: Divide polynomials using long division.

Dividing polynomials is a process very similar to long division of whole numbers. But before we look at that, we will first want to be able to master dividing a polynomial by a monomial. The way we do this is very similar to distributing, but the operation we distribute is the division, dividing each term by the monomial and reducing the resulting expression. This is shown in the following examples

Example 256.

$$\frac{9x^5+6x^4-18x^3-24x^2}{3x^2} \quad \text{Divide each term in the numerator by } 3x^2$$

$$\frac{9x^5}{3x^2}+\frac{6x^4}{3x^2}-\frac{18x^3}{3x^2}-\frac{24x^2}{3x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

$$3x^3+2x^2-6x-8 \quad \text{Our Solution}$$

Example 257.

$$\frac{8x^3+4x^2-2x+6}{4x^2} \quad \text{Divide each term in the numerator by } 4x^2$$

$$\frac{8x^3}{4x^2}+\frac{4x^2}{4x^2}-\frac{2x}{4x^2}+\frac{6}{4x^2} \quad \text{Reduce each fraction, subtracting exponents}$$

Remember negative exponents are moved to denominator

$$2x+1-\frac{1}{2x}+\frac{3}{2x^2} \quad \text{Our Solution}$$

The previous example illustrates that sometimes we will have fractions in our solution, as long as they are reduced this will be correct for our solution. Also interesting in this problem is the second term $\frac{4x^2}{4x^2}$ divided out completely. Remember that this means the reduced answer is 1 not 0.

Long division is required when we divide by more than just a monomial. Long division with polynomials works very similar to long division with whole numbers.

An example is given to review the (general) steps that are used with whole numbers that we will also use with polynomials

Example 258.

$4\vert\overline{631}$	Divide front numbers: $\frac{6}{4} = 1\ldots$
$\mathbf{1}$	
$4\vert\overline{631}$	Multiply this number by divisor: $1 \cdot 4 = 4$
$\underline{-\mathbf{4}}$	Change the sign of this number (make it subtract) and combine
$\mathbf{23}$	Bring down next number
$1\mathbf{5}$	Repeat, divide front numbers: $\frac{23}{4} = 5\ldots$
$4\vert\overline{631}$	
$\underline{-4}$	
23	Multiply this number by divisor: $5 \cdot 4 = 20$
$\underline{-\mathbf{20}}$	Change the sign of this number (make it subtract) and combine
$\mathbf{31}$	Bring down next number
$15\mathbf{7}$	Repeat, divide front numbers: $\frac{31}{4} = 7\ldots$
$4\vert\overline{631}$	
$\underline{-4}$	
23	
$\underline{-20}$	
31	Multiply this number by divisor: $7 \cdot 4 = 28$
$\underline{-\mathbf{28}}$	Change the sign of this number (make it subtract) and combine
$\mathbf{3}$	We will write our remainder as *a* fraction, over the divisor, added to the end
$157\frac{3}{4}$	Our Solution

This same process will be used to multiply polynomials. The only difference is we will replace the word "number" with the word "term"

Dividing Polynomials

1. Divide front terms

2. Multiply this term by the divisor

3. Change the sign of the terms and combine

4. Bring down the next term

5. Repeat

Step number 3 tends to be the one that students skip, not changing the signs of the terms would be equivalent to adding instead of subtracting on long division with whole numbers. Be sure not to miss this step! This process is illustrated in the following two examples.

Example 259.

$\dfrac{3x^3-5x^2-32x+7}{x-4}$ Rewrite problem as long division

$x-4\overline{)3x^3-5x^2-32x+7}$ Divide front terms: $\dfrac{3x^3}{x}=3x^2$

$$\begin{array}{rl}
\mathbf{3x^2} & \\
x-4\overline{)3x^3-5x^2-32x+7} & \text{Multiply this term by divisor: } 3x^2(x-4)=3x^3-12x^2 \\
\underline{\mathbf{-3x^3+12x^2}} & \text{Change the signs and combine} \\
\mathbf{7x^2-32x} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
3x^2+\mathbf{7x} & \text{Repeat, divide front terms: } \dfrac{7x^2}{x}=7x \\
x-4\overline{)3x^3-5x^2-32x+7} & \\
\underline{-3x^3+12x^2} & \\
7x^2-32x & \text{Multiply this term by divisor: } 7x(x-4)=7x^2-28x \\
\underline{\mathbf{-7x^2+28x}} & \text{Change the signs and combine} \\
\mathbf{-4x+7} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
3x^2+7x\mathbf{-4} & \text{Repeat, divide front terms: } \dfrac{-4x}{x}=-4 \\
x-4\overline{)3x^3-5x^2-32x+7} & \\
\underline{-3x^3+12x^2} & \\
7x^2-32x & \\
\underline{-7x^2+28x} & \\
-4x+7 & \text{Multiply this term by divisor: } -4(x-4)=-4x+16 \\
\underline{\mathbf{+4x-16}} & \text{Change the signs and combine} \\
\mathbf{-9} & \text{Remainder put over divisor and subtracted (due to negative)}
\end{array}$$

$3x^2+7x-4-\dfrac{9}{x-4}$ Our Solution

Example 260.

$\dfrac{6x^3-8x^2+10x+103}{2x+4}$ Rewrite problem as long division

$2x+4\overline{)6x^3-8x^2+10x+103}$ Divide front terms: $\dfrac{6x^3}{2x}=3x^2$

$$
\begin{array}{rl}
\mathbf{3x^2} & \\
2x+4\overline{)6x^3-8x^2+10x+103} & \text{Multiply term by divisor: } 3x^2(2x+4)=6x^3+12x^2 \\
\underline{\mathbf{-6x^3-12x^2}} & \text{Change the signs and combine} \\
\mathbf{-20x^2+10x} & \text{Bring down the next term}
\end{array}
$$

$$
\begin{array}{rl}
3x^2\mathbf{-10x} & \\
2x+4\overline{)6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \dfrac{-20x^2}{2x}=-10x \\
\underline{-6x^3-12x^2} & \text{Multiply this term by divisor:} \\
-20x^2+10x & -10x(2x+4)=-20x^2-40x \\
\underline{\mathbf{+20x^2+40x}} & \text{Change the signs and combine} \\
\mathbf{50x+103} & \text{Bring down the next term}
\end{array}
$$

$$
\begin{array}{rl}
3x^2-10x\mathbf{+25} & \\
2x+4\overline{)6x^3-8x^2+10x+103} & \text{Repeat, divide front terms: } \dfrac{50x}{2x}=25 \\
\underline{-6x^3-12x^2} & \\
-20x^2+10x & \\
\underline{+20x^2+40x} & \\
50x+103 & \text{Multiply this term by divisor: } 25(2x+4)=50x+100 \\
\underline{\mathbf{-50x-100}} & \text{Change the signs and combine} \\
3 & \text{Remainder is put over divsor and added (due to positive)}
\end{array}
$$

$3x^2-10x+25+\dfrac{3}{2x+4}$ Our Solution

In both of the previous example the dividends had the exponents on our variable counting down, no exponent skipped, third power, second power, first power, zero power (remember $x^0=1$ so there is no variable on zero power). This is very important in long division, the variables must count down and no exponent can be skipped. If they don't count down we must put them in order. If an exponent is skipped we will have to add a term to the problem, with zero for its coefficient. This is demonstrated in the following example.

Example 261.

$$\frac{2x^3+42-4x}{x+3} \quad \text{Reorder dividend, need } x^2 \text{ term, add } 0x^2 \text{ for this}$$

$$x+3\overline{|2x^3+\mathbf{0x^2}-4x+42} \quad \text{Divide front terms: } \frac{2x^3}{x}=2x^2$$

$$\begin{array}{rl}
\mathbf{2x^2} \qquad\qquad & \\
x+3\overline{|2x^3+0x^2-4x+42} & \text{Multiply this term by divisor: } 2x^2(x+3)=2x^3+6x^2 \\
\underline{\mathbf{-2x^3-6x^2}} \qquad\quad & \text{Change the signs and combine} \\
\mathbf{-6x^2-4x} \quad & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
2x^2\mathbf{-6x} \qquad & \\
x+3\overline{|2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{-6x^2}{x}=-6x \\
\underline{-2x^3-6x^2} \qquad\quad & \\
-6x^2-4x \quad & \text{Multiply this term by divisor: } -6x(x+3)=-6x^2-18x \\
\underline{\mathbf{+6x^2+18x}} \quad & \text{Change the signs and combine} \\
\mathbf{14x+42} & \text{Bring down the next term}
\end{array}$$

$$\begin{array}{rl}
2x^2-6x\mathbf{+14} \quad & \\
x+3\overline{|2x^3+0x^2-4x+42} & \text{Repeat, divide front terms: } \frac{14x}{x}=14 \\
\underline{-2x^3-6x^2} \qquad\quad & \\
-6x^2-4x \quad & \\
\underline{+6x^2+18x} \quad & \\
14x+42 & \text{Multiply this term by divisor: } 14(x+3)=14x+42 \\
\underline{\mathbf{-14x-42}} & \text{Change the signs and combine} \\
0 & \text{No remainder}
\end{array}$$

$$2x^2-6x+14 \quad \text{Our Solution}$$

It is important to take a moment to check each problem to verify that the exponents count down and no exponent is skipped. If so we will have to adjust the problem. Also, this final example illustrates, just as in regular long division, sometimes we have no remainder in a problem.

World View Note: Paolo Ruffini was an Italian Mathematician of the early 19th century. In 1809 he was the first to describe a process called synthetic division which could also be used to divide polynomials.

5.7 Practice - Divide Polynomials

Divide.

1) $\frac{20x^4+x^3+2x^2}{4x^3}$

2) $\frac{5x^4+45x^3+4x^2}{9x}$

3) $\frac{20n^4+n^3+40n^2}{10n}$

4) $\frac{3k^3+4k^2+2k}{8k}$

5) $\frac{12x^4+24x^3+3x^2}{6x}$

6) $\frac{5p^4+16p^3+16p^2}{4p}$

7) $\frac{10n^4+50n^3+2n^2}{10n^2}$

8) $\frac{3m^4+18m^3+27m^2}{9m^2}$

9) $\frac{x^2-2x-71}{x+8}$

10) $\frac{r^2-3r-53}{r-9}$

11) $\frac{n^2+13n+32}{n+5}$

12) $\frac{b^2-10b+16}{b-7}$

13) $\frac{v^2-2v-89}{v-10}$

14) $\frac{x^2+4x-26}{x+7}$

15) $\frac{a^2-4a-38}{a-8}$

16) $\frac{x^2-10x+22}{x-4}$

17) $\frac{45p^2+56p+19}{9p+4}$

18) $\frac{48k^2-70k+16}{6k-2}$

19) $\frac{10x^2-32x+9}{10x-2}$

20) $\frac{n^2+7n+15}{n+4}$

21) $\frac{4r^2-r-1}{4r+3}$

22) $\frac{3m^2+9m-9}{3m-3}$

23) $\frac{n^2-4}{n-2}$

24) $\frac{2x^2-5x-8}{2x+3}$

25) $\frac{27b^2+87b+35}{3b+8}$

26) $\frac{3v^2-32}{3v-9}$

27) $\frac{4x^2-33x+28}{4x-5}$

28) $\frac{4n^2-23n-38}{4n+5}$

29) $\frac{a^3+15a^2+49a-55}{a+7}$

30) $\frac{8k^3-66k^2+12k+37}{k-8}$

31) $\frac{x^3-26x-41}{x+4}$

32) $\frac{x^3-16x^2+71x-56}{x-8}$

33) $\frac{3n^3+9n^2-64n-68}{n+6}$

34) $\frac{k^3-4k^2-6k+4}{k-1}$

35) $\frac{x^3-46x+22}{x+7}$

36) $\frac{2n^3+21n^2+25n}{2n+3}$

37) $\frac{9p^3+45p^2+27p-5}{9p+9}$

38) $\frac{8m^3-57m^2+42}{8m+7}$

39) $\frac{r^3-r^2-16r+8}{r-4}$

40) $\frac{2x^3+12x^2+4x-37}{2x+6}$

41) $\frac{12n^3+12n^2-15n-4}{2n+3}$

42) $\frac{24b^3-38b^2+29b-60}{4b-7}$

43) $\frac{4v^3-21v^2+6v+19}{4v+3}$

Chapter 6 : Factoring

6.1

Factoring - Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The opposite of multiplying polynomials together is factoring polynomials. There are many benifits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials it is very important to have very strong factoring skills.

In this lesson we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$. In this lesson we will work the same problem backwards. We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to the $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what is the GCF of a polynomial. We will first introduce this by looking at finding the GCF of several numbers. To find a GCF of sevearal numbers we are looking for the largest number that can be divided by each of the numbers. This can often be done with quick mental math and it is shown in the following example

Example 262.

$$\begin{array}{rl}\text{Find the GCF of } 15, 24, \text{and } 27 & \\ \dfrac{15}{3} = 5,\ \dfrac{24}{3} = 6,\ \dfrac{27}{3} = 9 & \text{Each of the numbers can be divided by } 3 \\ \text{GCF} = 3 & \text{Our Solution}\end{array}$$

When there are variables in our problem we can first find the GCF of the numbers using mental math, then we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example

Example 263.

$$\begin{array}{rl} \text{GCF of } 24x^4y^2z, 18x^2y^4, \text{ and } 12x^3yz^5 & \\ \dfrac{24}{6}=4,\ \dfrac{18}{6}=3,\ \dfrac{12}{6}=2 & \text{Each number can be divided by 6} \\ x^2y & x \text{ and } y \text{ are in all 3, using lowest exponets} \\ \text{GCF} = 6x^2y & \text{Our Solution} \end{array}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parenthesis, then we divide each term by the GCF, the answer is what is left inside the parenthesis. This is shown in the following examples

Example 264.

$$\begin{array}{rl} 4x^2-20x+16 & \text{GCF is 4, divide each term by 4} \\ \dfrac{4x^2}{4}=x^2,\ \dfrac{-20x}{4}=-5x,\ \dfrac{16}{4}=4 & \text{This is what is left inside the parenthesis} \\ 4(x^2-5x+4) & \text{Our Solution} \end{array}$$

With factoring we can always check our solutions by multiplying (distributing in this case) out the answer and the solution should be the original equation.

Example 265.

$$\begin{array}{rl} 25x^4-15x^3+20x^2 & \text{GCF is } 5x^2\text{, divide each term by this} \\ \dfrac{25x^4}{5x^2}=5x^2,\ \dfrac{-15x^3}{5x^2}=-3x,\ \dfrac{20x^2}{5x^2}=4 & \text{This is what is left inside the parenthesis} \\ 5x^2(5x^2-3x+4) & \text{Our Solution} \end{array}$$

Example 266.

$$\begin{array}{rl} 3x^3y^2z+5x^4y^3z^5-4xy^4 & \text{GCF is } xy^2\text{, divide each term by this} \end{array}$$

$$\frac{3x^3y^2z}{xy^2}=3x^2z,\ \frac{5x^4y^3z^5}{xy^2}=5x^3yz^5,\ \frac{-4xy^4}{xy^2}=-4y^2 \quad \text{This is what is left in parenthesis}$$

$$xy^2(3x^2z+5x^3yz^5-4y^2) \quad \text{Our Solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from Greek mathematician Euclid around the year 300 BC!

Example 267.

$$21x^3+14x^2+7x \quad \text{GCF is } 7x\text{, divide each term by this}$$

$$\frac{21x^3}{7x}=3x^2,\ \frac{14x^2}{7x}=2x,\ \frac{7x}{7x}=1 \quad \text{This is what is left inside the parenthesis}$$

$$7x(3x^2+2x+1) \quad \text{Our Solution}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect, factoring will never make terms dissapear. Anything divided by itself is 1, be sure to not forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parenthesis as shown in the following two examples.

Example 268.

$$12x^5y^2-6x^4y^4+8x^3y^5 \quad \text{GCF is } 2x^3y^2\text{, put this in front of parenthesis and divide}$$

$$2x^3y^2(6x^2-3xy^2+4y^3) \quad \text{Our Solution}$$

Example 269.

$$18a^4b^3-27a^3b^3+9a^2b^3 \quad \text{GCF is } 9a^2b^3\text{, divide each term by this}$$

$$9a^2b^3(2a^2-3a+1) \quad \text{Our Solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

6.1 Practice - Greatest Common Factor

Factor the common factor out of each expression.

1) $9+8b^2$

2) $x-5$

3) $45x^2-25$

4) $1+2n^2$

5) $56-35p$

6) $50x-80y$

7) $7\text{ab}-35a^2b$

8) $27x^2y^5-72x^3y^2$

9) $-3a^2b+6a^3b^2$

10) $8x^3y^2+4x^3$

11) $-5x^2-5x^3-15x^4$

12) $-32n^9+32n^6+40n^5$

13) $20x^4-30x+30$

14) $21p^6+30p^2+27$

15) $28m^4+40m^3+8$

16) $-10x^4+20x^2+12x$

17) $30b^9+5ab-15a^2$

18) $27y^7+12y^2x+9y^2$

19) $-48a^2b^2-56a^3b-56a^5b$

20) $30m^6+15mn^2-25$

21) $20x^8y^2z^2+15x^5y^2z+35x^3y^3z$

22) $3p+12q-15q^2r^2$

23) $50x^2y+10y^2+70xz^2$

24) $30y^4z^3x^5+50y^4z^5-10y^4z^3x$

25) $30qpr-5qp+5q$

26) $28b+14b^2+35b^3+7b^5$

27) $-18n^5+3n^3-21n+3$

28) $30a^8+6a^5+27a^3+21a^2$

29) $-40x^{11}-20x^{12}+50x^{13}-50x^{14}$

30) $-24x^6-4x^4+12x^3+4x^2$

31) $-32mn^8+4m^6n+12mn^4+16mn$

32) $-10y^7+6y^{10}-4y^{10}x-8y^8x$

6.2

Factoring - Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a monomial like in the problem $5xy + 10xz$ the GCF is the monomial $5x$, so we would have $5x(y + 2z)$. However, a GCF does not have to be a monomial, it could be a binomial. To see this, consider the following two example.

Example 270.

$$\begin{array}{rl} 3ax - 7bx & \text{Both have } x \text{ in common, factor it out} \\ x(3a - 7b) & \text{Our Solution} \end{array}$$

Now the same problem, but instead of x we have $(2a + 5b)$.

Example 271.

$$\begin{array}{rl} 3a(2a+5b) - 7b(2a+5b) & \text{Both have } (2a+5b) \text{ in common, factor it out} \\ (2a+5b)(3a-7b) & \text{Our Solution} \end{array}$$

In the same way we factored out a GCF of x we can factor out a GCF which is a binomial, $(2a + 5b)$. This process can be extended to factor problems where there is no GCF to factor out, or after the GCF is factored out, there is more factoring that can be done. Here we will have to use another strategy to factor. We will use a process known as grouping. Grouping is how we will factor if there are four terms in the problem. Remember, factoring is like multiplying in reverse, so first we will look at a multiplication problem and then try to reverse the process.

Example 272.

$$\begin{array}{rl} (2a+3)(5b+2) & \text{Distribute } (2a+3) \text{ into second parenthesis} \\ 5b(2a+3) + 2(2a+3) & \text{Distribute each monomial} \\ 10\text{ab} + 15b + 4a + 6 & \text{Our Solution} \end{array}$$

The solution has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a + 3)$ and $2(2a + 3)$. When we are factoring by grouping we will always divide the problem into two parts, the first two terms and the last two terms. Then we can factor the GCF out of both the left and right sides. When we do this our hope is what is left in the parenthesis will match on both the left and right. If they match we can pull this matching GCF out front, putting the rest in parenthesis and we will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 273.

$$\begin{array}{rl} 10\text{ab}+15b+4a+6 & \text{Split problem into two groups} \\ \boxed{10\text{ab}+15b}\boxed{+4a+6} & \text{GCF on left is } 5b\text{, on the right is } 2 \\ \boxed{5b(2a+3)}\boxed{+2(2a+3)} & (2a+3) \text{ is the same! Factor out this GCF} \\ (2a+3)(5b+2) & \text{Our Solution} \end{array}$$

The key for grouping to work is after the GCF is factored out of the left and right, the two binomials must match exactly. If there is any difference between the two we either have to do some adjusting or it can't be factored using the grouping method. Consider the following example.

Example 274.

$$\begin{array}{rl} 6x^2+9xy-14x-21y & \text{Split problem into two groups} \\ \boxed{6x^2+9xy}\boxed{-14x-21y} & \text{GCF on left is } 3x\text{, on right is } 7 \\ \boxed{3x(2x+3y)}\boxed{+7(-2x-3y)} & \text{The signs in the parenthesis don't match!} \end{array}$$

when the signs don't match on both terms we can easily make them match by factoring the opposite of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second parenthesis.

$$\begin{array}{rl} \boxed{3x(2x+3y)}\boxed{-7(2x+3y)} & (2x+3y) \text{ is the same! Factor out this GCF} \\ (2x+3y)(3x-7) & \text{Our Solution} \end{array}$$

Often we can recognize early that we need to use the opposite of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If it is negative then we will use the opposite of the GCF to be sure they match.

Example 275.

$$\begin{array}{rl} 5xy-8x-10y+16 & \text{Split the problem into two groups} \\ \boxed{5xy-8x}\boxed{-10y+16} & \text{GCF on left is } x\text{, on right we need } a \text{ negative,} \\ & \text{so we use } -2 \\ \boxed{x(5y-8)}\boxed{-2(5y-8)} & (5y-8) \text{ is the same! Factor out this GCF} \\ (5y-8)(x-2) & \text{Our Solution} \end{array}$$

Sometimes when factoring the GCF out of the left or right side there is no GCF to factor out. In this case we will use either the GCF of 1 or -1. Often this is all we need to be sure the two binomials match.

Example 276.

$$\begin{array}{rl} 12ab-14a-6b+7 & \text{Split the problem into two groups} \\ \boxed{12\text{ab}-14a}\boxed{-6b+7} & \text{GCF on left is } 2a, \text{ on right, no GCF, use } -1 \\ \boxed{2a(6b-7)}\boxed{-1(6b-7)} & (6b-7) \text{ is the same! Factor out this GCF} \\ (6b-7)(2a-1) & \text{Our Solution} \end{array}$$

Example 277.

$$\begin{array}{rl} 6x^3-15x^2+2x-5 & \text{Split problem into two groups} \\ \boxed{6x^3-15x^2}\boxed{+2x-5} & \text{GCF on left is } 3x^2, \text{ on right, no GCF, use } 1 \\ \boxed{3x^2(2x-5)}\boxed{+1(2x-5)} & (2x-5) \text{ is the same! Factor out this GCF} \\ (2x-5)(3x^2+1) & \text{Our Solution} \end{array}$$

Another problem that may come up with grouping is after factoring out the GCF on the left and right, the binomials don't match, more than just the signs are different. In this case we may have to adjust the problem slightly. One way to do this is to change the order of the terms and try again. To do this we will move the second term to the end of the problem and see if that helps us use grouping.

Example 278.

$$\begin{array}{rl} 4a^2-21b^3+6ab-14ab^2 & \text{Split the problem into two groups} \\ \boxed{4a^2-21b^3}\boxed{+6ab-14ab^2} & \text{GCF on left is } 1, \text{ on right is } 2ab \\ \boxed{1(4a^2-21b^3)}\boxed{+2ab(3-7b)} & \text{Binomials don't match! Move second term to end} \\ 4a^2+6ab-14ab^2-21b^3 & \text{Start over, split the problem into two groups} \\ \boxed{4a^2+6\text{ab}}\boxed{-14ab^2-21b^3} & \text{GCF on left is } 2a, \text{ on right is } -7b^2 \\ \boxed{2a(2a+3b)}\boxed{-7b^2(2a+3b)} & (2a+3b) \text{ is the same! Factor out this GCF} \\ (2a+3b)(2a-7b^2) & \text{Our Solution} \end{array}$$

When rearranging terms the problem can still be out of order. Sometimes after factoring out the GCF the terms are backwards. There are two ways that this can happen, one with addition, one with subtraction. If it happens with addition, for example the binomials are $(a + b)$ and $(b + a)$, we don't have to do any extra work. This is because addition is the same in either order $(5+3=3+5=8)$.

Example 279.

$$\begin{array}{rl}
7+y-3xy-21x & \text{Split the problem into two groups} \\
\boxed{7+y}\boxed{-3xy-21x} & \text{GCF on left is } 1, \text{ on the right is } -3x \\
\boxed{1(7+y)}\boxed{-3x(y+7)} & y+7 \text{ and } 7+y \text{ are the same, use either one} \\
(y+7)(1-3x) & \text{Our Solution}
\end{array}$$

However, if the binomial has subtraction, then we need to be a bit more careful. For example, if the binomials are $(a-b)$ and $(b-a)$, we will factor out the opposite of the GCF on one part, usually the second. Notice what happens when we factor out -1.

Example 280.

$$\begin{array}{rl}
(b-a) & \text{Factor out } -1 \\
-1(-b+a) & \text{Addition can be in either order, switch order} \\
-1(a-b) & \text{The order of the subtraction has been switched!}
\end{array}$$

Generally we won't show all the above steps, we will simply factor out the opposite of the GCF and switch the order of the subtraction to make it match the other binomial.

Example 281.

$$\begin{array}{rl}
8xy-12y+15-10x & \text{Split the problem into two groups} \\
\boxed{8xy-12y}\boxed{15-10x} & \text{GCF on left is } 4y, \text{ on right, } 5 \\
\boxed{4y(2x-3)}\boxed{+5(3-2x)} & \text{Need to switch subtraction order, use } -5 \text{ in middle} \\
\boxed{4y(2y-3)}\boxed{-5(2x-3)} & \text{Now } 2x-3 \text{ match on both! Factor out this GCF} \\
(2x-3)(4y-5) & \text{Our Solution}
\end{array}$$

World View Note: Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19th century. She also did research on how the rings of Saturn rotated.

6.2 Practice - Grouping

Factor each completely.

1) $40r^3 - 8r^2 - 25r + 5$

2) $35x^3 - 10x^2 - 56x + 16$

3) $3n^3 - 2n^2 - 9n + 6$

4) $14v^3 + 10v^2 - 7v - 5$

5) $15b^3 + 21b^2 - 35b - 49$

6) $6x^3 - 48x^2 + 5x - 40$

7) $3x^3 + 15x^2 + 2x + 10$

8) $28p^3 + 21p^2 + 20p + 15$

9) $35x^3 - 28x^2 - 20x + 16$

10) $7n^3 + 21n^2 - 5n - 15$

11) $7xy - 49x + 5y - 35$

12) $42r^3 - 49r^2 + 18r - 21$

13) $32xy + 40x^2 + 12y + 15x$

14) $15ab - 6a + 5b^3 - 2b^2$

15) $16xy - 56x + 2y - 7$

16) $3mn - 8m + 15n - 40$

17) $2xy - 8x^2 + 7y^3 - 28y^2x$

18) $5mn + 2m - 25n - 10$

19) $40xy + 35x - 8y^2 - 7y$

20) $8xy + 56x - y - 7$

21) $32uv - 20u + 24v - 15$

22) $4uv + 14u^2 + 12v + 42u$

23) $10xy + 30 + 25x + 12y$

24) $24xy + 25y^2 - 20x - 30y^3$

25) $3uv + 14u - 6u^2 - 7v$

26) $56ab + 14 - 49a - 16b$

27) $16xy - 3x - 6x^2 + 8y$

6.3

Factoring - Trinomials where a = 1

Objective: Factor trinomials where the coefficient of x^2 is one.

Factoring with three terms, or trinomials, is the most important type of factoring to be able to master. As factoring is multiplication backwards we will start with a multipication problem and look at how we can reverse the process.

Example 282.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parenthesis
$x(x+6)-4(x+6)$	Distribute each monomial through parenthesis
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our Solution

You may notice that if you reverse the last three steps the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the GCF of the second two terms is -4. The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, the same problem worked backwards

Example 283.

$x^2+2x-24$	Split middle term into $+6x-4x$
$x^2+6x-4x-24$	Grouping: GCF on left is x, on right is -4
$x(x+6)-4(x+6)$	$(x+6)$ is the same, factor out this GCF
$(x+6)(x-4)$	Our Solution

The trick to make these problems work is how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that works! So how do we know what is the one combination that works? To find the correct way to split the middle term we will use what is called the ac method. In the next lesson we will discuss why it is called the ac method. The way the ac method works is we find a pair of numers that multiply to a certain number and add to another number. Here we will try to multiply to get the last term and add to get the coefficient of the middle term. In the previous example that would mean we wanted to multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4 ($6\cdot-4=-24$ and $6+(-4)=2$). This process is shown in the next few examples

Example 284.

$$\begin{array}{rl} x^2+9x+18 & \text{Want to multiply to } 18, \text{ add to } 9 \\ x^2+6x+3x+18 & 6 \text{ and } 3, \text{ split the middle term} \\ x(x+6)+3(x+6) & \text{Factor by grouping} \\ (x+6)(x+3) & \text{Our Solution} \end{array}$$

Example 285.

$$\begin{array}{rl} x^2-4x+3 & \text{Want to multiply to } 3, \text{ add to } -4 \\ x^2-3x-x+3 & -3 \text{ and } -1, \text{ split the middle term} \\ x(x-3)-1(x-3) & \text{Factor by grouping} \\ (x-3)(x-1) & \text{Our Solution} \end{array}$$

Example 286.

$$\begin{array}{rl} x^2-8x-20 & \text{Want to multiply to } -20, \text{ add to } -8 \\ x^2-10x+2x-20 & -10 \text{ and } 2, \text{ split the middle term} \\ x(x-10)+2(x-10) & \text{Factor by grouping} \\ (x-10)(x+2) & \text{Our Solution} \end{array}$$

Often when factoring we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term

Example 287.

$$\begin{array}{rl} a^2-9ab+14b^2 & \text{Want to multiply to } 14, \text{ add to } -9 \\ a^2-7ab-2ab+14b^2 & -7 \text{ and } -2, \text{ split the middle term} \\ a(a-7b)-2b(a-7b) & \text{Factor by grouping} \\ (a-7b)(a-2b) & \text{Our Solution} \end{array}$$

As the past few examples illustrate, it is very important to be aware of negatives as we find the pair of numbers we will use to split the middle term. Consier the following example, done incorrectly, ignoring negative signs

Warning 288.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } 6, \text{add } 5 \\ x^2+2x+3x-6 & 2 \text{ and } 3, \text{split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not use the negative sign with the six to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly.

Example 289.

$$\begin{array}{rl} x^2+5x-6 & \text{Want to multiply to } -6, \text{add to } 5 \\ x^2+6x-x-6 & 6 \text{ and } -1, \text{split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our Solution} \end{array}$$

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1, our factors turned out to be $(x+6)(x-1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it however, when we have no number (technically we have a 1) in front of x^2. In all the problems we have factored in this lesson there is no number in front of x^2. If this is the case then we can use this shortcut. This is shown in the next few examples.

Example 290.

$$\begin{array}{rl} x^2-7x-18 & \text{Want to multiply to } -18, \text{add to } -7 \\ & -9 \text{ and } 2, \text{write the factors} \\ (x-9)(x+2) & \text{Our Solution} \end{array}$$

Example 291.

$$\begin{array}{rl} m^2 - mn - 30n^2 & \text{Want to multiply to} - 30, \text{add to} - 1 \\ & 5 \text{ and} - 6, \text{write the factors, don't forget second variable} \\ (m+5n)(m-6n) & \text{Our Solution} \end{array}$$

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say we cannot factor the polynomial, or we say the polynomial is prime. This is shown in the following example.

Example 292.

$$\begin{array}{rl} x^2 + 2x + 6 & \text{Want to multiply to } 6, \text{add to } 2 \\ 1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to six, none add to } 2 \\ \text{Prime, can't factor} & \text{Our Solution} \end{array}$$

When factoring it is important not to forget about the GCF. If all the terms in a problem have a common factor we will want to first factor out the GCF before we factor using any other method.

Example 293.

$$\begin{array}{rl} 3x^2 - 24x + 45 & \text{GCF of all terms is } 3, \text{factor this out} \\ 3(x^2 - 8x + 15) & \text{Want to multiply to } 15, \text{add to} - 8 \\ & -5 \text{ and} - 3, \text{write the factors} \\ 3(x-5)(x-3) & \text{Our Solution} \end{array}$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if there is no coefficient on x^2. In the next lesson we will look at how this process changes slightly when we have a number in front of x^2. Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

6.3 Practice - Trinomials where a = 1

Factor each completely.

1) $p^2+17p+72$

2) x^2+x-72

3) n^2-9n+8

4) x^2+x-30

5) $x^2-9x-10$

6) $x^2+13x+40$

7) $b^2+12b+32$

8) $b^2-17b+70$

9) $x^2+3x-70$

10) $x^2+3x-18$

11) $n^2-8n+15$

12) $a^2-6a-27$

13) $p^2+15p+54$

14) $p^2+7p-30$

15) $n^2-15n+56$

16) $m^2-15mn+50n^2$

17) $u^2-8uv+15v^2$

18) $m^2-3mn-40n^2$

19) $m^2+2mn-8n^2$

20) $x^2+10xy+16y^2$

21) $x^2-11xy+18y^2$

22) $u^2-9uv+14v^2$

23) $x^2+xy-12y^2$

24) $x^2+14xy+45y^2$

25) $x^2+4xy-12y^2$

26) $4x^2+52x+168$

27) $5a^2+60a+100$

28) $5n^2-45n+40$

29) $6a^2+24a-192$

30) $5v^2+20v-25$

31) $6x^2+18xy+12y^2$

32) $5m^2+30mn-90n^2$

33) $6x^2+96xy+378y^2$

34) $6m^2-36mn-162n^2$

6.4

Factoring - Trinomials where a $\neq$ 1

Objective: Factor trinomials using the ac method when the coefficient of x^2 is not one.

When factoring trinomials we used the ac method to split the middle term and then factor by grouping. The ac method gets it's name from the general trinomial equation, $ax^2 + bx + c$, where a, b, and c are the numbers in front of x^2, x and the constant at the end respectively.

World View Note: It was French philosopher Rene Descartes who first used letters from the beginning of the alphabet to represent values we know (a, b, c) and letters from the end to represent letters we don't know and are solving for (x, y, z).

The ac method is named ac because we multiply $a \cdot c$ to find out what we want to multiply to. In the previous lesson we always multiplied to just c because there was no number in front of x^2. This meant the number was 1 and we were multiplying to 1c or just c. Now we will have a number in front of x^2 so we will be looking for numbers that multiply to ac and add to b. Other than this, the process will be the same.

Example 294.

$$\begin{array}{rl} 3x^2+11x+6 & \text{Multiply to } ac \text{ or } (3)(6)=18, \text{add to } 11 \\ 3x^2+9x+2x+6 & \text{The numbers are 9 and 2, split the middle term} \\ 3x(x+3)+2(x+3) & \text{Factor by grouping} \\ (x+3)(3x+2) & \text{Our Solution} \end{array}$$

When $a = 1$, or no coefficient in front of x^2, we were able to use a shortcut, using the numbers that split the middle term in the factors. The previous example illustrates an important point, the shortcut does not work when $a \neq 1$. We must go through all the steps of grouping in order to factor the problem.

Example 295.

$$\begin{array}{rl} 8x^2-2x-15 & \text{Multiply to } ac \text{ or } (8)(-15)=-120, \text{add to } -2 \\ 8x^2-12x+10x-15 & \text{The numbers are } -12 \text{ and } 10, \text{split the middle term} \\ 4x(2x-3)+5(2x-3) & \text{Factor by grouping} \\ (2x-3)(4x+5) & \text{Our Solution} \end{array}$$

Example 296.

$$\begin{array}{rl} 10x^2-27x+5 & \text{Multiply to } ac \text{ or } (10)(5)=50, \text{add to} -27 \\ 10x^2-25x-2x+5 & \text{The numbers are} -25 \text{ and} -2, \text{split the middle term} \\ 5x(2x-5)-1(2x-5) & \text{Factor by grouping} \\ (2x-5)(5x-1) & \text{Our Solution} \end{array}$$

The same process works with two variables in the problem

Example 297.

$$\begin{array}{rl} 4x^2-xy-5y^2 & \text{Multiply to } ac \text{ or } (4)(-5)=-20, \text{add to} -1 \\ 4x^2+4xy-5xy-5y^2 & \text{The numbers are 4 and} -5, \text{split the middle term} \\ 4x(x+y)-5y(x+y) & \text{Factor by grouping} \\ (x+y)(4x-5y) & \text{Our Solution} \end{array}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac method. Factoring out the GCF first also has the added bonus of making the numbers smaller so the ac method becomes easier.

Example 298.

$$\begin{array}{rl} 18x^3+33x^2-30x & \text{GCF} = 3x, \text{factor this out first} \\ 3x[6x^2+11x-10] & \text{Multiply to } ac \text{ or } (6)(-10)=-60, \text{add to } 11 \\ 3x[6x^2+15x-4x-10] & \text{The numbers are 15 and} -4, \text{split the middle term} \\ 3x[3x(2x+5)-2(2x+5)] & \text{Factor by grouping} \\ 3x(2x+5)(3x-2) & \text{Our Solution} \end{array}$$

As was the case with trinomials when $a=1$, not all trinomials can be factored. If there is no combinations that multiply and add correctly then we can say the trinomial is prime and cannot be factored.

Example 299.

$$\begin{array}{rl} 3x^2+2x-7 & \text{Multiply to } ac \text{ or } (3)(-7)=-21, \text{add to } 2 \\ -3(7) \text{ and} -7(3) & \text{Only two ways to multiply to} -21, \text{it doesn}'t \text{ add to } 2 \\ \text{Prime, cannot be factored} & \text{Our Solution} \end{array}$$

6.4 Practice - Trinomials where a $\neq$ 1

Factor each completely.

1) $7x^2-48x+36$

2) $7n^2-44n+12$

3) $7b^2+15b+2$

4) $7v^2-24v-16$

5) $5a^2-13a-28$

6) $5n^2-4n-20$

7) $2x^2-5x+2$

8) $3r^2-4r-4$

9) $2x^2+19x+35$

10) $7x^2+29x-30$

11) $2b^2-b-3$

12) $5k^2-26k+24$

13) $5k^2+13k+6$

14) $3r^2+16r+21$

15) $3x^2-17x+20$

16) $3u^2+13uv-10v^2$

17) $3x^2+17xy+10y^2$

18) $7x^2-2xy-5y^2$

19) $5x^2+28xy-49y^2$

20) $5u^2+31uv-28v^2$

21) $6x^2-39x-21$

22) $10a^2-54a-36$

23) $21k^2-87k-90$

24) $21n^2+45n-54$

25) $14x^2-60x+16$

26) $4r^2+r-3$

27) $6x^2+29x+20$

28) $6p^2+11p-7$

29) $4k^2-17k+4$

30) $4r^2+3r-7$

31) $4x^2+9xy+2y^2$

32) $4m^2+6mn+6n^2$

33) $4m^2-9mn-9n^2$

34) $4x^2-6xy+30y^2$

35) $4x^2+13xy+3y^2$

36) $18u^2-3uv-36v^2$

37) $12x^2+62xy+70y^2$

38) $16x^2+60xy+36y^2$

39) $24x^2-52xy+8y^2$

40) $12x^2+50xy+28y^2$

6.5

Factoring - Factoring Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring there are a few special products that, if we can recognize them, can help us factor polynomials. The first is one we have seen before. When multiplying special products we found that a sum and a difference could multiply to a difference of squares. Here we will use this special product to help us factor

Difference of Squares: $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

If we are subtracting two perfect squares then it will always factor to the sum and difference of the square roots.

Example 300.

$x^2 - 16$	Subtracting two perfect squares, the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 301.

$9a^2 - 25b^2$	Subtracting two perfect squares, the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

It is important to note, that a sum of squares will never factor. It is always prime. This can be seen if we try to use the ac method to factor $x^2 + 36$.

Example 302.

$x^2 + 36$	No bx term, we use $0x$.
$x^2 + 0x + 36$	Multiply to 36, add to 0
$1 \cdot 36, 2 \cdot 18, 3 \cdot 12, 4 \cdot 9, 6 \cdot 6$	No combinations that multiply to 36 add to 0
Prime, cannot factor	Our Solution

It turns out that a sum of squares is always prime.

$$\textbf{Sum of Squares: } \boldsymbol{a^2 + b^2 = \textbf{Prime}}$$

A great example where we see a sum of squares comes from factoring a difference of 4th powers. Because the square root of a fourth power is a square ($\sqrt{a^4} = a^2$), we can factor a difference of fourth powers just like we factor a difference of squares, to a sum and difference of the square roots. This will give us two factors, one which will be a prime sum of squares, and a second which will be a difference of squares which we can factor again. This is shown in the following examples.

Example 303.

$a^4 - b^4$	Difference of squares with roots a^2 and b^2
$(a^2 + b^2)(a^2 - b^2)$	The first factor is prime, the second is a difference of squares!
$(a^2 + b^2)(a + b)(a - b)$	Our Solution

Example 304.

$x^4 - 16$	Difference of squares with roots x^2 and 4
$(x^2 + 4)(x^2 - 4)$	The first factor is prime, the second is a difference of squares!
$(x^2 + 4)(x + 2)(x - 2)$	Our Solution

Another factoring shortcut is the perfect square. We had a shortcut for multiplying a perfect square which can be reversed to help us factor a perfect square

$$\textbf{Perfect Square: } \boldsymbol{a^2 + 2\textbf{ab} + b^2 = (a + b)^2}$$

A perfect square can be difficult to recognize at first glance, but if we use the ac method and get two of the same numbers we know we have a perfect square. Then we can just factor using the square roots of the first and last terms and the sign from the middle. This is shown in the following examples.

Example 305.

$x^2 - 6x + 9$	Multiply to 9, add to -6
	The numbers are -3 and -3, the same! Perfect square
$(x - 3)^2$	Use square roots from first and last terms and sign from the middle

Example 306.

$$\begin{array}{rl} 4x^2+20xy+25y^2 & \text{Multiply to 100, add to 20} \\ & \text{The numbers are 10 and 10, the same! Perfect square} \\ (2x+5y)^2 & \text{Use square roots from first and last terms and sign from the middle} \end{array}$$

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) $3x+2y+z=29$.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas

$$\textbf{Sum of Cubes: } \boldsymbol{a^3+b^3=(a+b)(a^2-\text{ab}+b^2)}$$

$$\textbf{Difference of Cubes: } \boldsymbol{a^3-b^3=(a-b)(a^2+\text{ab}+b^2)}$$

Comparing the formulas you may notice that the only difference is the signs in between the terms. One way to keep these two formulas straight is to think of SOAP. S stands for Same sign as the problem. If we have a sum of cubes, we add first, a difference of cubes we subtract first. O stands for Opposite sign. If we have a sum, then subtraction is the second sign, a difference would have addition for the second sign. Finally, AP stands for Always Positive. Both formulas end with addition. The following examples show factoring with cubes.

Example 307.

$$\begin{array}{rl} m^3-27 & \text{We have cube roots } m \text{ and } 3 \\ (m\quad 3)(m^2\quad 3m\quad 9) & \text{Use formula, use SOAP to fill in signs} \\ (m-3)(m^2+3m+9) & \text{Our Solution} \end{array}$$

Example 308.

$$\begin{array}{rl} 125p^3+8r^3 & \text{We have cube roots } 5p \text{ and } 2r \\ (5p\quad 2r)(25p^2\quad 10r\quad 4r^2) & \text{Use formula, use SOAP to fill in signs} \\ (5p+2r)(25p^2-10r+4r^2) & \text{Our Solution} \end{array}$$

The previous example illustrates an important point. When we fill in the trinomial's first and last terms we square the cube roots $5p$ and $2r$. Often students forget to square the number in addition to the variable. Notice that when done correctly, both get cubed.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial further. As a general rule, this factor will always be prime (unless there is a GCF which should have been factored out before using cubes rule).

The following table sumarizes all of the shortcuts that we can use to factor special products

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares	$a^2 + b^2 = \text{Prime}$
Perfect Square	$a^2 + 2ab + b^2 = (a+b)^2$
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This is shown in the following examples

Example 309.

$72x^2 - 2$	GCF is 2
$2(36x^2 - 1)$	Difference of Squares, square roots are $6x$ and 1
$2(6x+1)(6x-1)$	Our Solution

Example 310.

$48x^2y - 24xy + 3y$	GCF is $3y$
$3y(16x^2 - 8x + 1)$	Multiply to 16 add to 8
	The numbers are 4 and 4, the same! Perfect Square
$3y(4x-1)^2$	Our Solution

Example 311.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$
$2ab^2(64a^3 + 27b^3)$	Sum of cubes! Cube roots are $4a$ and $3b$
$2\text{ab}^2(4a+3b)(16a^2 - 12ab + 9b^2)$	Our Solution

6.5 Practice - Factoring Special Products

Factor each completely.

1) r^2-16

2) x^2-9

3) v^2-25

4) x^2-1

5) p^2-4

6) $4v^2-1$

7) $9k^2-4$

8) $9a^2-1$

9) $3x^2-27$

10) $5n^2-20$

11) $16x^2-36$

12) $125x^2+45y^2$

13) $18a^2-50b^2$

14) $4m^2+64n^2$

15) a^2-2a+1

16) k^2+4k+4

17) x^2+6x+9

18) $n^2-8n+16$

19) x^2-6x+9

20) k^2-4k+4

21) $25p^2-10p+1$

22) x^2+2x+1

23) $25a^2+30ab+9b^2$

24) $x^2+8xy+16y^2$

25) $4a^2-20ab+25b^2$

26) $18m^2-24mn+8n^2$

27) $8x^2-24xy+18y^2$

28) $20x^2+20xy+5y^2$

29) $8-m^3$

30) x^3+64

31) x^3-64

32) x^3+8

33) $216-u^3$

34) $125x^3-216$

35) $125a^3-64$

36) $64x^3-27$

37) $64x^3+27y^3$

38) $32m^3-108n^3$

39) $54x^3+250y^3$

40) $375m^3+648n^3$

41) a^4-81

42) x^4-256

43) $16-z^4$

44) n^4-1

45) x^4-y^4

46) $16a^4-b^4$

47) m^4-81b^4

48) $81c^4-16d^4$

6.6

Factoring - Factoring Strategy

Objective: Idenfity and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which tool to use when. Here we will attempt to organize all the different factoring types we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!!)

- **2 terms:** sum or difference of squares or cubes:

  $a^2 - b^2 = (a+b)(a-b)$

  $a^2 + b^2 = \text{Prime}$

  $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  $a^3 - b^3 = (a-b)(a^2 + \text{ab} + b^2)$

- **3 terms:** ac method, watch for perfect square!

  $a^2 + 2ab + b^2 = (a+b)^2$

  Multiply to ac and add to b

- **4 terms:** grouping

We will use the above strategy to factor each of the following examples. Here the emphasis will be on which strategy to use rather than the steps used in that method.

Example 312.

$$\begin{array}{rl} 4x^2 + 56xy + 196y^2 & \text{GCF first, 4} \\ 4(x^2 + 14xy + 49y^2) & \text{Three terms, try ac method, multiply to 49, add to 14} \\ & \text{7 and 7, perfect square!} \end{array}$$

$$4(x+7y)^2 \quad \text{Our Solution}$$

Example 313.

$$\begin{array}{rl} 5x^2y+15xy-35x^2-105x & \text{GCF first, } 5x \\ 5x(xy+3y-7x-21) & \text{Four terms, try grouping} \\ 5x[y(x+3)-7(x+3)] & (x+3) \text{ match!} \\ 5x(x+3)(y-7) & \text{Our Solution} \end{array}$$

Example 314.

$$\begin{array}{rl} 100x^2-400 & \text{GCF first, } 100 \\ 100(x^2-4) & \text{Two terms, difference of squares} \\ 100(x+4)(x-4) & \text{Our Solution} \end{array}$$

Example 315.

$$\begin{array}{rl} 108x^3y^2-39x^2y^2+3xy^2 & \text{GCF first, } 3xy^2 \\ 3xy^2(36x^2-13x+1) & \text{Thee terms, ac method, multiply to 36, add to } -13 \\ 3xy^2(36x^2-9x-4x+1) & -9 \text{ and } -4\text{, split middle term} \\ 3xy^2[9x(4x-1)-1(4x-1)] & \text{Factor by grouping} \\ 3xy^2(4x-1)(9x-1) & \text{Our Solution} \end{array}$$

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

Example 316.

$$\begin{array}{rl} 5+625y^3 & \text{GCF first, } 5 \\ 5(1+125y^3) & \text{Two terms, sum of cubes} \\ 5(1+5y)(1-5y+25y^2) & \text{Our Solution} \end{array}$$

It is important to be comfortable and confident not just with using all the factoring methods, but decided on which method to use. This is why practice is very important!

6.6 Practice - Factoring Strategy

Factor each completely.

1) $24az - 18ah + 60yz - 45yh$

2) $2x^2 - 11x + 15$

3) $5u^2 - 9uv + 4v^2$

4) $16x^2 + 48xy + 36y^2$

5) $-2x^3 + 128y^3$

6) $20uv - 60u^3 - 5xv + 15xu^2$

7) $5n^3 + 7n^2 - 6n$

8) $2x^3 + 5x^2y + 3y^2x$

9) $54u^3 - 16$

10) $54 - 128x^3$

11) $n^2 - n$

12) $5x^2 - 22x - 15$

13) $x^2 - 4xy + 3y^2$

14) $45u^2 - 150uv + 125v^2$

15) $9x^2 - 25y^2$

16) $x^3 - 27y^3$

17) $m^2 - 4n^2$

18) $12ab - 18a + 6nb - 9n$

19) $36b^2c - 16xd - 24b^2d + 24xc$

20) $3m^3 - 6m^2n - 24n^2m$

21) $128 + 54x^3$

22) $64m^3 + 27n^3$

23) $2x^3 + 6x^2y - 20y^2x$

24) $3ac + 15ad^2 + x^2c + 5x^2d^2$

25) $n^3 + 7n^2 + 10n$

26) $64m^3 - n^3$

27) $27x^3 - 64$

28) $16a^2 - 9b^2$

29) $5x^2 + 2x$

30) $2x^2 - 10x + 12$

31) $3k^3 - 27k^2 + 60k$

32) $32x^2 - 18y^2$

33) $mn - 12x + 3m - 4xn$

34) $2k^2 + k - 10$

35) $16x^2 - 8\text{xy} + y^2$

36) $v^2 + v$

37) $27m^2 - 48n^2$

38) $x^3 + 4x^2$

39) $9x^3 + 21x^2y - 60y^2x$

40) $9n^3 - 3n^2$

41) $2m^2 + 6mn - 20n^2$

42) $2u^2v^2 - 11uv^3 + 15v^4$

6.7

Factoring - Solve by Factoring

Objective: Solve quadratic equation by factoring and using the zero product rule.

When solving linear equations such as $2x - 5 = 21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x) we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the varaible is known as the zero product rule

Zero Product Rule: If $ab = 0$ then either $a = 0$ or $b = 0$

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 317.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero}\\
2x-3=0 \;\text{ or }\; 5x+1=0 & \text{Set each factor equal to zero}\\
\underline{+3+3} \qquad\quad \underline{\;-1-1} & \text{Solve each equation}\\
2x=3 \;\text{ or }\; 5x=-1 & \\
\overline{\;2\;}\;\overline{\;2\;} \qquad \overline{\;5\;}\quad\overline{\;5\;} & \\
x=\dfrac{3}{2} \;\text{ or }\; \dfrac{-1}{5} & \text{Our Solution}
\end{array}$$

For the zero product rule to work we must have factors to set equal to zero. This means if the problem is not already factored we will factor it first.

Example 318.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the ac method, multiply to} -12\text{, add to }1\\
4x^2-3x+4x-3=0 & \text{The numbers are} -3 \text{ and } 4\text{, split the middle term}\\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}\\
(4x-3)(x+1)=0 & \text{One factor must be zero}\\
4x-3=0 \;\text{ or }\; x+1=0 & \text{Set each factor equal to zero}
\end{array}$$

$$\begin{array}{rlr}
\underline{+3+3} \quad \underline{-1-1} & \text{Solve each equation} \\
4x=3 \text{ or } x=-1 & \\
\overline{4}\ \overline{4} & \\
x=\dfrac{3}{4} \text{ or } -1 & \text{Our Solution}
\end{array}$$

Another important part of the zero product rule is that before we factor, the equation must equal zero. If it does not, we must move terms around so it does equal zero. Generally we like the x^2 term to be positive.

Example 319.

$$\begin{array}{rl}
x^2=8x-15 & \text{Set equal to zero by moving terms to the left} \\
\underline{-8x+15} \quad \underline{-8x+15} & \\
x^2-8x+15=0 & \text{Factor using the ac method, multiply to 15, add to} -8 \\
(x-5)(x-3)=0 & \text{The numbers are} -5 \text{ and} -3 \\
x-5=0 \text{ or } x-3=0 & \text{Set each factor equal to zero} \\
\underline{+5+5} \qquad \underline{+3+3} & \text{Solve each equation} \\
x=5 \text{ or } x=3 & \text{Our Solution}
\end{array}$$

Example 320.

$$\begin{array}{rl}
(x-7)(x+3)=-9 & \text{Not equal to zero, multiply first, use FOIL} \\
x^2-7x+3x-21=-9 & \text{Combine like terms} \\
x^2-4x-21=-9 & \text{Move} -9 \text{ to other side so equation equals zero} \\
\underline{+9 \quad +9} & \\
x^2-4x-12=0 & \text{Factor using the ac method, mutiply to} -12\text{, add to} -4 \\
(x-6)(x+2)=0 & \text{The numbers are 6 and} -2 \\
x-6=0 \text{ or } x+2=0 & \text{Set each factor equal to zero} \\
\underline{+6+6} \qquad \underline{-2-2} & \text{Solve each equation} \\
x=6 \text{ or } -2 & \text{Our Solution}
\end{array}$$

Example 321.

$$\begin{array}{rl}
3x^2+4x-5=7x^2+4x-14 & \text{Set equal to zero by moving terms to the right} \\
\underline{-3x^2-4x+5} \quad \underline{-3x^2-4x+5} & \\
0=4x^2-9 & \text{Factor using difference of squares}
\end{array}$$

$$\begin{array}{rl}
0 = (2x+3)(2x-3) & \text{One factor must be zero} \\
2x+3=0 \text{ or } 2x-3=0 & \text{Set each factor equal to zero} \\
\underline{-3-3} \qquad \underline{+3+3} & \text{Solve each equation} \\
2x=-3 \text{ or } 2x=3 & \\
\overline{2} \quad \overline{2} \qquad \overline{2} \quad \overline{2} & \\
x = \frac{-3}{2} \text{ or } \frac{3}{2} & \text{Our Solution}
\end{array}$$

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 322.

$$\begin{array}{rl}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to left} \\
\underline{-12x+9} \quad \underline{-12x+9} & \\
4x^2 - 12x + 9 = 0 & \text{Factor using the ac method, multiply to 36, add to} -12 \\
(2x-3)^2 = 0 & -6 \text{ and } -6, a \text{ perfect square!} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\underline{+3+3} & \text{Solve the equation} \\
2x = 3 & \\
\overline{2} \quad \overline{2} & \\
x = \frac{3}{2} & \text{Our Solution}
\end{array}$$

As always it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solution. The next few examples illustrate this.

Example 323.

$$\begin{array}{rl}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
-8x - 8x & \text{Be careful, on the right side, they are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x-2) = 0 & \text{One factor must be zero} \\
4x = 0 \text{ or } x - 2 = 0 & \text{Set each factor equal to zero} \\
\overline{4} \quad \overline{4} \qquad \underline{+2+2} & \text{Solve each equation} \\
x = 0 \text{ or } 2 & \text{Our Solution}
\end{array}$$

Example 324.

$$\begin{array}{rl}
2x^3-14x^2+24x=0 & \text{Factor out the GCF of } 2x\\
2x(x^2-7x+12)=0 & \text{Factor with ac method, multiply to } 12, \text{ add to } -7\\
2x(x-3)(x-4)=0 & \text{The numbers are } -3 \text{ and } -4\\
2x=0 \ \text{ or } \ x-3=0 \ \text{ or } \ x-4=0 & \text{Set each factor equal to zero}\\
\overline{2}\ \ \overline{2} \qquad \underline{+3+3} \qquad \underline{+4+4} & \text{Solve each equation}\\
x=0 \ \text{ or } \ 3 \ \text{ or } \ 4 & \text{Our Solutions}
\end{array}$$

Example 325.

$$\begin{array}{rl}
6x^2+21x-27=0 & \text{Factor out the GCF of } 3\\
3(2x^2+7x-9)=0 & \text{Factor with ac method, multiply to } -18, \text{ add to } 7\\
3(2x^2+9x-2x-9)=0 & \text{The numbers are } 9 \text{ and } -2\\
3[x(2x+9)-1(2x+9)]=0 & \text{Factor by grouping}\\
3(2x+9)(x-1)=0 & \text{One factor must be zero}\\
3=0 \ \text{ or } \ 2x+9=0 \ \text{ or } \ x-1=0 & \text{Set each factor equal to zero}\\
3\neq 0 \qquad \underline{-9-9} \qquad \underline{+1+1} & \text{Solve each equation}\\
2x=-9 \ \text{ or } \ x=1 & \\
\overline{2}\ \ \overline{2} \qquad\qquad & \\
x=-\dfrac{9}{2} \ \text{ or } \ 1 & \text{Our Solution}
\end{array}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. Often a student will skip setting the GCF factor equal to zero if there is no variables in the GCF.

Just as not all polynomials cannot factor, all equations cannot be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another section.

World View Note: While factoring works great to solve problems with x^2, Tartaglia, in 16th century Italy, developed a method to solve problems with x^3. He kept his method a secret until another mathematician, Cardan, talked him out of his secret and published the results. To this day the formula is known as Cardan's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that we have not covered yet and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only give us one of the two answers. When we talk about roots we will come back to problems like these and see how we can solve using square roots in a method called completing the square. For now, **never** take the square root of both sides!

6.7 Practice - Solve by Factoring

Solve each equation by factoring.

1) $(k-7)(k+2)=0$

2) $(a+4)(a-3)=0$

3) $(x-1)(x+4)=0$

4) $(2x+5)(x-7)=0$

5) $6x^2-150=0$

6) $p^2+4p-32=0$

7) $2n^2+10n-28=0$

8) $m^2-m-30=0$

9) $7x^2+26x+15=0$

10) $40r^2-285r-280=0$

11) $5n^2-9n-2=0$

12) $2b^2-3b-2=0$

13) $x^2-4x-8=-8$

14) $v^2-8v-3=-3$

15) $x^2-5x-1=-5$

16) $a^2-6a+6=-2$

17) $49p^2+371p-163=5$

18) $7k^2+57k+13=5$

19) $7x^2+17x-20=-8$

20) $4n^2-13n+8=5$

21) $7r^2+84=-49r$

22) $7m^2-224=28m$

23) $x^2-6x=16$

24) $7n^2-28n=0$

25) $3v^2+7v=40$

26) $6b^2=5+7b$

27) $35x^2+120x=-45$

28) $9n^2+39n=-36$

29) $4k^2+18k-23=6k-7$

30) $a^2+7a-9=-3+6a$

31) $9x^2-46+7x=7x+8x^2+3$

32) $x^2+10x+30=6$

33) $2m^2+19m+40=-2m$

34) $5n^2+41n+40=-2$

35) $40p^2+183p-168=p+5p^2$

36) $24x^2+11x-80=3x$

Chapter 7 : Rational Expressions

7.1

Rational Expressions - Reduce Rational Expressions

Objective: Reduce rational expressions by dividing out common factors.

Rational expressions are expressions written as a quotient of polynomials. Examples of rational expressions include:

$$\frac{x^2 - x - 12}{x^2 - 9x + 20} \quad \text{and} \quad \frac{3}{x - 2} \quad \text{and} \quad \frac{a - b}{b - a} \quad \text{and} \quad \frac{3}{2}$$

As rational expressions are a special type of fraction, it is important to remember with fractions we cannot have zero in the denominator of a fraction. For this reason, rational expressions may have one more excluded values, or values that the variable cannot be or the expression would be undefined.

Example 326.

$$\begin{array}{rl}
\text{State the excluded value}(s)\text{:}\ \dfrac{x^2 - 1}{3x^2 + 5x} & \text{Denominator cannot be zero} \\
3x^2 + 5x \neq 0 & \text{Factor} \\
x(3x + 5) \neq 0 & \text{Set each factor not equal to zero} \\
x \neq 0 \ \text{ or } \ 3x + 5 \neq 0 & \text{Subtract 5 from second equation} \\
\underline{-5 \ -5} & \\
3x \neq -5 & \text{Divide by 3} \\
\overline{3} \quad \overline{3} & \\
x \neq \dfrac{-5}{3} & \text{Second equation is solved} \\
x \neq 0 \ \text{ or } \ \dfrac{-5}{3} & \text{Our Solution}
\end{array}$$

This means we can use any value for x in the equation except for 0 and $\frac{-5}{3}$. We can however, evaluate any other value in the expression.

World View Note: The number zero was not widely accepted in mathematical thought around the world for many years. It was the Mayans of Central America who first used zero to aid in the use of their base 20 system as a place holder!

Rational expressions are easily evaluated by simply substituting the value for the variable and using order of operations.

Example 327.

$$\frac{x^2-4}{x^2+6x+8} \text{ when } x=-6 \quad \text{Substitute} -5 \text{ in for each variable}$$

$$\frac{(-6)^2-4}{(-6)^2+6(-6)+8} \quad \text{Exponents first}$$

$$\frac{36-4}{36+6(-6)+8} \quad \text{Multiply}$$

$$\frac{36-4}{36-36+8} \quad \text{Add and subtract}$$

$$\frac{32}{8} \quad \text{Reduce}$$

$$4 \quad \text{Our Solution}$$

Just as we reduced the previous example, often a rational expression can be reduced, even without knowing the value of the variable. When we reduce we divide out common factors. We have already seen this with monomials when we discussed properties of exponents. If the problem only has monomials we can reduce the coefficients, and subtract exponents on the variables.

Example 328.

$$\frac{15x^4y^2}{25x^2y^6} \quad \text{Reduce, subtract exponents. Negative exponents move to denominator}$$

$$\frac{3x^2}{5y^4} \quad \text{Our Solution}$$

However, if there is more than just one term in either the numerator or denominator, we can't divide out common factors unless we first factor the numerator and denominator.

Example 329.

$$\frac{28}{8x^2-16} \quad \text{Denominator has } a \text{ common factor of } 8$$

$$\frac{28}{8(x^2-2)} \quad \text{Reduce by dividing 24 and 8 by 4}$$

$$\frac{7}{2(x^2-2)} \quad \text{Our Solution}$$

Example 330.

$$\frac{9x-3}{18x-6} \quad \text{Numerator has } a \text{ common factor of } 3\text{, denominator of } 6$$

$$\frac{3(3x-1)}{6(3x-1)} \quad \text{Divide out common factor } (3x-1) \text{ and divide 3 and 6 by 3}$$

$$\frac{1}{2} \quad \text{Our Solution}$$

Example 331.

$$\frac{x^2-25}{x^2+8x+15} \quad \text{Numerator is difference of squares, denominator is factored using ac}$$

$$\frac{(x+5)(x-5)}{(x+3)(x+5)} \quad \text{Divide out common factor } (x+5)$$

$$\frac{x-5}{x+3} \quad \text{Our Solution}$$

It is important to remember we cannot reduce terms, only factors. This means if there are any $+$ or $-$ between the parts we want to reduce we cannot. In the previous example we had the solution $\frac{x-5}{x+3}$, we cannot divide out the x's because they are terms (separated by $+$ or $-$) not factors (separated by multiplication).

7.1 Practice - Reduce Rational Expressions

Evaluate

1) $\frac{4v+2}{6}$ when $v=4$

2) $\frac{b-3}{3b-9}$ when $b=-2$

3) $\frac{x-3}{x^2-4x+3}$ when $x=-4$

4) $\frac{a+2}{a^2+3a+2}$ when $a=-1$

5) $\frac{b+2}{b^2+4b+4}$ when $b=0$

6) $\frac{n^2-n-6}{n-3}$ when $n=4$

State the excluded values for each.

7) $\frac{3k^2+30k}{k+10}$

8) $\frac{27p}{18p^2-36p}$

9) $\frac{15n^2}{10n+25}$

10) $\frac{x+10}{8x^2+80x}$

11) $\frac{10m^2+8m}{10m}$

12) $\frac{10x+16}{6x+20}$

13) $\frac{r^2+3r+2}{5r+10}$

14) $\frac{6n^2-21n}{6n^2+3n}$

15) $\frac{b^2+12b+32}{b^2+4b-32}$

16) $\frac{10v^2+30v}{35v^2-5v}$

Simplify each expression.

17) $\frac{21x^2}{18x}$

18) $\frac{12n}{4n^2}$

19) $\frac{24a}{40a^2}$

20) $\frac{21k}{24k^2}$

21) $\frac{32x^3}{8x^4}$

22) $\frac{90x^2}{20x}$

23) $\frac{18m-24}{60}$

24) $\frac{10}{81n^3+36n^2}$

25) $\frac{20}{4p+2}$

26) $\frac{n-9}{9n-81}$

27) $\frac{x+1}{x^2+8x+7}$

28) $\frac{28m+12}{36}$

29) $\frac{32x^2}{28x^2+28x}$

30) $\frac{49r+56}{56r}$

31) $\frac{n^2+4n-12}{n^2-7n+10}$

32) $\frac{b^2+14b+48}{b^2+15b+56}$

33) $\frac{9v+54}{v^2-4v-60}$

34) $\frac{30x-90}{50x+40}$

35) $\frac{12x^2-42x}{30x^2-42x}$

36) $\frac{k^2-12k+32}{k^2-64}$

37) $\frac{6a-10}{10a+4}$

38) $\frac{9p+18}{p^2+4p+4}$

39) $\frac{2n^2+19n-10}{9n+90}$

40) $\frac{3x^2-29x+40}{5x^2-30x-80}$

41) $\frac{8m+16}{20m-12}$

42) $\frac{56x-48}{24x^2+56x+32}$

43) $\frac{2x^2-10x+8}{3x^2-7x+4}$

44) $\frac{50b-80}{50b+20}$

45) $\frac{7n^2-32n+16}{4n-16}$

46) $\frac{35v+35}{21v+7}$

47) $\frac{n^2-2n+1}{6n+6}$

48) $\frac{56x-48}{24x^2+56x+32}$

49) $\frac{7a^2-26a-45}{6a^2-34a+20}$

50) $\frac{4k^3-2k^2-2k}{9k^3-18k^2+9k}$

7.2

Rational Expressions - Multiply & Divide

Objective: Multiply and divide rational expressions.

Multiplying and dividing rational expressions is very similar to the process we use to multiply and divide fractions.

Example 332.

$\frac{15}{49}\cdot\frac{14}{45}$ First reduce common factors from numerator and denominator (15 and 7)

$\frac{1}{7}\cdot\frac{2}{3}$ Multiply numerators across and denominators across

$\frac{2}{21}$ Our Solution

The process is identical for division with the extra first step of multiplying by the reciprocal. When multiplying with rational expressions we follow the same process, first divide out common factors, then multiply straight across.

Example 333.

$\dfrac{25x^2}{9y^8}\cdot\dfrac{24y^4}{55x^7}$	Reduce coefficients by dividing out common factors (3 and 5) Reduce, subtracting exponents, negative exponents in denominator
$\dfrac{5}{3y^4}\cdot\dfrac{8}{11x^5}$	Multiply across
$\dfrac{40}{33x^5y^4}$	Our Solution

Division is identical in process with the extra first step of multiplying by the reciprocal.

Example 334.

$\dfrac{a^4b^2}{a}\div\dfrac{b^4}{4}$	Multiply by the reciprocal
$\dfrac{a^4b^2}{a}\cdot\dfrac{4}{b^4}$	Subtract exponents on variables, negative exponents in denominator
$\dfrac{a^3}{1}\cdot\dfrac{4}{b^2}$	Multiply across
$\dfrac{4a^3}{b^2}$	Our Solution

Just as with reducing rational expressions, before we reduce a multiplication problem, it must be factored first.

Example 335.

$\dfrac{x^2-9}{x^2+x-20}\cdot\dfrac{x^2-8x+16}{3x+9}$	Factor each numerator and denominator
$\dfrac{(x+3)(x-3)}{(x-4)(x+5)}\cdot\dfrac{(x-4)(x-4)}{3(x+3)}$	Divide out common factors $(x+3)$ and $(x-4)$
$\dfrac{x-3}{x+5}\cdot\dfrac{x-4}{3}$	Multiply across

$$\frac{(x-3)(x-4)}{3(x+5)} \quad \text{Our Solution}$$

Again we follow the same pattern with division with the extra first step of multiplying by the reciprocal.

Example 336.

$$\frac{x^2-x-12}{x^2-2x-8} \div \frac{5x^2+15x}{x^2+x-2} \quad \text{Multiply by the reciprocal}$$

$$\frac{x^2-x-12}{x^2-2x-8} \cdot \frac{x^2+x-2}{5x^2+15x} \quad \text{Factor each numerator and denominator}$$

$$\frac{(x-4)(x+3)}{(x+2)(x-4)} \cdot \frac{(x+2)(x-1)}{5x(x+3)} \quad \text{Divide out common factors:}$$

$(x-4)$ and $(x+3)$ and $(x+2)$

$$\frac{1}{1} \cdot \frac{x-1}{5x} \quad \text{Multiply across}$$

$$\frac{x-1}{5x} \quad \text{Our Solution}$$

We can combine multiplying and dividing of fractions into one problem as shown below. To solve we still need to factor, and we use the reciprocal of the divided fraction.

Example 337.

$$\frac{a^2+7a+10}{a^2+6a+5} \cdot \frac{a+1}{a^2+4a+4} \div \frac{a-1}{a+2} \quad \text{Factor each expression}$$

$$\frac{(a+5)(a+2)}{(a+5)(a+1)} \cdot \frac{(a+1)}{(a+2)(a+2)} \div \frac{(a-1)}{(a+2)} \quad \text{Reciprocal of last fraction}$$

$$\frac{(a+5)(a+2)}{(a+5)(a+1)} \cdot \frac{(a+1)}{(a+2)(a+2)} \cdot \frac{(a+2)}{(a-1)} \quad \text{Divide out common factors}$$

$(a+2), (a+2), (a+1), (a+5)$

$$\frac{1}{a-1} \quad \text{Our Solution}$$

World View Note: Indian mathematician Aryabhata, in the 6th century, published a work which included the rational expression $\frac{n(n+1)(n+2)}{6}$ for the sum of the first n squares $(1^1+2^2+3^2+\ldots.+n^2)$

7.2 Practice - Multiply and Divide

Simplify each expression.

1) $\frac{8x^2}{9}\cdot\frac{9}{2}$

2) $\frac{8x}{3x}\div\frac{4}{7}$

3) $\frac{9n}{2n}\cdot\frac{7}{5n}$

4) $\frac{9m}{5m^2}\cdot\frac{7}{2}$

5) $\frac{5x^2}{4}\cdot\frac{6}{5}$

6) $\frac{10p}{5}\div\frac{8}{10}$

7) $\frac{7\,(m-6)}{m-6}\cdot\frac{5m(7m-5)}{7(7m-5)}$

8) $\frac{7}{10(n+3)}\div\frac{n-2}{(n+3)(n-2)}$

9) $\frac{7r}{7r(r+10)}\div\frac{r-6}{(r-6)^2}$

10) $\frac{6x(x+4)}{x-3}\cdot\frac{(x-3)(x-6)}{6x(x-6)}$

11) $\frac{25n+25}{5}\cdot\frac{4}{30n+30}$

12) $\frac{9}{b^2-b-12}\div\frac{b-5}{b^2-b-12}$

13) $\frac{x-10}{35x+21}\div\frac{7}{35x+21}$

14) $\frac{v-1}{4}\cdot\frac{4}{v^2-11v+10}$

15) $\frac{x^2-6x-7}{x+5}\cdot\frac{x+5}{x-7}$

16) $\frac{1}{a-6}\cdot\frac{8a+80}{8}$

17) $\frac{8k}{24k^2-40k}\div\frac{1}{15k-25}$

18) $\frac{p-8}{p^2-12p+32}\div\frac{1}{p-10}$

19) $(n-8)\cdot\frac{6}{10n-80}$

20) $\frac{x^2-7x+10}{x-2}\cdot\frac{x+10}{x^2-x-20}$

21) $\frac{4m+36}{m+9}\cdot\frac{m-5}{5m^2}$

22) $\frac{2r}{r+6}\div\frac{2r}{7r+42}$

23) $\frac{3x-6}{12x-24}(x+3)$

24) $\frac{2n^2-12n-54}{n+7}\div(2n+6)$

25) $\frac{b+2}{40b^2-24b}(5b-3)$

26) $\frac{21v^2+16v-16}{3v+4}\div\frac{35v-20}{v-9}$

27) $\frac{n-7}{6n-12}\cdot\frac{12-6n}{n^2-13n+42}$

28) $\frac{x^2+11x+24}{6x^3+18x^2}\cdot\frac{6x^3+6x^2}{x^2+5x-24}$

29) $\frac{27a+36}{9a+63}\div\frac{6a+8}{2}$

30) $\frac{k-7}{k^2-k-12}\cdot\frac{7k^2-28k}{8k^2-56k}$

31) $\frac{x^2-12x+32}{x^2-6x-16}\cdot\frac{7x^2+14x}{7x^2+21x}$

32) $\frac{9x^3+54x^2}{x^2+5x-14}\cdot\frac{x^2+5x-14}{10x^2}$

33) $(10m^2+100m)\cdot\frac{18m^3-36m^2}{20m^2-40m}$

34) $\frac{n-7}{n^2-2n-35}\div\frac{9n+54}{10n+50}$

35) $\frac{7p^2+25p+12}{6p+48}\cdot\frac{3p-8}{21p^2-44p-32}$

36) $\frac{7x^2-66x+80}{49x^2+7x-72}\div\frac{7x^2+39x-70}{49x^2+7x-72}$

37) $\frac{10b^2}{30b+20}\cdot\frac{30b+20}{2b^2+10b}$

38) $\frac{35n^2-12n-32}{49n^2-91n+40}\cdot\frac{7n^2+16n-15}{5n+4}$

39) $\frac{7r^2-53r-24}{7r+2}\div\frac{49r+21}{49r+14}$

40) $\frac{12x+24}{10x^2+34x+28}\cdot\frac{15x+21}{5}$

41) $\dfrac{x^2-1}{2x-4} \cdot \dfrac{x^2-4}{x^2-x-2} \div \dfrac{x^2+x-2}{3x-6}$

42) $\dfrac{a^3+b^3}{a^2+3\text{ab}+2b^2} \cdot \dfrac{3a-6b}{3a^2-3\text{ab}+3b^2} \div \dfrac{a^2-4b^2}{a+2b}$

43) $\dfrac{x^2+3x+9}{x^2+x-12} \cdot \dfrac{x^2+2x-8}{x^3-27} \div \dfrac{x^2-4}{x^2-6x+9}$

44) $\dfrac{x^2+3x-10}{x^2+6x+5} \cdot \dfrac{2x^2-x-3}{2x^2+x-6} \div \dfrac{8x+20}{6x+15}$

7.3

Rational Expressions - Least Common Denominators

Objective: Idenfity the least common denominator and build up denominators to match this common denominator.

As with fractions, the least common denominator or LCD is very important to working with rational expressions. The process we use to find the LCD is based on the process used to find the LCD of intergers.

Example 338.

Find the LCD of 8 and 6	Consider multiples of the larger number
$8, 16, 24....$	24 is the first multiple of 8 that is also divisible by 6
24	Our Solution

When finding the LCD of several monomials we first find the LCD of the coefficients, then use all variables and attach the highest exponent on each variable.

Example 339.

Find the LCD of $4x^2y^5$ and $6x^4y^3z^6$

	First find the LCD of coefficients 4 and 6
12	12 is the LCD of 4 and 6
$x^4y^5z^6$	Use all variables with highest exponents on each variable
$12x^4y^5z^6$	Our Solution

The same pattern can be used on polynomials that have more than one term. However, we must first factor each polynomial so we can identify all the factors to be used (attaching highest exponent if necessary).

Example 340.

Find the LCD of x^2+2x-3 and x^2-x-12	Factor each polynomial
$(x-1)(x+3)$ and $(x-4)(x+3)$	LCD uses all unique factors
$(x-1)(x+3)(x-4)$	Our Solution

Notice we only used $(x+3)$ once in our LCD. This is because it only appears as a factor once in either polynomial. The only time we need to repeat a factor or use an exponent on a factor is if there are exponents when one of the polynomials is factored

Example 341.

Find the LCD of $x^2 - 10x + 25$ and $x^2 - 14x + 45$

	Factor each polynomial
$(x-5)^2$ and $(x-5)(x-9)$	LCD uses all unique factors with highest exponent
$(x-5)^2(x-9)$	Our Solution

The previous example could have also been done with factoring the first polynomial to $(x-5)(x-5)$. Then we would have used $(x-5)$ twice in the LCD because it showed up twice in one of the polynomials. However, it is the author's suggestion to use the exponents in factored form so as to use the same pattern (highest exponent) as used with monomials.

Once we know the LCD, our goal will be to build up fractions so they have matching denominators. In this lesson we will not be adding and subtracting fractions, just building them up to a common denominator. We can build up a fraction's denominator by multipliplying the numerator and denoinator by any factors that are not already in the denominator.

Example 342.

$\dfrac{5a}{3a^2b} = \dfrac{?}{6a^5b^3}$	Idenfity what factors we need to match denominators
$2a^3b^2$	$3\cdot 2 = 6$ and we need three more $a's$ and two more $b's$
$\dfrac{5a}{3a^2b}\left(\dfrac{2a^3b^2}{2a^3b^2}\right)$	Multiply numerator and denominator by this
$\dfrac{10a^4b^2}{6a^5b^3}$	Our Solution

Example 343.

$\dfrac{x-2}{x+4} = \dfrac{?}{x^2+7x+12}$ $(x+4)(x+3)$	Factor to idenfity factors we need to match denominators
$(x+3)$	The missing factor
$\dfrac{x-2}{x+4}\left(\dfrac{x+3}{x+3}\right)$	Multiply numerator and denominator by missing factor,
	FOIL numerator
$\dfrac{x^2+x-6}{(x+4)(x+3)}$	Our Solution

As the above example illustrates, we will multiply out our numerators, but keep our denominators factored. The reason for this is to add and subtract fractions we will want to be able to combine like terms in the numerator, then when we reduce at the end we will want our denominators factored.

Once we know how to find the LCD and how to build up fractions to a desired denominator we can combine them together by finding a common denominator and building up those fractions.

Example 344.

Build up each fraction so they have a common denominator

$$\frac{5a}{4b^3c} \text{ and } \frac{3c}{6a^2b} \quad \text{First identify LCD}$$

$$12a^2b^3c \quad \text{Determine what factors each fraction is missing}$$

$$\text{First: } 3a^2 \quad \text{Second: } 2b^2c \quad \text{Multiply each fraction by missing factors}$$

$$\frac{5a}{4b^3c}\left(\frac{3a^2}{3a^2}\right) \text{ and } \frac{3c}{6a^2b}\left(\frac{2b^2c}{2b^2c}\right)$$

$$\frac{15a^3}{12a^2b^3c} \text{ and } \frac{6b^2c^2}{12a^2b^3c} \quad \text{Our Solution}$$

Example 345.

Build up each fraction so they have a common denominator

$$\frac{5x}{x^2-5x-6} \text{ and } \frac{x-2}{x^2+4x+3} \quad \text{Factor to find LCD}$$

$$(x-6)(x+1) \qquad (x+1)(x+3) \quad \text{Use factors to find LCD}$$

$$\text{LCD: } (x-6)(x+1)(x+3) \quad \text{Identify which factors are missing}$$

$$\text{First: } (x+3) \quad \text{Second: } (x-6) \quad \text{Multiply fractions by missing factors}$$

$$\frac{5x}{(x-6)(x+1)}\left(\frac{x+3}{x+3}\right) \text{ and } \frac{x-2}{(x+1)(x+3)}\left(\frac{x-6}{x-6}\right) \quad \text{Multiply numerators}$$

$$\frac{5x^2+15x}{(x-6)(x+1)(x+3)} \text{ and } \frac{x^2-8x+12}{(x-6)(x+1)(x+3)} \quad \text{Our Solution}$$

World View Note: When the Egyptians began working with fractions, they expressed all fractions as a sum of unit fraction. Rather than $\frac{4}{5}$, they would write the fraction as the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$. An interesting problem with this system is this is not a unique solution, $\frac{4}{5}$ is also equal to the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{10}$.

7.3 Practice - Least Common Denominator

Build up denominators.

1) $\frac{3}{8} = \frac{?}{48}$

2) $\frac{a}{5} = \frac{?}{5a}$

3) $\frac{a}{x} = \frac{?}{xy}$

4) $\frac{5}{2x^2} = \frac{?}{8x^3y}$

5) $\frac{2}{3a^3b^2c} = \frac{?}{9a^5b^2c^4}$

6) $\frac{4}{3a^5b^2c^4} = \frac{?}{9a^5b^2c^4}$

7) $\frac{2}{x+4} = \frac{?}{x^2-16}$

8) $\frac{x+1}{x-3} = \frac{?}{x^2-6x+9}$

9) $\frac{x-4}{x+2} = \frac{?}{x^2+5x+6}$

10) $\frac{x-6}{x+3} = \frac{?}{x^2-2x-15}$

Find Least Common Denominators

11) $2a^3, 6a^4b^2, 4a^3b^5$

12) $5x^2y, 25x^3y^5z$

13) $x^2-3x, x-3, x$

14) $4x-8, x-2, 4$

15) $x+2, x-4$

16) $x, x-7, x+1$

17) $x^2-25, x+5$

18) x^2-9, x^2-6x+9

19) x^2+3x+2, x^2+5x+6

20) $x^2-7x+10, x^2-2x-15, x^2+x-6$

Find LCD and build up each fraction

21) $\frac{3a}{5b^2}, \frac{2}{10a^3b}$

22) $\frac{3x}{x-4}, \frac{2}{x+2}$

23) $\frac{x+2}{x-3}, \frac{x-3}{x+2}$

24) $\frac{5}{x^2-6x}, \frac{2}{x}, \frac{-3}{x-6}$

25) $\frac{x}{x^2-16}, \frac{3x}{x^2-8x+16}$

26) $\frac{5x+1}{x^2-3x-10}, \frac{4}{x-5}$

27) $\frac{x+1}{x^2-36}, \frac{2x+3}{x^2+12x+36}$

28) $\frac{3x+1}{x^2-x-12}, \frac{2x}{x^2+4x+3}$

29) $\frac{4x}{x^2-x-6}, \frac{x+2}{x-3}$

30) $\frac{3x}{x^2-6x+8}, \frac{x-2}{x^2+x-20}, \frac{5}{x^2+3x-10}$

7.4

Rational Expressions - Add & Subtract

Objective: Add and subtract rational expressions with and without common denominators.

Adding and subtracting rational expressions is identical to adding and subtracting with integers. Recall that when adding with a common denominator we add the numerators and keep the denominator. This is the same process used with rational expressions. Remember to reduce, if possible, your final answer.

Example 346.

$\dfrac{x-4}{x^2-2x-8}+\dfrac{x+8}{x^2-2x-8}$	Same denominator, add numerators, combine like terms
$\dfrac{2x+4}{x^2-2x-8}$	Factor numerator and denominator
$\dfrac{2(x+2)}{(x+2)(x-4)}$	Divide out $(x+2)$
$\dfrac{2}{x-4}$	Our Solution

Subtraction with common denominator follows the same pattern, though the subtraction can cause problems if we are not careful with it. To avoid sign errors we will first distribute the subtraction through the numerator. Then we can treat it like an addition problem. This process is the same as "add the opposite" we saw when subtracting with negatives.

Example 347.

$\dfrac{6x-12}{3x-6}-\dfrac{15x-6}{3x-6}$	Add the opposite of the second fraction (distribute negative)
$\dfrac{6x-12}{3x-6}+\dfrac{-15x+6}{3x-6}$	Add numerators, combine like terms
$\dfrac{-9x-6}{3x-6}$	Factor numerator and denominator
$\dfrac{-3(3x+2)}{3(x-2)}$	Divide out common factor of 3
$\dfrac{-(3x+2)}{x-2}$	Our Solution

World View Note: The Rhind papyrus of Egypt from 1650 BC gives some of the earliest known symbols for addition and subtraction, a pair of legs walking in the direction one reads for addition, and a pair of legs walking in the opposite direction for subtraction..

When we don't have a common denominator we will have to find the least common denominator (LCD) and build up each fraction so the denominators match. The following example shows this process with integers.

Example 348.

$\dfrac{5}{6}+\dfrac{1}{4}$	The LCD is 12. Build up, multiply 6 by 2 and 4 by 3
$\left(\dfrac{2}{2}\right)\dfrac{5}{6}+\dfrac{1}{4}\left(\dfrac{3}{3}\right)$	Multiply
$\dfrac{10}{12}+\dfrac{3}{12}$	Add numerators

$$\frac{13}{12} \quad \text{Our Solution}$$

The same process is used with variables.

Example 349.

$$\frac{7a}{3a^2b} + \frac{4b}{6ab^4} \quad \text{The LCD is } 6a^2b^4. \text{ We will then build up each fraction}$$

$$\left(\frac{2b^3}{2b^3}\right)\frac{7a}{3a^2b} + \frac{4b}{6ab^4}\left(\frac{a}{a}\right) \quad \text{Multiply first fraction by } 2b^3 \text{ and second by } a$$

$$\frac{14ab^3}{6a^2b^4} + \frac{4ab}{6a^2b^4} \quad \text{Add numerators, no like terms to combine}$$

$$\frac{14ab^3 + 4ab}{6a^2b^4} \quad \text{Factor numerator}$$

$$\frac{2ab(7b^3 + 2)}{6a^2b^4} \quad \text{Reduce, dividing out factors } 2, a, \text{ and } b$$

$$\frac{7b^3 + 2}{3ab^3} \quad \text{Our Solution}$$

The same process can be used for subtraction, we will simply add the first step of adding the opposite.

Example 350.

$$\frac{4}{5a} - \frac{7b}{4a^2} \quad \text{Add the opposite}$$

$$\frac{4}{5a} + \frac{-7b}{4a^2} \quad \text{LCD is } 20a^2. \text{ Build up denominators}$$

$$\left(\frac{4a}{4a}\right)\frac{4}{5a} + \frac{-7b}{4a^2}\left(\frac{5}{5}\right) \quad \text{Multiply first fraction by } 4a, \text{ second by } 5$$

$$\frac{16a - 35b}{20a^2} \quad \text{Our Solution}$$

If our denominators have more than one term in them we will need to factor first to find the LCD. Then we build up each denominator using the factors that are missing on each fraction.

Example 351.

$$\frac{6}{8a+4}+\frac{3a}{8} \quad \text{Factor denominators to find LCD}$$

$$4(2a+1) \quad 8 \quad \text{LCD is } 8(2a+1)\text{, build up each fraction}$$

$$\left(\frac{2}{2}\right)\frac{6}{4(2a+1)}+\frac{3a}{8}\left(\frac{2a+1}{2a+1}\right) \quad \text{Multiply first fraction by 2, second by } 2a+1$$

$$\frac{12}{8(2a+1)}+\frac{6a^2+3a}{8(2a+1)} \quad \text{Add numerators}$$

$$\frac{6a^2+3a+12}{8(2a+1)} \quad \text{Our Solution}$$

With subtraction remember to add the opposite.

Example 352.

$$\frac{x+1}{x-4}-\frac{x+1}{x^2-7x+12} \quad \text{Add the opposite (distribute negative)}$$

$$\frac{x+1}{x-4}+\frac{-x-1}{x^2-7x+12} \quad \text{Factor denominators to find LCD}$$

$$x-4 \quad (x-4)(x-3) \quad \text{LCD is } (x-4)(x-3)\text{, build up each fraction}$$

$$\left(\frac{x-3}{x-3}\right)\frac{x+1}{x-4}+\frac{-x-1}{x^2-7x+12} \quad \text{Only first fraction needs to be multiplied by } x-3$$

$$\frac{x^2-2x-3}{(x-3)(x-4)}+\frac{-x-1}{(x-3)(x-4)} \quad \text{Add numerators, combine like terms}$$

$$\frac{x^2-3x-4}{(x-3)(x-4)} \quad \text{Factor numerator}$$

$$\frac{(x-4)(x+1)}{(x-3)(x-4)} \quad \text{Divide out } x-4 \text{ factor}$$

$$\frac{x+1}{x-3} \quad \text{Our Solution}$$

7.4 Practice - Add and Subtract

Add or subtract the rational expressions. Simplify your answers whenever possible.

1) $\frac{2}{a+3}+\frac{4}{a+3}$

2) $\frac{x^2}{x-2}-\frac{6x-8}{x-2}$

3) $\frac{t^2+4t}{t-1}+\frac{2t-7}{t-1}$

4) $\frac{a^2+3a}{a^2+5a-6}-\frac{4}{a^2+5a-6}$

5) $\frac{2x^2+3}{x^2-6x+5}-\frac{x^2-5x+9}{x^2-6x+5}$

6) $\frac{3}{x}+\frac{4}{x^2}$

7) $\frac{5}{6r}-\frac{5}{8r}$

8) $\frac{7}{xy^2}+\frac{3}{x^2y}$

9) $\frac{8}{9t^3}+\frac{5}{6t^2}$

10) $\frac{x+5}{8}+\frac{x-3}{12}$

11) $\frac{a+2}{2}-\frac{a-4}{4}$

12) $\frac{2a-1}{3a^2}+\frac{5a+1}{9a}$

13) $\frac{x-1}{4x}-\frac{2x+3}{x}$

14) $\frac{2c-d}{c^2d}-\frac{c+d}{cd^2}$

15) $\frac{5x+3y}{2x^2y}-\frac{3x+4y}{xy^2}$

16) $\frac{2}{x-1}+\frac{2}{x+1}$

17) $\frac{2z}{z-1}-\frac{3z}{z+1}$

18) $\frac{2}{x-5}+\frac{3}{4x}$

19) $\frac{8}{x^2-4}-\frac{3}{x+2}$

20) $\frac{4x}{x^2-25}+\frac{x}{x+5}$

21) $\frac{t}{t-3}-\frac{5}{4t-12}$

22) $\frac{2}{x+3}+\frac{4}{(x+3)^2}$

23) $\frac{2}{5x^2+5x}-\frac{4}{3x+3}$

24) $\frac{3a}{4a-20}+\frac{9a}{6a-30}$

25) $\frac{t}{y-t}-\frac{y}{y+t}$

26) $\frac{x}{x-5}+\frac{x-5}{x}$

27) $\frac{x}{x^2+5x+6}-\frac{2}{x^2+3x+2}$

28) $\frac{2x}{x^2-1}-\frac{3}{x^2+5x+4}$

29) $\frac{x}{x^2+15x+56}-\frac{7}{x^2+13x+42}$

30) $\frac{2x}{x^2-9}+\frac{5}{x^2+x-6}$

31) $\frac{5x}{x^2-x-6}-\frac{18}{x^2-9}$

32) $\frac{4x}{x^2-2x-3}-\frac{3}{x^2-5x+6}$

33) $\frac{2x}{x^2-1}-\frac{4}{x^2+2x-3}$

34) $\frac{x-1}{x^2+3x+2}+\frac{x+5}{x^2+4x+3}$

35) $\frac{x+1}{x^2-2x-35}+\frac{x+6}{x^2+7x+10}$

36) $\frac{3x+2}{3x+6}+\frac{x}{4-x^2}$

37) $\frac{4-a^2}{a^2-9}-\frac{a-2}{3-a}$

38) $\frac{4y}{y^2-1}-\frac{2}{y}-\frac{2}{y+1}$

39) $\frac{2z}{1-2z}+\frac{3z}{2z+1}-\frac{3}{4z^2-1}$

40) $\frac{2r}{r^2-s^2}+\frac{1}{r+s}-\frac{1}{r-s}$

41) $\frac{2x-3}{x^2+3x+2}+\frac{3x-1}{x^2+5x+6}$

42) $\frac{x+2}{x^2-4x+3}+\frac{4x+5}{x^2+4x-5}$

43) $\frac{2x+7}{x^2-2x-3}-\frac{3x-2}{x^2+6x+5}$

44) $\frac{3x-8}{x^2+6x+8}+\frac{2x-3}{x^2+3x+2}$

7.5

Rational Expressions - Complex Fractions

Objective: Simplify complex fractions by multiplying each term by the least common denominator.

Complex fractions have fractions in either the numerator, or denominator, or usually both. These fractions can be simplified in one of two ways. This will be illustrated first with integers, then we will consider how the process can be expanded to include expressions with variables.

The first method uses order of operations to simplify the numerator and denominator first, then divide the two resulting fractions by multiplying by the reciprocal.

Example 353.

$$\frac{\frac{2}{3}-\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}} \quad \text{Get common denominator in top and bottom fractions}$$

$$\frac{\frac{8}{12}-\frac{3}{12}}{\frac{5}{6}+\frac{3}{6}} \quad \text{Add and subtract fractions, reducing solutions}$$

$$\frac{\frac{5}{12}}{\frac{4}{3}} \quad \text{To divide fractions we multiply by the reciprocal}$$

$$\left(\frac{5}{12}\right)\left(\frac{3}{4}\right) \quad \text{Reduce}$$

$$\left(\frac{5}{4}\right)\left(\frac{1}{4}\right) \quad \text{Multiply}$$

$$\frac{5}{16} \quad \text{Our Solution}$$

The process above works just fine to simplify, but between getting common denominators, taking reciprocals, and reducing, it can be a very involved process. Generally we prefer a different method, to multiply the numerator and denominator of the large fraction (in effect each term in the complex fraction) by the least common denominator (LCD). This will allow us to reduce and clear the small fractions. We will simplify the same problem using this second method.

Example 354.

$$\frac{\frac{2}{3}-\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}} \quad \text{LCD is 12, multiply each term}$$

$$\frac{\frac{2(12)}{3}-\frac{1(12)}{4}}{\frac{5(12)}{6}+\frac{1(12)}{2}} \quad \text{Reduce each fraction}$$

$$\frac{2(4)-1(3)}{5(2)+1(6)} \quad \text{Multiply}$$

$$\frac{8-3}{10+6} \quad \text{Add and subtract}$$

$$\frac{5}{16} \quad \text{Our Solution}$$

Clearly the second method is a much cleaner and faster method to arrive at our solution. It is the method we will use when simplifying with variables as well. We will first find the LCD of the small fractions, and multiply each term by this LCD so we can clear the small fractions and simplify.

Example 355.

$$\frac{1-\frac{1}{x^2}}{1-\frac{1}{x}} \quad \text{Identify LCD (use highest exponent)}$$

$$\text{LCD}=x^2 \quad \text{Multiply each term by LCD}$$

$$\frac{1(x^2)-\frac{1(x^2)}{x^2}}{1(x^2)-\frac{1(x^2)}{x}} \quad \text{Reduce fractions (subtract exponents)}$$

$$\frac{1(x^2)-1}{1(x^2)-x} \quad \text{Multiply}$$

$$\frac{x^2-1}{x^2-x} \quad \text{Factor}$$

$$\frac{(x+1)(x-1)}{x(x-1)} \quad \text{Divide out } (x-1) \text{ factor}$$

$$\frac{x+1}{x} \quad \text{Our Solution}$$

The process is the same if the LCD is a binomial, we will need to distribute

$$\frac{\frac{3}{x+4}-2}{5+\frac{2}{x+4}} \quad \text{Multiply each term by LCD, } (x+4)$$

$$\frac{\frac{3(x+4)}{x+4}-2(x+4)}{5(x+4)+\frac{2(x+4)}{x+4}} \quad \text{Reduce fractions}$$

$$\frac{3-2(x+4)}{5(x+4)+2} \quad \text{Distribute}$$

$$\frac{3-2x-8}{5x+20+2} \quad \text{Combine like terms}$$

$$\frac{-2x-5}{5x+22} \quad \text{Our Solution}$$

The more fractions we have in our problem, the more we repeat the same process.

Example 356.

$$\frac{\frac{2}{\mathrm{ab}^2}-\frac{3}{ab^3}+\frac{1}{ab}}{\frac{4}{a^2b}+ab-\frac{1}{ab}} \quad \text{Idenfity LCD (highest exponents)}$$

$$\text{LCD}=a^2b^3 \quad \text{Multiply each term by LCD}$$

$$\frac{\frac{2(a^2b^3)}{\mathrm{ab}^2}-\frac{3(a^2b^3)}{ab^3}+\frac{1(a^2b^3)}{ab}}{\frac{4(a^2b^3)}{a^2b}+ab(a^2b^3)-\frac{1(a^2b^3)}{ab}} \quad \text{Reduce each fraction (subtract exponents)}$$

$$\frac{2ab-3a+ab^2}{4b^2+a^3b^4-ab^2} \quad \text{Our Solution}$$

World View Note: Sophie Germain is one of the most famous women in mathematics, many primes, which are important to finding an LCD, carry her name. Germain primes are prime numbers where one more than double the prime number is also prime, for example 3 is prime and so is $2\cdot 3+1=7$ prime. The largest known Germain prime (at the time of printing) is $183027\cdot 2^{265440}-1$ which has 79911 digits!

Some problems may require us to FOIL as we simplify. To avoid sign errors, if there is a binomial in the numerator, we will first distribute the negative through the numerator.

Example 357.

$$\frac{\frac{x-3}{x+3}-\frac{x+3}{x-3}}{\frac{x-3}{x+3}+\frac{x+3}{x-3}} \quad \text{Distribute the subtraction to numerator}$$

$$\frac{\frac{x-3}{x+3}+\frac{-x-3}{x-3}}{\frac{x-3}{x+3}+\frac{x+3}{x-3}} \quad \text{Identify LCD}$$

$$\text{LCD} = (x+3)(x-3) \qquad \text{Multiply each term by LCD}$$

$$\frac{\frac{(x-3)(x+3)(x-3)}{x+3} + \frac{(-x-3)(x+3)(x-3)}{x-3}}{\frac{(x-3)(x+3)(x-3)}{x+3} + \frac{(x+3)(x+3)(x-3)}{x-3}} \qquad \text{Reduce fractions}$$

$$\frac{(x-3)(x-3) + (-x-3)(x+3)}{(x-3)(x-3) + (x+3)(x+3)} \qquad \text{FOIL}$$

$$\frac{x^2 - 6x + 9 - x^2 - 6x - 9}{x^2 - 6x + 9 + x^2 + 6x - 9} \qquad \text{Combine like terms}$$

$$\frac{-12x}{2x^2 + 18} \qquad \text{Factor out 2 in denominator}$$

$$\frac{-12x}{2(x^2 + 9)} \qquad \text{Divide out common factor 2}$$

$$\frac{-6x}{x^2 - 9} \qquad \text{Our Solution}$$

If there are negative exponents in an expression we will have to first convert these negative exponents into fractions. Remember, the exponent is only on the factor it is attached to, not the whole term.

Example 358.

$$\frac{m^{-2} + 2m^{-1}}{m + 4m^{-2}} \qquad \text{Make each negative exponent into } a \text{ fraction}$$

$$\frac{\frac{1}{m^2} + \frac{2}{m}}{m + \frac{4}{m^2}} \qquad \text{Multiply each term by LCD, } m^2$$

$$\frac{\frac{1(m^2)}{m^2} + \frac{2(m^2)}{m}}{m(m^2) + \frac{4(m^2)}{m^2}} \qquad \text{Reduce the fractions}$$

$$\frac{1 + 2m}{m^3 + 4} \qquad \text{Our Solution}$$

Once we convert each negative exponent into a fraction, the problem solves exactly like the other complex fraction problems.

7.5 Practice - Complex Fractions

Solve.

1) $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}}$

2) $\dfrac{\frac{1}{y^2}-1}{1+\frac{1}{y}}$

3) $\dfrac{a-2}{\frac{4}{a}-a}$

4) $\dfrac{\frac{25}{a}-a}{5+a}$

5) $\dfrac{\frac{1}{a^2}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{a}}$

6) $\dfrac{\frac{1}{b}+\frac{1}{2}}{\frac{4}{b^2-1}}$

7) $\dfrac{2-\frac{4}{x+2}}{5-\frac{10}{x+2}}$

8) $\dfrac{4+\frac{12}{2x-3}}{5+\frac{15}{2x-3}}$

9) $\dfrac{\frac{3}{2a-3}+2}{\frac{-6}{2a-3}-4}$

10) $\dfrac{\frac{-5}{b-5}-3}{\frac{10}{b-5}+6}$

11) $\dfrac{\frac{x}{x+1}-\frac{1}{x}}{\frac{x}{x+1}+\frac{1}{x}}$

12) $\dfrac{\frac{2a}{a-1}-\frac{3}{a}}{\frac{-6}{a-1}-4}$

13) $\dfrac{\frac{3}{x}}{\frac{9}{x^2}}$

14) $\dfrac{\frac{x}{3x-2}}{\frac{x}{9x^2-4}}$

15) $\dfrac{\frac{a^2-b^2}{4a^2b}}{\frac{a+b}{16\mathrm{ab}^2}}$

16) $\dfrac{1-\frac{1}{x}-\frac{6}{x^2}}{1-\frac{4}{x}+\frac{3}{x^2}}$

17) $\dfrac{1-\frac{3}{x}-\frac{10}{x^2}}{1+\frac{11}{x}+\frac{18}{x^2}}$

18) $\dfrac{\frac{15}{x^2}-\frac{2}{x}-1}{\frac{4}{x^2}-\frac{5}{x}+4}$

19) $\dfrac{1-\frac{2x}{3x-4}}{x-\frac{32}{3x-4}}$

20) $\dfrac{1-\frac{12}{3x+10}}{x-\frac{8}{3x+10}}$

21) $\dfrac{x-1+\frac{2}{x-4}}{x+3+\frac{6}{x-4}}$

22) $\dfrac{x-5-\frac{18}{x+2}}{x+7+\frac{6}{x+2}}$

23) $\dfrac{x-4+\frac{9}{2x+3}}{x+3-\frac{5}{2x+3}}$

24) $\dfrac{\frac{1}{a}-\frac{3}{a-2}}{\frac{2}{a}+\frac{5}{a-2}}$

25) $\dfrac{\frac{2}{b}-\frac{5}{b+3}}{\frac{3}{b}+\frac{3}{b+3}}$

26) $\dfrac{\frac{1}{y^2}-\frac{1}{xy}-\frac{2}{x^2}}{\frac{1}{y^2}-\frac{3}{xy}+\frac{2}{x^2}}$

27) $\dfrac{\frac{2}{b^2}-\frac{5}{\mathrm{ab}}-\frac{3}{a^2}}{\frac{2}{b^2}+\frac{7}{\mathrm{ab}}+\frac{3}{a^2}}$

28) $\dfrac{\frac{x-1}{x+1}-\frac{x+1}{x-1}}{\frac{x-1}{x+1}+\frac{x+1}{x-1}}$

29) $\dfrac{\frac{y}{y+2}-\frac{y}{y-2}}{\frac{y}{y+2}+\frac{y}{y-2}}$

30) $\dfrac{\frac{x+1}{x-1}-\frac{1-x}{1+x}}{\frac{1}{(x+1)^2}+\frac{1}{(x-1)^2}}$

Simplify each of the following fractional expressions.

31) $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$

32) $\dfrac{x^{-2}y+xy^{-2}}{x^{-2}-y^{-2}}$

33) $\dfrac{x^{-3}y-xy^{-3}}{x^{-2}-y^{-2}}$

34) $\dfrac{4-4x^{-1}+x^{-2}}{4-x^{-2}}$

35) $\dfrac{x^{-2}-6x^{-1}+9}{x^2-9}$

36) $\dfrac{x^{-3}+y^{-3}}{x^{-2}-x^{-1}y^{-1}+y^{-2}}$

7.6

Rational Expressions - Proportions

Objective: Solve proportions using the cross product and use proportions to solve application problems

When two fractions are equal, they are called a proportion. This definition can be generalized to two equal rational expressions. Proportions have an important property called the cross-product.

$$\textbf{Cross Product: If } \boldsymbol{\frac{a}{b} = \frac{c}{d}} \textbf{ then ad = bc}$$

The cross product tells us we can multiply diagonally to get an equation with no fractions that we can solve.

Example 359.

$$\frac{20}{6} = \frac{x}{9} \quad \text{Calculate cross product}$$

$$
\begin{aligned}
(20)(9) &= 6x && \text{Multiply}\\
180 &= 6x && \text{Divide both sides by } 6\\
\overline{6}\quad & \quad\overline{6} &&\\
30 &= x && \text{Our Solution}
\end{aligned}
$$

World View Note: The first clear definition of a proportion and the notation for a proportion came from the German Leibniz who wrote, "I write dy: $x=$ dt: a; for dy is to x as dt is to a, is indeed the same as, dy divided by x is equal to dt divided by a. From this equation follow then all the rules of proportion."

If the proportion has more than one term in either numerator or denominator, we will have to distribute while calculating the cross product.

Example 360.

$$
\begin{aligned}
\frac{x+3}{4} &= \frac{2}{5} && \text{Calculate cross product}\\
5(x+3) &= (4)(2) && \text{Multiply and distribute}\\
5x+15 &= 8 && \text{Solve}\\
\underline{-15} & \underline{-15} && \text{Subtract 15 from both sides}\\
5x &= -7 && \text{Divide both sides by } 5\\
\overline{5}\quad & \quad\overline{5} &&\\
x &= -\frac{7}{5} && \text{Our Solution}
\end{aligned}
$$

This same idea can be seen when the variable appears in several parts of the proportion.

Example 361.

$$
\begin{aligned}
\frac{4}{x} &= \frac{6}{3x+2} && \text{Calculate cross product}\\
4(3x+2) &= 6x && \text{Distribute}\\
12x+8 &= 6x && \text{Move variables to one side}\\
\underline{-12x} & \underline{\quad -12x} && \text{Subtract } 12x \text{ from both sides}\\
8 &= -6x && \text{Divide both sides by } -6\\
\overline{-6}\quad & \quad\overline{-6} &&\\
-\frac{4}{3} &= x && \text{Our Solution}
\end{aligned}
$$

Example 362.

$$\begin{array}{rl}
\dfrac{2x-3}{7x+4}=\dfrac{2}{5} & \text{Calculate cross product}\\
5(2x-3)=2(7x+4) & \text{Distribute}\\
10x-15=14x+8 & \text{Move variables to one side}\\
\underline{-10x \qquad -10x} & \text{Subtract } 10x \text{ from both sides}\\
-15=4x+8 & \text{Subtract 8 from both sides}\\
\underline{-8 \qquad -8} & \\
-23=4x & \text{Divide both sides by 4}\\
\overline{4} \quad \overline{4} & \\
-\dfrac{23}{4}=x & \text{Our Solution}
\end{array}$$

As we solve proportions we may end up with a quadratic that we will have to solve. We can solve this quadratic in the same way we solved quadratics in the past, either factoring, completing the square or the quadratic formula. As with solving quadratics before, we will generally end up with two solutions.

Example 363.

$$\begin{array}{rl}
\dfrac{k+3}{3}=\dfrac{8}{k-2} & \text{Calculate cross product}\\
(k+3)(k-2)=(8)(3) & \text{FOIL and multiply}\\
k^2+k-6=24 & \text{Make equation equal zero}\\
\underline{-24-24} & \text{Subtract 24 from both sides}\\
k^2+k-30=0 & \text{Factor}\\
(k+6)(k-5)=0 & \text{Set each factor equal to zero}\\
k+6=0 \text{ or } k-5=0 & \text{Solve each equation}\\
\underline{-6-6} \qquad \underline{+5=5} & \text{Add or subtract}\\
k=-6 \text{ or } k=5 & \text{Our Solutions}
\end{array}$$

Proportions are very useful in how they can be used in many different types of applications. We can use them to compare different quantities and make conclusions about how quantities are related. As we set up these problems it is important to remember to stay organized, if we are comparing dogs and cats, and the number of dogs is in the numerator of the first fraction, then the numerator of the second fraction should also refer to the dogs. This consistency of the numerator and denominator is essential in setting up our proportions.

Example 364.

A six foot tall man casts a shadow that is 3.5 feet long. If the shadow of a flag pole is 8 feet long, how tall is the flag pole?

$$\begin{array}{rl}
\dfrac{\text{shadow}}{\text{height}} & \text{We will put shadows in numerator, heights in denomintor}
\end{array}$$

$$\frac{3.5}{6} \quad \text{The man has } a \text{ shadow of } 3.5 \text{ feet and } a \text{ height of } 6 \text{ feet}$$

$$\frac{8}{x} \quad \text{The flagpole has } a \text{ shadow of } 8 \text{ feet, but we don}'t \text{ know the height}$$

$$\begin{array}{rl}
\dfrac{3.5}{6} = \dfrac{8}{x} & \text{This gives us our proportion, calculate cross product} \\
3.5x = (8)(6) & \text{Multiply} \\
3.5x = 48 & \text{Divide both sides by } 3.5 \\
\overline{3.5} \quad \overline{3.5} & \\
x = 13.7\,\text{ft} & \text{Our Solution}
\end{array}$$

Example 365.

In a basketball game, the home team was down by 9 points at the end of the game. They only scored 6 points for every 7 points the visiting team scored. What was the final score of the game?

$$\frac{\text{home}}{\text{visiter}} \quad \text{We will put home in numerator, visitor in denominator}$$

$$\frac{x-9}{x} \quad \text{Don}'t \text{ know visitor score, but home is 9 points less}$$

$$\frac{6}{7} \quad \text{Home team scored 6 for every 7 the visitor scored}$$

$$\begin{array}{rl}
\dfrac{x-9}{x} = \dfrac{6}{7} & \text{This gives our proportion, calculate the cross product} \\
7(x-9) = 6x & \text{Distribute} \\
7x - 63 = 6x & \text{Move variables to one side} \\
\underline{-7x \qquad -7x} & \text{Subtract } 7x \text{ from both sides} \\
-63 = -x & \text{Divide both sides by } -1 \\
\overline{-1} \quad \overline{-1} & \\
63 = x & \text{We used } x \text{ for the visitor score.} \\
63 - 9 = 54 & \text{Subtract 9 to get the home score} \\
63 \text{ to } 54 & \text{Our Solution}
\end{array}$$

7.6 Practice - Proportions

Solve each proportion.

1) $\frac{10}{a}=\frac{6}{8}$

2) $\frac{7}{9}=\frac{n}{6}$

3) $\frac{7}{6}=\frac{2}{k}$

4) $\frac{8}{x}=\frac{4}{8}$

5) $\frac{6}{x}=\frac{8}{2}$

6) $\frac{n-10}{8}=\frac{9}{3}$

7) $\frac{m-1}{5}=\frac{8}{2}$

8) $\frac{8}{5}=\frac{3}{x-8}$

9) $\frac{2}{9}=\frac{10}{p-4}$

10) $\frac{9}{n+2}=\frac{3}{9}$

11) $\frac{b-10}{7}=\frac{b}{4}$

12) $\frac{9}{4}=\frac{r}{r-4}$

13) $\frac{x}{5}=\frac{x+2}{9}$

14) $\frac{n}{8}=\frac{n-4}{3}$

15) $\frac{3}{10}=\frac{a}{a+2}$

16) $\frac{x+1}{9}=\frac{x+2}{2}$

17) $\frac{v-5}{v+6}=\frac{4}{9}$

18) $\frac{n+8}{10}=\frac{n-9}{4}$

19) $\frac{7}{x-1}=\frac{4}{x-6}$

20) $\frac{k+5}{k-6}=\frac{8}{5}$

21) $\frac{x+5}{5}=\frac{6}{x-2}$

22) $\frac{4}{x-3}=\frac{x+5}{5}$

23) $\frac{m+3}{4}=\frac{11}{m-4}$

24) $\frac{x-5}{8}=\frac{4}{x-1}$

25) $\frac{2}{p+4}=\frac{p+5}{3}$

26) $\frac{5}{n+1}=\frac{n-4}{10}$

27) $\frac{n+4}{3}=\frac{-3}{n-2}$

28) $\frac{1}{n+3}=\frac{n+2}{2}$

29) $\frac{3}{x+4}=\frac{x+2}{5}$

30) $\frac{x-5}{4}=\frac{-3}{x+3}$

Answer each question. Round your answer to the nearest tenth. Round dollar amounts to the nearest cent.

31) The currency in Western Samoa is the Tala. The exchange rate is approximately \$0.70 to 1 Tala. At this rate, how many dollars would you get if you exchanged 13.3 Tala?

32) If you can buy one plantain for \$0.49 then how many can you buy with \$7.84?

33) Kali reduced the size of a painting to a height of 1.3 in. What is the new width if it was originally 5.2 in. tall and 10 in. wide?

34) A model train has a scale of 1.2 in : 2.9 ft. If the model train is 5 in tall then how tall is the real train?

35) A bird bath that is 5.3 ft tall casts a shadow that is 25.4 ft long. Find the length of the shadow that a 8.2 ft adult elephant casts.

36) Victoria and Georgetown are 36.2 mi from each other. How far apart would the cities be on a map that has a scale of 0.9 in : 10.5 mi?

37) The Vikings led the Timberwolves by 19 points at the half. If the Vikings scored 3 points for every 2 points the Timberwolves scored, what was the half time score?

38) Sarah worked 10 more hours than Josh. If Sarah worked 7 hr for every 2 hr Josh worked, how long did they each work?

39) Computer Services Inc. charges $8 more for a repair than Low Cost Computer Repair. If the ratio of the costs is 3 : 6, what will it cost for the repair at Low Cost Computer Repair?

40) Kelsey's commute is 15 minutes longer than Christina's. If Christina drives 12 minutes for every 17 minutes Kelsey drives, how long is each commute?

7.7

Rational Expressions - Solving Rational Equations

Objective: Solve rational equations by identifying and multiplying by the least common denominator.

When solving equations that are made up of rational expressions we will solve them using the same strategy we used to solve linear equations with fractions. When we solved problems like the next example, we cleared the fraction by multiplying by the least common denominator (LCD)

Example 366.

$$
\begin{array}{rl}
\frac{2}{3}x - \frac{5}{6} = \frac{3}{4} & \text{Multiply each term by LCD, 12} \\[2ex]
\frac{2(12)}{3}x - \frac{5(12)}{6} = \frac{3(12)}{4} & \text{Reduce fractions} \\[2ex]
2(4)x - 5(2) = 3(3) & \text{Multiply} \\
8x - 10 = 9 & \text{Solve} \\
\underline{+10 + 10} & \text{Add 10 to both sides} \\
8x = 19 & \text{Divide both sides by 8} \\
\overline{8} \quad \overline{8} & \\
x = \frac{19}{8} & \text{Our Solution}
\end{array}
$$

We will use the same process to solve rational equations, the only difference is our

LCD will be more involved. We will also have to be aware of domain issues. If our LCD equals zero, the solution is undefined. We will always check our solutions in the LCD as we may have to remove a solution from our solution set.

Example 367.

$$\frac{5x+5}{x+2}+3x=\frac{x^2}{x+2} \quad \text{Multiply each term by LCD, } (x+2)$$

$$\frac{(5x+5)(x+2)}{x+2}+3x(x+2)=\frac{x^2(x+2)}{x+2} \quad \text{Reduce fractions}$$

$$\begin{array}{rl}
5x+5+3x(x+2)=x^2 & \text{Distribute} \\
5x+5+3x^2+6x=x^2 & \text{Combine like terms} \\
3x^2+11x+5=x^2 & \text{Make equation equal zero} \\
\underline{-x^2 \qquad\qquad -x^2} & \text{Subtract } x^2 \text{ from both sides} \\
2x^2+11x+5=0 & \text{Factor} \\
(2x+1)(x+5)=0 & \text{Set each factor equal to zero} \\
2x+1=0 \text{ or } x+5=0 & \text{Solve each equation} \\
\underline{-1-1} \qquad \underline{-5-5} & \\
2x=-1 \text{ or } x=-5 & \\
\overline{2} \quad \overline{2} \qquad\qquad & \\
x=-\frac{1}{2} \text{ or } -5 & \text{Check solutions, LCD can}'t\text{ be zero} \\
-\frac{1}{2}+2=\frac{3}{2} \qquad -5+2=-3 & \text{Neither make LCD zero, both are solutions} \\
x=-\frac{1}{2} \text{ or } -5 & \text{Our Solution}
\end{array}$$

The LCD can be several factors in these problems. As the LCD gets more complex, it is important to remember the process we are using to solve is still the same.

Example 368.

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)} \quad \text{Multiply terms by LCD, } (x+1)(x+2)$$

$$\frac{x(x+1)(x+2)}{x+2}+\frac{1(x+1)(x+2)}{x+1}=\frac{5(x+1)(x+2)}{(x+1)(x+2)} \quad \text{Reduce fractions}$$

$$
\begin{array}{rl}
x(x+1)+1(x+2)=5 & \text{Distribute} \\
x^2+x+x+2=5 & \text{Combine like terms} \\
x^2+2x+2=5 & \text{Make equatino equal zero} \\
\underline{-5-5} & \text{Subtract 6 from both sides} \\
x^2+2x-3=0 & \text{Factor} \\
(x+3)(x-1)=0 & \text{Set each factor equal to zero} \\
x+3=0 \text{ or } x-1=0 & \text{Solve each equation} \\
\underline{-3-3} \qquad \underline{+1+1} & \\
x=-3 \text{ or } x=1 & \text{Check solutions, LCD can}'t\text{ be zero} \\
(-3+1)(-3+2)=(-2)(-1)=2 & \text{Check } -3 \text{ in } (x+1)(x+2), \text{it works} \\
(1+1)(1+2)=(2)(3)=6 & \text{Check 1 in } (x+1)(x+2), \text{it works} \\
x=-3 \text{ or } 1 & \text{Our Solution}
\end{array}
$$

In the previous example the denominators were factored for us. More often we will need to factor before finding the LCD

Example 369.

$$
\begin{array}{rl}
\dfrac{x}{x-1}-\dfrac{1}{x-2}=\dfrac{11}{x^2-3x+2} & \text{Factor denominator} \\
(x-1)(x-2) & \\
\text{LCD}=(x-1)(x-2) & \text{Identify LCD} \\
\dfrac{x(x-1)(x-2)}{x-1}-\dfrac{1(x-1)(x-2)}{x-2}=\dfrac{11(x-1)(x-2)}{(x-1)(x-2)} & \text{Multiply each term by LCD, reduce} \\
x(x-2)-1(x-1)=11 & \text{Distribute} \\
x^2-2x-x+1=11 & \text{Combine like terms} \\
x^2-3x+1=11 & \text{Make equation equal zero} \\
\underline{-11-11} & \text{Subtract 11 from both sides} \\
x^2-3x-10=0 & \text{Factor} \\
(x-5)(x+2)=0 & \text{Set each factor equal to zero} \\
x-5=0 \text{ or } x+2=0 & \text{Solve each equation} \\
\underline{+5+5} \qquad \underline{-2-2} & \\
x=5 \text{ or } x=-2 & \text{Check answers, LCD can}'t\text{ be 0} \\
(5-1)(5-2)=(4)(3)=12 & \text{Check 5 in } (x-1)(x-2), \text{it works} \\
(-2-1)(-2-2)=(-3)(-4)=12 & \text{Check } -2 \text{ in } (x-1)(x-2), \text{it works}
\end{array}
$$

$$x=5 \text{ or } -2 \quad \text{Our Solution}$$

World View Note: Maria Agnesi was the first women to publish a math textbook in 1748, it took her over 10 years to write! This textbook covered everything from arithmetic thorugh differential equations and was over 1,000 pages!

If we are subtracting a fraction in the problem, it may be easier to avoid a future sign error by first distributing the negative through the numerator.

Example 370.

$$\frac{x-2}{x-3}-\frac{x+2}{x+2}=\frac{5}{8} \quad \text{Distribute negative through numerator}$$

$$\frac{x-2}{x-3}+\frac{-x-2}{x+2}=\frac{5}{8} \quad \text{Identify LCD, } 8(x-3)(x+2)\text{, multiply each term}$$

$$\frac{(x-2)8(x-3)(x+2)}{x-3}+\frac{(-x-2)8(x-3)(x+2)}{x+2}=\frac{5\cdot 8(x-3)(x+2)}{8} \quad \text{Reduce}$$

$$\begin{aligned}
8(x-2)(x+2)+8(-x-2)(x-3)&=5(x-3)(x+2) && \text{FOIL}\\
8(x^2-4)+8(-x^2+x+6)&=5(x^2-x-6) && \text{Distribute}\\
8x^2-32-8x^2+8x+48&=5x^2-5x-30 && \text{Combine like terms}\\
8x+16&=5x^2-5x-30 && \text{Make equation equal zero}\\
\underline{-8x-16}\qquad&\qquad\underline{-8x-16} && \text{Subtract } 8x \text{ and } 16\\
0&=5x^2-13x-46 && \text{Factor}\\
0&=(5x-23)(x+2) && \text{Set each factor equal to zero}\\
5x-23=0 \text{ or } x+2&=0 && \text{Solve each equation}\\
\underline{+23+23}\qquad&\underline{-2-2} &&\\
5x=23 \text{ or } x&=-2 &&\\
\overline{5}\quad\overline{5}\qquad& &&\\
x=\frac{23}{5} \text{ or } &-2 && \text{Check solutions, LCD can't be 0}\\
8\left(\frac{23}{5}-3\right)\left(\frac{23}{5}+2\right)=8\left(\frac{8}{5}\right)\left(\frac{33}{5}\right)&=\frac{2112}{25} && \text{Check } \frac{23}{5} \text{ in } 8(x-3)(x+2)\text{, it works}\\
8(-2-3)(-2+2)=8(-5)(0)&=0 && \text{Check } -2 \text{ in } 8(x-3)(x+2)\text{, can't be 0!}\\
x&=\frac{23}{5} && \text{Our Solution}
\end{aligned}$$

In the previous example, one of the solutions we found made the LCD zero. When this happens we ignore this result and only use the results that make the rational expressions defined.

7.7 Practice - Solving Rational Equations

Solve the following equations for the given variable:

1) $3x - \frac{1}{2} - \frac{1}{x} = 0$

2) $x + 1 = \frac{4}{x+1}$

3) $x + \frac{20}{x-4} = \frac{5x}{x-4} - 2$

4) $\frac{x^2+6}{x-1} + \frac{x-2}{x-1} = 2x$

5) $x + \frac{6}{x-3} = \frac{2x}{x-3}$

6) $\frac{x-4}{x-1} = \frac{12}{3-x} + 1$

7) $\frac{2x}{3x-4} = \frac{4x+5}{6x-1} - \frac{3}{3x-4}$

8) $\frac{6x+5}{2x^2-2x} - \frac{2}{1-x^2} = \frac{3x}{x^2-1}$

9) $\frac{3m}{2m-5} - \frac{7}{3m+1} = \frac{3}{2}$

10) $\frac{4x}{2x-6} - \frac{4}{5x-15} = \frac{1}{2}$

11) $\frac{4-x}{1-x} = \frac{12}{3-x}$

12) $\frac{7}{3-x} + \frac{1}{2} = \frac{3}{4-x}$

13) $\frac{7}{y-3} - \frac{1}{2} = \frac{y-2}{y-4}$

14) $\frac{2}{3-x} - \frac{6}{8-x} = 1$

15) $\frac{1}{x+2} - \frac{1}{2-x} = \frac{3x+8}{x^2-4}$

16) $\frac{x+2}{3x-1} - \frac{1}{x} = \frac{3x-3}{3x^2-x}$

17) $\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{5}{6}$

18) $\frac{x-1}{x-3} + \frac{x+2}{x+3} = \frac{3}{4}$

19) $\frac{3}{2x+1} + \frac{2x+1}{1-2x} = 1 - \frac{8x^2}{4x^2-1}$

20) $\frac{3x-5}{5x-5} + \frac{5x-1}{7x-7} - \frac{x-4}{1-x} = 2$

21) $\frac{x-2}{x+3} - \frac{1}{x-2} = \frac{1}{x^2+x-6}$

22) $\frac{x-1}{x-2} + \frac{x+4}{2x+1} = \frac{1}{2x^2-3x-2}$

23) $\frac{3}{x+2} + \frac{x-1}{x+5} = \frac{5x+20}{6x+24}$

24) $\frac{x}{x+3} - \frac{4}{x-2} = \frac{-5x^2}{x^2+x-6}$

25) $\frac{x}{x-1} - \frac{2}{x+1} = \frac{4x^2}{x^2-1}$

26) $\frac{2x}{x+2} + \frac{2}{x-4} = \frac{3x}{x^2-2x-8}$

27) $\frac{2x}{x+1} - \frac{3}{x+5} = \frac{-8x^2}{x^2+6x+5}$

28) $\frac{x}{x+1} - \frac{3}{x+3} = \frac{-2x^2}{x^2+4x+3}$

29) $\frac{x-5}{x-9} + \frac{x+3}{x-3} = \frac{-4x^2}{x^2-12x+27}$

30) $\frac{x-3}{x+6} + \frac{x-2}{x-3} = \frac{x^2}{x^2+3x-18}$

31) $\frac{x-3}{x-6} + \frac{x+5}{x+3} = \frac{-2x^2}{x^2-3x-18}$

32) $\frac{x+3}{x-2} + \frac{x-2}{x+1} = \frac{9x^2}{x^2-x-2}$

33) $\frac{4x+1}{x+3} + \frac{5x-3}{x-1} = \frac{8x^2}{x^2+2x-3}$

34) $\frac{3x-1}{x+6} - \frac{2x-3}{x-3} = \frac{-3x^2}{x^2+3x-18}$

7.8

Rational Expressions - Dimensional Analysis

Objective: Use dimensional analysis to preform single unit, dual unit, square unit, and cubed unit conversions.

One application of rational expressions deals with converting units. When we convert units of measure we can do so by multiplying several fractions together in a process known as dimensional analysis. The trick will be to decide what fractions to multiply. When multiplying, if we multiply by 1, the value of the expression does not change. One written as a fraction can look like many different things as long as the numerator and denominator are identical in value. Notice the numerator and denominator are not identical in appearance, but rather identical in value. Below are several fractions, each equal to one where numerator and denominator are identical in value.

$$\frac{1}{1}=\frac{4}{4}=\frac{\frac{1}{2}}{\frac{2}{4}}=\frac{100\,\text{cm}}{1m}=\frac{1\,\text{lb}}{16\,\text{oz}}=\frac{1\,\text{hr}}{60\,\text{min}}=\frac{60\,\text{min}}{1\,\text{hr}}$$

The last few fractions that include units are called conversion factors. We can make a conversion factor out of any two measurements that represent the same distance. For example, 1 mile = 5280 feet. We could then make a conversion factor $\frac{1\,\text{mi}}{5280\,\text{ft}}$ because both values are the same, the fraction is still equal to one. Similarly we could make a conversion factor $\frac{5280\,\text{ft}}{1\,\text{mi}}$. The trick for conversions will be to use the correct fractions.

The idea behind dimensional analysis is we will multiply by a fraction in such a way that the units we don't want will divide out of the problem. We found out when multiplying rational expressions that if a variable appears in the numerator and denominator we can divide it out of the expression. It is the same with units. Consider the following conversion.

Example 371.

$$\begin{array}{rl} 17.37\,\text{miles to feet} & \text{Write 17.37 miles as } a \text{ fraction, put it over 1} \\ \left(\dfrac{17.37\,\text{mi}}{1}\right) & \text{To divide out the miles we need miles in the denominator} \\ \left(\dfrac{17.37\,\text{mi}}{1}\right)\left(\dfrac{??\,\text{ft}}{??\,\text{mi}}\right) & \text{We are converting to feet, so this will go in the numerator} \\ \left(\dfrac{17.37\,\text{mi}}{1}\right)\left(\dfrac{5280\,\text{ft}}{1\,\text{mi}}\right) & \text{Fill in the relationship described above, 1 mile} = 5280 \text{ feet} \\ \left(\dfrac{17.37}{1}\right)\left(\dfrac{5280\,\text{ft}}{1}\right) & \text{Divide out the miles and multiply across} \\ 91,713.6\,\text{ft} & \text{Our Solution} \end{array}$$

In the previous example, we had to use the conversion factor $\frac{5280\,\text{ft}}{1\,\text{mi}}$ so the miles would divide out. If we had used $\frac{1\,\text{mi}}{5280\,\text{ft}}$ we would not have been able to divide out the miles. This is why when doing dimensional analysis it is very important to use units in the set-up of the problem, so we know how to correctly set up the conversion factor.

Example 372.

If 1 pound = 16 ounces, how many pounds 435 ounces?

$$\begin{array}{rl} \left(\dfrac{435\,\text{oz}}{1}\right) & \text{Write 435 as } a \text{ fraction, put it over 1} \\ \left(\dfrac{435\,\text{oz}}{1}\right)\left(\dfrac{??\,\text{lbs}}{??\,\text{oz}}\right) & \text{To divide out oz,} \\ & \text{put it in the denominator and lbs in numerator} \\ \left(\dfrac{435\,\text{oz}}{1}\right)\left(\dfrac{1\,\text{lbs}}{16\,\text{oz}}\right) & \text{Fill in the given relationship, 1 pound} = 16 \text{ ounces} \end{array}$$

$$\left(\frac{435}{1}\right)\left(\frac{1\,\text{lbs}}{16}\right)=\frac{435\,\text{lbs}}{16} \quad \text{Divide out oz, multiply across. Divide result}$$

$$27.1875\,\text{lbs} \quad \text{Our Solution}$$

The same process can be used to convert problems with several units in them. Consider the following example.

Example 373.

A student averaged 45 miles per hour on a trip. What was the student's speed in feet per second?

$$\left(\frac{45\,\text{mi}}{\text{hr}}\right) \quad \text{"per" is the fraction bar, put hr in denominator}$$

$$\left(\frac{45\,\text{mi}}{\text{hr}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mi}}\right) \quad \text{To clear mi they must go in denominator and become ft}$$

$$\left(\frac{45\,\text{mi}}{\text{hr}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mi}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{sec}}\right) \quad \text{To clear hr they must go in numerator and become sec}$$

$$\left(\frac{45}{1}\right)\left(\frac{5280\,\text{ft}}{1}\right)\left(\frac{1}{3600\,\text{sec}}\right) \quad \text{Divide out mi and hr. Multiply across}$$

$$\frac{237600\,\text{ft}}{3600\,\text{sec}} \quad \text{Divide numbers}$$

$$66\,\text{ft per sec} \quad \text{Our Solution}$$

If the units are two-dimensional (such as square inches - in^2) or three-dimensional (such as cubic feet - ft^3) we will need to put the same exponent on the conversion factor. So if we are converting square inches (in^2) to square ft (ft^2), the conversion factor would be squared, $\left(\frac{1\,\text{ft}}{12\text{in}}\right)^2$. Similarly if the units are cubed, we will cube the convesion factor.

Example 374.

$$\text{Convert 8 cubic feet to yd}^3 \quad \text{Write } 8\text{ft}^3 \text{ as fraction, put it over 1}$$

$$\left(\frac{8\,\text{ft}^3}{1}\right) \quad \text{To clear ft, put them in denominator, yard in numerator}$$

$$\left(\frac{8\,\text{ft}^3}{1}\right)\left(\frac{??\,\text{yd}}{??\,\text{ft}}\right)^3 \quad \text{Because the units are cubed,}$$

$$\text{we cube the conversion factor}$$

$$\left(\frac{8\,\text{ft}^3}{1}\right)\left(\frac{1\,\text{yd}}{3\,\text{ft}}\right)^3 \quad \text{Evaluate exponent, cubing all numbers and units}$$

$$\left(\frac{8\,\text{ft}^3}{1}\right)\left(\frac{1\,\text{yd}^3}{27\,\text{ft}^3}\right) \quad \text{Divide out ft}^3$$

$$\left(\frac{8}{1}\right)\left(\frac{1\,\text{yd}^3}{27}\right)=\frac{8\,\text{yd}^3}{27} \quad \text{Multiply across and divide}$$

$$0.296296\,\text{yd}^3 \quad \text{Our Solution}$$

When calculating area or volume, be sure to use the units and multiply them as well.

Example 375.

A room is 10 ft by 12 ft. How many square yards are in the room?

$$A=lw=(10\text{ft})(12\,\text{ft})=120\,\text{ft}^2 \quad \text{Multiply length by width, also multiply units}$$

$$\left(\frac{120\,\text{ft}^2}{1}\right) \quad \text{Write area as } a \text{ fraction, put it over 1}$$

$$\left(\frac{120\,\text{ft}^2}{1}\right)\left(\frac{??\,\text{yd}}{??\,\text{ft}}\right)^2 \quad \text{Put ft in denominator to clear,}$$

$$\text{square conversion factor}$$

$$\left(\frac{120\,\text{ft}^2}{1}\right)\left(\frac{1\,\text{yd}}{3\,\text{ft}}\right)^2 \quad \text{Evaluate exponent, squaring all numbers and units}$$

$$\left(\frac{120\,\text{ft}^2}{1}\right)\left(\frac{1\,\text{yd}^2}{9\,\text{ft}^2}\right) \quad \text{Divide out ft}^2$$

$$\left(\frac{120}{1}\right)\left(\frac{1\,\text{yd}^2}{9}\right)=\frac{120\,\text{yd}^2}{9} \quad \text{Multiply across and divide}$$

$$13.33\,\text{yd}^2 \quad \text{Our solution}$$

To focus on the process of conversions, a conversion sheet has been included at the end of this lesson which includes several conversion factors for length, volume, mass and time in both English and Metric units.

The process of dimensional analysis can be used to convert other types of units as well. If we can identify relationships that represent the same value we can make them into a conversion factor.

Example 376.

A child is perscribed a dosage of 12 mg of a certain drug and is allowed to refill his prescription twice. If a there are 60 tablets in a prescription, and each tablet has 4 mg, how many doses are in the 3 prescriptions (original + 2 refills)?

Convert 3 Rx to doses	Identify what the problem is asking
$1\,\text{Rx} = 60\,\text{tab}, 1\,\text{tab} = 4\,\text{mg}, 1\,\text{dose} = 12\,\text{mg}$	Identify given conversion factors
$\left(\frac{3\,\text{Rx}}{1}\right)$	Write 3Rx as fraction, put over 1
$\left(\frac{3\,\text{Rx}}{1}\right)\left(\frac{60\,\text{tab}}{1\text{Rx}}\right)$	Convert Rx to tab, put Rx in denominator
$\left(\frac{3\,\text{Rx}}{1}\right)\left(\frac{60\,\text{tab}}{1\text{Rx}}\right)\left(\frac{4\,\text{mg}}{1\,\text{tab}}\right)$	Convert tab to mg, put tab in denominator
$\left(\frac{3\,\text{Rx}}{1}\right)\left(\frac{60\,\text{tab}}{1\text{Rx}}\right)\left(\frac{4\,\text{mg}}{1\,\text{tab}}\right)\left(\frac{1\,\text{dose}}{12\,\text{mg}}\right)$	Convert mg to dose, put mg in denominator
$\left(\frac{3}{1}\right)\left(\frac{60}{1}\right)\left(\frac{4}{1}\right)\left(\frac{1\,\text{dose}}{12}\right)$	Divide out Rx, tab, and mg, multiply across
$\frac{720\,\text{dose}}{12}$	Divide
60 doses	Our Solution

World View Note: Only three countries in the world still use the English system commercially: Liberia (Western Africa), Myanmar (between India and Vietnam), and the USA.

Conversion Factors

Length

English	Metric
12 in = 1 ft	1000 mm = 1 m
1 yd = 3 ft	10 mm = 1 cm
1 yd = 36 in	100 cm = 1 m
1 mi = 5280 ft	10 dm = 1 m
	1 dam = 10 m
	1 hm = 100 m
	1 km = 1000 m

English/Metric
2.54 cm = 1 in
1 m = 3.28 ft
1.61 km = 1 mi

Area

English	Metric
1 ft^2 = 144 in^2	1 a = 100 m^2
1 yd^2 = 9 ft^2	1 ha = 100 a
1 acre = 43,560 ft^2	
640 acres = 1 mi^2	

English/Metric
1 ha = 2.47 acres

Volume

English	Metric
1 c = 8 oz	1 mL = 1 cc = 1 cm^3
1 pt = 2 c	1 L = 1000 mL
1 qt = 2 pt	1 L = 100 cL
1 gal = 4 qt	1 L = 10 dL
	1000 L = 1 kL

English/Metric
16.39 mL = 1 in^3
1.06 qt = 1 L
3.79 L = 1gal

Weight (Mass)

English	Metric
1 lb = 16 oz	1 g = 1000 mg
1 T = 2000 lb	1 g = 100 cg
	1000 g = 1 kg
	1000 kg = 1 t

English/Metric
28.3 g = 1 oz
2.2 lb = 1 kg

Time

60 sec = 1 min
60 min = 1 hr
3600 sec = 1 hr
24 hr = 1 day

7.8 Practice - Dimensional Analysis

Use dimensional analysis to convert the following:

1) 7 mi. to yards

2) 234 oz. to tons

3) 11.2 mg to grams

4) 1.35 km to centimeters

5) 9,800,000 mm (milimeters) to miles

6) 4.5 ft^2 to square yards

7) 435,000 m^2 to sqaure kilometers

8) 8 km^2 to square feet

9) 0.0065 km^3 to cubic meters

10) 14.62 in^3 to cubic centimeters

11) 5,500 cm^3 to cubic yards

12) 3.5 mph (miles per hour) to feet per second

13) 185 yd. per min. to miles per hour

14) 153 ft/s (feet per second) to miles per hour

15) 248 mph to meters per second

16) 186,000 mph to kilometers per year

17) 7.50 T/yd^2 (tons per square yard) to pounds per square inch

18) 16 ft/s^2 to kilometers per hour squared

Use dimensional analysis to solve the following:

19) On a recent trip, Jan traveled 260 miles using 8 gallons of gas. How many miles per 1-gallon did she travel? How many yards per 1-ounce?

20) A chair lift at the Divide ski resort in Cold Springs, WY is 4806 feet long and takes 9 minutes. What is the average speed in miles per hour? How many feet per second does the lift travel?

21) A certain laser printer can print 12 pages per minute. Determine this printer's output in pages per day, and reams per month. (1 ream = 5000 pages)

22) An average human heart beats 60 times per minute. If an average person lives to the age of 75, how many times does the average heart beat in a lifetime?

23) Blood sugar levels are measured in miligrams of gluclose per deciliter of blood volume. If a person's blood sugar level measured 128 mg/dL, how much is this in grams per liter?

24) You are buying carpet to cover a room that measures 38 ft by 40 ft. The carpet cost $18 per square yard. How much will the carpet cost?

25) A car travels 14 miles in 15 minutes. How fast is it going in miles per hour? in meters per second?

26) A cargo container is 50 ft long, 10 ft wide, and 8 ft tall. Find its volume in cubic yards and cubic meters.

27) A local zoning ordinance says that a house's "footprint" (area of its ground floor) cannot occupy more than $\frac{1}{4}$ of the lot it is built on. Suppose you own a $\frac{1}{3}$ acre lot, what is the maximum allowed footprint for your house in square feet? in square inches? (1 acre = 43560 ft^2)

28) Computer memory is measured in units of bytes, where one byte is enough memory to store one character (a letter in the alphabet or a number). How many typical pages of text can be stored on a 700-megabyte compact disc? Assume that one typical page of text contains 2000 characters. (1 megabyte = 1,000,000 bytes)

29) In April 1996, the Department of the Interior released a "spike flood" from the Glen Canyon Dam on the Colorado River. Its purpose was to restore the river and the habitants along its bank. The release from the dam lasted a week at a rate of 25,800 cubic feet of water per second. About how much water was released during the 1-week flood?

30) The largest single rough diamond ever found, the Cullinan diamond, weighed 3106 carats; how much does the diamond weigh in miligrams? in pounds? (1 carat - 0.2 grams)

Chapter 8 : Radicals

8.1

Radicals - Square Roots

Objective: Simplify expressions with square roots.

Square roots are the most common type of radical used. A square root "unsquares" a number. For example, because $5^2 = 25$ we say the square root of 25 is 5. The square root of 25 is written as $\sqrt{25}$.

World View Note: The radical sign, when first used was an R with a line through the tail, similar to our perscription symbol today. The R came from the latin, "radix", which can be translated as "source" or "foundation". It wasn't until the 1500s that our current symbol was first used in Germany (but even then it was just a check mark with no bar over the numbers!

The following example gives several square roots:

Example 377.

$\sqrt{1} = 1$	$\sqrt{121} = 11$
$\sqrt{4} = 2$	$\sqrt{625} = 25$
$\sqrt{9} = 3$	$\sqrt{-81} = \text{Undefined}$

The final example, $\sqrt{-81}$ is currently undefined as negatives have no square root. This is because if we square a positive or a negative, the answer will be positive. Thus we can only take square roots of positive numbers. In another lesson we will define a method we can use to work with and evaluate negative square roots, but for now we will simply say they are undefined.

Not all numbers have a nice even square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.828427124746190097603377448419... and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals

Product Rule of Square Roots: $\boldsymbol{\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}}$

We can use the product rule to simplify an expression such as $\sqrt{36 \cdot 5}$ by spliting it into two roots, $\sqrt{36} \cdot \sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to translate a problem like $\sqrt{180}$ into $\sqrt{36\cdot 5}$. There are several ways this can be done. The most common and, with a bit of practice, the fastest method, is to find perfect squares that divide evenly into the radicand, or number under the radical. This is shown in the next example.

Example 378.

$$\begin{array}{rl}\sqrt{75} & \text{75 is divisible by 25, } a \text{ perfect square}\\ \sqrt{25\cdot 3} & \text{Split into factors}\\ \sqrt{25}\cdot\sqrt{3} & \text{Product rule, take the square root of 25}\\ 5\sqrt{3} & \text{Our Solution}\end{array}$$

If there is a coefficient in front of the radical to begin with, the problem merely becomes a big multiplication problem.

Example 379.

$$\begin{array}{rl}5\sqrt{63} & \text{63 is divisible by 9, } a \text{ perfect square}\\ 5\sqrt{9\cdot 7} & \text{Split into factors}\\ 5\sqrt{9}\cdot\sqrt{7} & \text{Product rule, take the square root of 9}\\ 5\cdot 3\sqrt{7} & \text{Multiply coefficients}\\ 15\sqrt{7} & \text{Our Solution}\end{array}$$

As we simplify radicals using this method it is important to be sure our final answer can be simplified no more.

Example 380.

$$\begin{array}{rl}\sqrt{72} & \text{72 is divisible by 9, } a \text{ perfect square}\\ \sqrt{9\cdot 8} & \text{Split into factors}\\ \sqrt{9}\cdot\sqrt{8} & \text{Product rule, take the square root of 9}\\ 3\sqrt{8} & \text{But 8 is also divisible by } a \text{ perfect square, 4}\\ 3\sqrt{4\cdot 2} & \text{Split into factors}\\ 3\sqrt{4}\cdot\sqrt{2} & \text{Product rule, take the square root of 4}\\ 3\cdot 2\sqrt{2} & \text{Multiply}\end{array}$$

$$6\sqrt{2} \quad \text{Our Solution.}$$

The previous example could have been done in fewer steps if we had noticed that $72 = 36 \cdot 2$, but often the time it takes to discover the larger perfect square is more than it would take to simplify in several steps.

Variables often are part of the radicand as well. When taking the square roots of variables, we can divide the exponent by 2. For example, $\sqrt{x^8} = x^4$, because we divide the exponent of 8 by 2. This follows from the power of a power rule of expoennts, $(x^4)^2 = x^8$. When squaring, we multiply the exponent by two, so when taking a square root we divide the exponent by 2. This is shown in the following example.

Example 381.

$$\begin{array}{rl}
-5\sqrt{18x^4y^6z^{10}} & \text{18 is divisible by 9, } a \text{ perfect square} \\
-5\sqrt{9 \cdot 2x^4y^6z^{10}} & \text{Split into factors} \\
-5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}} & \text{Product rule, simplify roots, divide exponents by 2} \\
-5 \cdot 3x^2y^3z^5\sqrt{2} & \text{Multiply coefficients} \\
-15x^2y^3z^5\sqrt{2} & \text{Our Solution}
\end{array}$$

We can't always evenly divide the exponent on a variable by 2. Sometimes we have a remainder. If there is a remainder, this means the remainder is left inside the radical, and the whole number part is how many are outside the radical. This is shown in the following example.

Example 382.

$$\begin{array}{rl}
\sqrt{20x^5y^9z^6} & \text{20 is divisible by 4, } a \text{ perfect square} \\
\sqrt{4 \cdot 5x^5y^9z^6} & \text{Split into factors} \\
\sqrt{4} \cdot \sqrt{5} \cdot \sqrt{x^5} \cdot \sqrt{y^9} \cdot \sqrt{z^6} & \text{Simplify, divide exponents by 2, remainder is left inside} \\
2x^2y^4z^3\sqrt{5xy} & \text{Our Solution}
\end{array}$$

8.1 Practice - Square Roots

Simplify.

1) $\sqrt{245}$

2) $\sqrt{125}$

3) $\sqrt{36}$

4) $\sqrt{196}$

5) $\sqrt{12}$

6) $\sqrt{72}$

7) $3\sqrt{12}$

8) $5\sqrt{32}$

9) $6\sqrt{128}$

10) $7\sqrt{128}$

11) $-8\sqrt{392}$

12) $-7\sqrt{63}$

13) $\sqrt{192n}$

14) $\sqrt{343b}$

15) $\sqrt{196v^2}$

16) $\sqrt{100n^3}$

17) $\sqrt{252x^2}$

18) $\sqrt{200a^3}$

19) $-\sqrt{100k^4}$

20) $-4\sqrt{175p^4}$

21) $-7\sqrt{64x^4}$

22) $-2\sqrt{128n}$

23) $-5\sqrt{36m}$

24) $8\sqrt{112p^2}$

25) $\sqrt{45x^2y^2}$

26) $\sqrt{72a^3b^4}$

27) $\sqrt{16x^3y^3}$

28) $\sqrt{512a^4b^2}$

29) $\sqrt{320x^4y^4}$

30) $\sqrt{512m^4n^3}$

31) $6\sqrt{80xy^2}$

32) $8\sqrt{98mn}$

33) $5\sqrt{245x^2y^3}$

34) $2\sqrt{72x^2y^2}$

35) $-2\sqrt{180u^3v}$

36) $-5\sqrt{72x^3y^4}$

37) $-8\sqrt{180x^4y^2z^4}$

38) $6\sqrt{50a^4bc^2}$

39) $2\sqrt{80hj^4k}$

40) $-\sqrt{32xy^2z^3}$

41) $-4\sqrt{54mnp^2}$

42) $-8\sqrt{32m^2p^4q}$

8.2

Radicals - Higher Roots

Objective: Simplify radicals with an index greater than two.

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. Following is a definition of radicals.

$$\boldsymbol{\sqrt[m]{a} = b \text{ if } b^m = a}$$

The small letter m inside the radical is called the index. It tells us which root we are taking, or which power we are "un-doing". For square roots the index is 2. As this is the most common root, the two is not usually written.

World View Note: The word for root comes from the French mathematician Franciscus Vieta in the late 16th century.

The following example includes several higher roots.

Example 383.

$\sqrt[3]{125} = 5$	$\sqrt[3]{-64} = -4$
$\sqrt[4]{81} = 3$	$\sqrt[7]{-128} = -2$
$\sqrt[5]{32} = 2$	$\sqrt[4]{-16} = \text{undefined}$

We must be careful of a few things as we work with higher roots. First its important not to forget to check the index on the root. $\sqrt{81} = 9$ but $\sqrt[4]{81} = 3$. This is because $9^2 = 81$ and $3^4 = 81$. Another thing to watch out for is negatives under roots. We can take an odd root of a negative number, because a negative number raised to an odd power is still negative. However, we cannot take an even root of a negative number, this we will say is undefined. In a later section we will discuss how to work with roots of negative, but for now we will simply say they are undefined.

We can simplify higher roots in much the same way we simplified square roots, using the product property of radicals.

$$\textbf{Product Property of Radicals: } \boldsymbol{\sqrt[m]{ab} = \sqrt[m]{a} \cdot \sqrt[m]{b}}$$

Often we are not as familiar with higher powers as we are with squares. It is important to remember what index we are working with as we try and work our way to the solution.

Example 384.

$\sqrt[3]{54}$	We are working with a cubed root, want third powers
$2^3 = 8$	Test 2, $2^3 = 8$, 54 is not divisible by 8.
$3^3 = 27$	Test 3, $3^3 = 27$, 54 is divisible by 27!
$\sqrt[3]{27 \cdot 2}$	Write as factors
$\sqrt[3]{27} \cdot \sqrt[3]{2}$	Product rule, take cubed root of 27
$3\sqrt[3]{2}$	Our Solution

Just as with square roots, if we have a coefficient, we multiply the new coefficients together.

Example 385.

$3\sqrt[4]{48}$	We are working with a fourth root, want fourth powers
$2^4 = 16$	Test 2, $2^4 = 16$, 48 is divisible by 16!
$3\sqrt[4]{16 \cdot 3}$	Write as factors
$3\sqrt[4]{16} \cdot \sqrt[4]{3}$	Product rule, take fourth root of 16
$3 \cdot 2\sqrt[4]{3}$	Multiply coefficients
$6\sqrt[4]{3}$	Our Solution

We can also take higher roots of variables. As we do, we will divide the exponent on the variable by the index. Any whole answer is how many of that varible will come out of the square root. Any remainder is how many are left behind inside the square root. This is shown in the following examples.

Example 386.

$\sqrt[5]{x^{25}y^{17}z^3}$	Divide each exponent by 5, whole number outside, remainder inside
$x^5y^3\sqrt[5]{y^2z^3}$	Our Solution

In the previous example, for the x, we divided $\frac{25}{5} = 5\,R\,0$, so x^5 came out, no x's remain inside. For the y, we divided $\frac{17}{5} = 3R2$, so y^3 came out, and y^2 remains inside. For the z, when we divided $\frac{3}{5} = 0R3$, all three or z^3 remained inside. The following example includes integers in our problem.

Example 387.

$2\sqrt[3]{40a^4b^8}$	Looking for cubes that divide into 40. The number 8 works!
$2\sqrt[3]{8 \cdot 5a^4b^8}$	Take cube root of 8, dividing exponents on variables by 3
$2 \cdot 2ab^2\sqrt[3]{5ab^2}$	Remainders are left in radical. Multiply coefficients
$4ab^2\sqrt[3]{5ab^2}$	Our Solution

8.2 Practice - Higher Roots

Simplify.

1) $\sqrt[3]{625}$

2) $\sqrt[3]{375}$

3) $\sqrt[3]{750}$

4) $\sqrt[3]{250}$

5) $\sqrt[3]{875}$

6) $\sqrt[3]{24}$

7) $-4\sqrt[4]{96}$

8) $-8\sqrt[4]{48}$

9) $6\sqrt[4]{112}$

10) $3\sqrt[4]{48}$

11) $-\sqrt[4]{112}$

12) $5\sqrt[4]{243}$

13) $\sqrt[4]{648a^2}$

14) $\sqrt[4]{64n^3}$

15) $\sqrt[5]{224n^3}$

16) $\sqrt[5]{-96x^3}$

17) $\sqrt[5]{224p^5}$

18) $\sqrt[6]{256x^6}$

19) $-3\sqrt[7]{896r}$

20) $-8\sqrt[7]{384b^8}$

21) $-2\sqrt[3]{-48v^7}$

22) $4\sqrt[3]{250a^6}$

23) $-7\sqrt[3]{320n^6}$

24) $-\sqrt[3]{512n^6}$

25) $\sqrt[3]{-135x^5y^3}$

26) $\sqrt[3]{64u^5v^3}$

27) $\sqrt[3]{-32x^4y^4}$

28) $\sqrt[3]{1000a^4b^5}$

29) $\sqrt[3]{256x^4y^6}$

30) $\sqrt[3]{189x^3y^6}$

31) $7\sqrt[3]{-81x^3y^7}$

32) $-4\sqrt[3]{56x^2y^8}$

33) $2\sqrt[3]{375u^2v^8}$

34) $8\sqrt[3]{-750xy}$

35) $-3\sqrt[3]{192ab^2}$

36) $3\sqrt[3]{135xy^3}$

37) $6\sqrt[3]{-54m^8n^3p^7}$

38) $-6\sqrt[4]{80m^4p^7q^4}$

39) $6\sqrt[4]{648x^5y^7z^2}$

40) $-6\sqrt[4]{405a^5b^8c}$

41) $7\sqrt[4]{128h^6j^8k^8}$

42) $-6\sqrt[4]{324x^7yz^7}$

8.3

Radicals - Adding Radicals

Objective: Add like radicals by first simplifying each radical.

Adding and subtracting radicals is very similar to adding and subtracting with variables. Consider the following example.

Example 388.

$$
\begin{array}{rl}
5x+3x-2x & \text{Combine like terms} \\
6x & \text{Our Solution}
\end{array}
$$

$$
\begin{array}{rl}
5\sqrt{11}+3\sqrt{11}-2\sqrt{11} & \text{Combine like terms} \\
6\sqrt{11} & \text{Our Solution}
\end{array}
$$

Notice that when we combined the terms with $\sqrt{11}$ it was just like combining terms with x. When adding and subtracting with radicals we can combine like radicals just as like terms. We add and subtract the coefficients in front of the radical, and the radical stays the same. This is shown in the following example.

Example 389.

$$\begin{array}{rl} 7\sqrt[5]{6}+4\sqrt[5]{3}-9\sqrt[5]{3}+\sqrt[5]{6} & \text{Combine like radicals } 7\sqrt[5]{6}+\sqrt[5]{6} \text{ and } 4\sqrt[5]{3}-8\sqrt[5]{3} \\ 8\sqrt[5]{6}-5\sqrt[5]{3} & \text{Our Solution} \end{array}$$

We cannot simplify this expression any more as the radicals do not match. Often problems we solve have no like radicals, however, if we simplify the radicals first we may find we do in fact have like radicals.

Example 390.

$$\begin{array}{rl} 5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20} & \text{Simplify radicals, find perfect square factors} \\ 5\sqrt{9\cdot 5}+6\sqrt{9\cdot 2}-2\sqrt{49\cdot 2}+\sqrt{4\cdot 5} & \text{Take roots where possible} \\ 5\cdot 3\sqrt{5}+6\cdot 3\sqrt{2}-2\cdot 7\sqrt{2}+2\sqrt{5} & \text{Multiply coefficients} \\ 15\sqrt{5}+18\sqrt{2}-14\sqrt{2}+2\sqrt{5} & \text{Combine like terms} \\ 17\sqrt{5}+4\sqrt{2} & \text{Our Solution} \end{array}$$

World View Note: The Arab writers of the 16th century used the symbol similar to the greater than symbol with a dot underneath for radicals.

This exact process can be used to add and subtract radicals with higher indices

Example 391.

$$\begin{array}{rl} 4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9} & \text{Simplify each radical, finding perfect cube factors} \\ 4\sqrt[3]{27\cdot 2}-9\sqrt[3]{8\cdot 2}+5\sqrt[3]{9} & \text{Take roots where possible} \\ 4\cdot 3\sqrt[3]{2}-9\cdot 2\sqrt[3]{2}+5\sqrt[3]{9} & \text{Multiply coefficients} \\ 12\sqrt[3]{2}-18\sqrt[3]{2}+5\sqrt[3]{9} & \text{Combine like terms } 12\sqrt[3]{2}-18\sqrt[3]{2} \\ -6\sqrt[3]{2}+5\sqrt[3]{9} & \text{Our Solution} \end{array}$$

8.3 Practice - Adding Radicals

Simiplify

1) $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2) $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

3) $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

4) $-2\sqrt{6}-\sqrt{3}-3\sqrt{6}$

5) $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

6) $-3\sqrt{3}+2\sqrt{3}-2\sqrt{3}$

7) $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$

8) $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

9) $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

10) $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

11) $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$

12) $-\sqrt{5}-\sqrt{5}-2\sqrt{54}$

13) $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

14) $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

15) $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

16) $-3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

17) $-3\sqrt{6}-3\sqrt{6}-\sqrt{3}+3\sqrt{6}$

18) $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

19) $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

20) $-3\sqrt{18}-\sqrt{8}+2\sqrt{8}+2\sqrt{8}$

21) $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$

22) $-3\sqrt{8}-\sqrt{5}-3\sqrt{6}+2\sqrt{18}$

23) $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$

24) $2\sqrt{6}-\sqrt{54}-3\sqrt{27}-\sqrt{3}$

25) $-2\sqrt[3]{16}+2\sqrt[3]{16}+2\sqrt[3]{2}$

26) $3\sqrt[3]{135}-\sqrt[3]{81}-\sqrt[3]{135}$

27) $2\sqrt[4]{243}-2\sqrt[4]{243}-\sqrt[4]{3}$

28) $-3\sqrt[4]{4}+3\sqrt[4]{324}+2\sqrt[4]{64}$

29) $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$

30) $2\sqrt[4]{6}+2\sqrt[4]{4}+3\sqrt[4]{6}$

31) $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

32) $-2\sqrt[4]{243}-\sqrt[4]{96}+2\sqrt[4]{96}$

33) $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

34) $2\sqrt[4]{48}-3\sqrt[4]{405}-3\sqrt[4]{48}-\sqrt[4]{162}$

35) $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{64}$

36) $-3\sqrt[7]{3}-3\sqrt[7]{768}+2\sqrt[7]{384}+3\sqrt[7]{5}$

37) $2\sqrt[5]{160}-2\sqrt[5]{192}-\sqrt[5]{160}-\sqrt[5]{-160}$

38) $-2\sqrt[7]{256}-2\sqrt[7]{256}-3\sqrt[7]{2}-\sqrt[7]{640}$

39) $-\sqrt[6]{256}-2\sqrt[6]{4}-3\sqrt[6]{320}-2\sqrt[6]{128}$

8.4

Radicals - Multiply and Divide Radicals

Objective: Multiply and divide radicals using the product and quotient rules of radicals.

Multiplying radicals is very simple if the index on all the radicals match. The prodcut rule of radicals which we have already been using can be generalized as follows:

$$\textbf{Product Rule of Radicals: } \boldsymbol{a\sqrt[m]{b}\cdot c\sqrt[m]{d} = ac\sqrt[m]{bd}}$$

Another way of stating this rule is we are allowed to multiply the factors outside the radical and we are allowed to multiply the factors inside the radicals, as long as the index matches. This is shown in the following example.

Example 392.

$$\begin{array}{rl} -5\sqrt{14}\cdot 4\sqrt{6} & \text{Multiply outside and inside the radical} \\ -20\sqrt{84} & \text{Simplify the radical, divisible by 4} \\ -20\sqrt{4\cdot 21} & \text{Take the square root where possible} \\ -20\cdot 2\sqrt{21} & \text{Multiply coefficients} \\ -40\sqrt{21} & \text{Our Solution} \end{array}$$

The same process works with higher roots

Example 393.

$$\begin{array}{rl} 2\sqrt[3]{18}\cdot 6\sqrt[3]{15} & \text{Multiply outside and inside the radical} \\ 12\sqrt[3]{270} & \text{Simplify the radical, divisible by 27} \\ 12\sqrt[3]{27\cdot 10} & \text{Take cube root where possible} \\ 12\cdot 3\sqrt[3]{10} & \text{Multiply coefficients} \\ 36\sqrt[3]{10} & \text{Our Solution} \end{array}$$

When multiplying with radicals we can still use the distributive property or FOIL just as we could with variables.

Example 394.

$$\begin{array}{rl} 7\sqrt{6}(3\sqrt{10}-5\sqrt{15}) & \text{Distribute, following rules for multiplying radicals} \\ 21\sqrt{60}-35\sqrt{90} & \text{Simplify each radical, finding perfect square factors} \\ 21\sqrt{4\cdot 15}-35\sqrt{9\cdot 10} & \text{Take square root where possible} \\ 21\cdot 2\sqrt{15}-35\cdot 3\sqrt{10} & \text{Multiply coefficients} \\ 42\sqrt{15}-105\sqrt{10} & \text{Our Solution} \end{array}$$

Example 395.

$$\begin{array}{rl} (\sqrt{5}-2\sqrt{3})(4\sqrt{10}+6\sqrt{6}) & \text{FOIL, following rules for multiplying radicals} \end{array}$$

$$
\begin{array}{rl}
4\sqrt{50}+6\sqrt{30}-8\sqrt{30}-12\sqrt{18} & \text{Simplify radicals, find perfect square factors} \\
4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2} & \text{Take square root where possible} \\
4\cdot 5\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2} & \text{Multiply coefficients} \\
20\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2} & \text{Combine like terms} \\
-16\sqrt{2}-2\sqrt{30} & \text{Our Solution}
\end{array}
$$

World View Note: Clay tablets have been discovered revealing much about Babylonian mathematics dating back from 1800 to 1600 BC. In one of the tables there is an approximation of $\sqrt{2}$ accurate to five decimal places (1.41421)

Example 396.

$$
\begin{array}{rl}
(2\sqrt{5}-3\sqrt{6})(7\sqrt{2}-8\sqrt{7}) & \text{FOIL, following rules for multiplying radicals} \\
14\sqrt{10}-16\sqrt{35}-21\sqrt{12}-24\sqrt{42} & \text{Simplify radicals, find perfect square factors} \\
14\sqrt{10}-16\sqrt{35}-21\sqrt{4\cdot 3}-24\sqrt{42} & \text{Take square root where possible} \\
14\sqrt{10}-16\sqrt{35}-21\cdot 2\sqrt{3}-24\sqrt{42} & \text{Multiply coefficient} \\
14\sqrt{10}-16\sqrt{35}-42\sqrt{3}-24\sqrt{42} & \text{Our Solution}
\end{array}
$$

As we are multiplying we always look at our final solution to check if all the radicals are simplified and all like radicals or like terms have been combined.

Division with radicals is very similar to multiplication, if we think about division as reducing fractions, we can reduce the coefficients outside the radicals and reduce the values inside the radicals to get our final solution.

$$\textbf{Quotient Rule of Radicals: } \boldsymbol{\frac{a\sqrt[m]{b}}{c\sqrt[m]{d}}=\frac{a}{c}\sqrt[m]{\frac{b}{d}}}$$

Example 397.

$$
\begin{array}{rl}
\dfrac{15\sqrt[3]{108}}{20\sqrt[3]{2}} & \text{Reduce } \dfrac{15}{20} \text{ and } \dfrac{\sqrt[3]{108}}{\sqrt{2}} \text{ by dividing by 5 and 2 respectively} \\[2ex]
\dfrac{3\sqrt[3]{54}}{4} & \text{Simplify radical, 54 is divisible by 27} \\[2ex]
\dfrac{3\sqrt[3]{27\cdot 2}}{4} & \text{Take the cube root of 27} \\[2ex]
\dfrac{3\cdot 3\sqrt[3]{2}}{4} & \text{Multiply coefficients} \\[2ex]
\dfrac{9\sqrt[3]{2}}{4} & \text{Our Solution}
\end{array}
$$

There is one catch to dividing with radicals, it is considered bad practice to have a radical in the denominator of our final answer. If there is a radical in the denominator we will rationalize it, or clear out any radicals in the denominator.

We do this by multiplying the numerator and denominator by the same thing. The problems we will consider here will all have a monomial in the denominator. The way we clear a monomial radical in the denominator is to focus on the index. The index tells us how many of each factor we will need to clear the radical. For example, if the index is 4, we will need 4 of each factor to clear the radical. This is shown in the following examples.

Example 398.

$$\frac{\sqrt{6}}{\sqrt{5}} \quad \text{Index is 2, we need two fives in denominator, need 1 more}$$

$$\frac{\sqrt{6}}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) \quad \text{Multiply numerator and denominator by } \sqrt{5}$$

$$\frac{\sqrt{30}}{5} \quad \text{Our Solution}$$

Example 399.

$$\frac{3\sqrt[4]{11}}{\sqrt[4]{2}} \quad \text{Index is 4, we need four twos in denominator, need 3 more}$$

$$\frac{3\sqrt[4]{11}}{\sqrt[4]{2}}\left(\frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[4]{2^3}$$

$$\frac{3\sqrt[4]{88}}{2} \quad \text{Our Solution}$$

Example 400.

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{25}} \quad \text{The 25 can be written as } 5^2\text{. This will help us keep the numbers small}$$

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{5^2}} \quad \text{Index is 3, we need three fives in denominator, need 1 more}$$

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{5^2}}\left(\frac{\sqrt[3]{5}}{\sqrt[3]{5}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[3]{5}$$

$$\frac{4\sqrt[3]{10}}{7\cdot 5} \quad \text{Multiply out denominator}$$

$$\frac{4\sqrt[3]{10}}{35} \quad \text{Our Solution}$$

The previous example could have been solved by multiplying numerator and denominator by $\sqrt[3]{25^2}$. However, this would have made the numbers very large and we would have needed to reduce our soultion at the end. This is why re-writing the radical as $\sqrt[3]{5^2}$ and multiplying by just $\sqrt[3]{5}$ was the better way to simplify.

We will also always want to reduce our fractions (inside and out of the radical) before we rationalize.

Example 401.

$$\frac{6\sqrt{14}}{12\sqrt{22}} \quad \text{Reduce coefficients and inside radical}$$

$$\frac{\sqrt{7}}{2\sqrt{11}} \quad \text{Index is 2, need two elevens, need 1 more}$$

$$\frac{\sqrt{7}}{2\sqrt{11}}\left(\frac{\sqrt{11}}{\sqrt{11}}\right) \quad \text{Multiply numerator and denominator by } \sqrt{11}$$

$$\frac{\sqrt{77}}{2\cdot 11} \quad \text{Multiply denominator}$$

$$\frac{\sqrt{77}}{22} \quad \text{Our Solution}$$

The same process can be used to rationalize fractions with variables.

Example 402.

$$\frac{18\sqrt[4]{6x^3y^4z}}{8\sqrt[4]{10xy^6z^3}} \quad \text{Reduce coefficients and inside radical}$$

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^3}} \quad \text{Index is 4. We need four of everything to rationalize, three more fives, two more } y's \text{ and one more } z.$$

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^3}}\left(\frac{\sqrt[4]{5^3y^2z}}{\sqrt[4]{5^3y^2z}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[4]{5^3y^2z}$$

$$\frac{9\sqrt[4]{375x^2y^2z}}{4\cdot 5yz} \quad \text{Multiply denominator}$$

$$\frac{9\sqrt[4]{375x^2y^2z}}{20yz} \quad \text{Our Solution}$$

8.4 Practice - Multiply and Divide Radicals

Multiply or Divide and Simplify.

1) $3\sqrt{5}\cdot-4\sqrt{16}$

2) $-5\sqrt{10}\cdot\sqrt{15}$

3) $\sqrt{12m}\cdot\sqrt{15m}$

4) $\sqrt{5r^3}\cdot-5\sqrt{10r^2}$

5) $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$

6) $3\sqrt[3]{4a^4}\cdot\sqrt[3]{10a^3}$

7) $\sqrt{6}(\sqrt{2}+2)$

8) $\sqrt{10}(\sqrt{5}+\sqrt{2})$

9) $-5\sqrt{15}(3\sqrt{3}+2)$

10) $5\sqrt{15}(3\sqrt{3}+2)$

11) $5\sqrt{10}(5n+\sqrt{2})$

12) $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$

13) $(2+2\sqrt{2})(-3+\sqrt{2})$

14) $(-2+\sqrt{3})(-5+2\sqrt{3})$

15) $(\sqrt{5}-5)(2\sqrt{5}-1)$

16) $(2\sqrt{3}+\sqrt{5})(5\sqrt{3}+2\sqrt{4})$

17) $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$

18) $(-2\sqrt{2p}+5\sqrt{5})(\sqrt{5p}+\sqrt{5p})$

19) $(-5-4\sqrt{3})(-3-4\sqrt{3})$

20) $(5\sqrt{2}-1)(-\sqrt{2m}+5)$

21) $\frac{\sqrt{12}}{5\sqrt{100}}$

22) $\frac{\sqrt{15}}{2\sqrt{4}}$

23) $\frac{\sqrt{5}}{4\sqrt{125}}$

24) $\frac{\sqrt{12}}{\sqrt{3}}$

25) $\frac{\sqrt{10}}{\sqrt{6}}$

26) $\frac{\sqrt{2}}{3\sqrt{5}}$

27) $\frac{2\sqrt{4}}{3\sqrt{3}}$

28) $\frac{4\sqrt{3}}{\sqrt{15}}$

29) $\frac{5x^2}{4\sqrt{3x^3y^3}}$

30) $\frac{4}{5\sqrt{3xy^4}}$

31) $\frac{\sqrt{2p^2}}{\sqrt{3p}}$

32) $\frac{\sqrt{8n^2}}{\sqrt{10n}}$

33) $\frac{3\sqrt[3]{10}}{5\sqrt[3]{27}}$

34) $\frac{\sqrt[3]{15}}{\sqrt[3]{64}}$

35) $\frac{\sqrt[3]{5}}{4\sqrt[3]{4}}$

36) $\frac{\sqrt[4]{2}}{2\sqrt[4]{64}}$

37) $\frac{5\sqrt[4]{5r^4}}{\sqrt[4]{8r^2}}$

38) $\frac{4}{\sqrt[4]{64m^4n^2}}$

8.5

Radicals - Rationalize Denominators

Objective: Rationalize the denominators of radical expressions.

It is considered bad practice to have a radical in the denominator of a fraction. When this happens we multiply the numerator and denominator by the same thing in order to clear the radical. In the lesson on dividing radicals we talked about how this was done with monomials. Here we will look at how this is done with binomials.

If the binomial is in the numerator the process to rationalize the denominator is essentially the same as with monomials. The only difference is we will have to distribute in the numerator.

Example 403.

$$\frac{\sqrt{3}-9}{2\sqrt{6}} \quad \text{Want to clear } \sqrt{6} \text{ in denominator, multiply by } \sqrt{6}$$

$$\frac{(\sqrt{3}-9)}{2\sqrt{6}}\left(\frac{\sqrt{6}}{\sqrt{6}}\right) \quad \text{We will distribute the } \sqrt{6} \text{ through the numerator}$$

$$\frac{\sqrt{18}-9\sqrt{6}}{2\cdot 6} \quad \text{Simplify radicals in numerator, multiply out denominator}$$

$$\frac{\sqrt{9\cdot 2}-9\sqrt{6}}{12} \quad \text{Take square root where possible}$$

$$\frac{3\sqrt{2}-9\sqrt{6}}{12} \quad \text{Reduce by dividing each term by 3}$$

$$\frac{\sqrt{2}-3\sqrt{6}}{4} \quad \text{Our Solution}$$

It is important to remember that when reducing the fraction we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator we must divide all terms by the same number.

The problem can often be made easier if we first simplify any radicals in the problem.

$$\frac{2\sqrt{20x^5}-\sqrt{12x^2}}{\sqrt{18x}} \quad \text{Simplify radicals by finding perfect squares}$$

$$\frac{2\sqrt{4\cdot 5x^3}-\sqrt{4\cdot 3x^2}}{\sqrt{9\cdot 2x}} \quad \text{Simplify roots, divide exponents by 2.}$$

$$\frac{2\cdot 2x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}} \quad \text{Multiply coefficients}$$

$$\frac{4x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}} \quad \text{Multiplying numerator and denominator by } \sqrt{2x}$$

$$\frac{(4x^2\sqrt{5x}-2x\sqrt{3})}{3\sqrt{2x}}\left(\frac{\sqrt{2x}}{\sqrt{2x}}\right) \quad \text{Distribute through numerator}$$

$$\frac{4x^2\sqrt{10x^2}-2x\sqrt{6x}}{3\cdot 2x} \quad \begin{array}{l}\text{Simplify roots in numerator,}\\ \text{multiply coefficients in denominator}\end{array}$$

$$\frac{4x^3\sqrt{10}-2x\sqrt{6x}}{6x} \quad \text{Reduce, dividing each term by } 2x$$

$$\frac{2x^2\sqrt{10}-\sqrt{6x}}{3x} \quad \text{Our Solution}$$

As we are rationalizing it will always be important to constantly check our problem to see if it can be simplified more. We ask ourselves, can the fraction be reduced? Can the radicals be simplified? These steps may happen several times on our way to the solution.

If the binomial occurs in the denominator we will have to use a different strategy to clear the radical. Consider $\frac{2}{\sqrt{3}-5}$, if we were to multiply the denominator by $\sqrt{3}$ we would have to distribute it and we would end up with $3-5\sqrt{3}$. We have not cleared the radical, only moved it to another part of the denominator. So our current method will not work. Instead we will use what is called a conjugate. A **conjugate** is made up of the same terms, with the opposite sign in the middle. So for our example with $\sqrt{3}-5$ in the denominator, the conjugate would be $\sqrt{3}+5$. The advantage of a conjugate is when we multiply them together we have $(\sqrt{3}-5)(\sqrt{3}+5)$, which is a sum and a difference. We know when we multiply these we get a difference of squares. Squaring $\sqrt{3}$ and 5, with subtraction in the middle gives the product $3-25=-22$. Our answer when multiplying conjugates will no longer have a square root. This is exactly what we want.

Example 404.

$$\frac{2}{\sqrt{3}-5} \quad \text{Multiply numerator and denominator by conjugate}$$

$$\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right) \quad \text{Distribute numerator, difference of squares in denominator}$$

$$\frac{2\sqrt{3}+10}{3-25} \quad \text{Simplify denoinator}$$

$$\frac{2\sqrt{3}+10}{-22} \quad \text{Reduce by dividing all terms by } -2$$

$$\frac{-\sqrt{3}-5}{11} \quad \text{Our Solution}$$

In the previous example, we could have reduced by dividng by 2, giving the solution $\frac{\sqrt{3}+5}{-11}$, both answers are correct.

Example 405.

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}} \quad \text{Multiply by conjugate, } \sqrt{5}-\sqrt{3}$$

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{\sqrt{75}-\sqrt{45}}{5-3} \quad \text{Simplify radicals in numerator, subtract in denominator}$$

$$\frac{\sqrt{25\cdot 3}-\sqrt{9\cdot 5}}{2} \quad \text{Take square roots where possible}$$

$$\frac{5\sqrt{3}-3\sqrt{5}}{2} \quad \text{Our Solution}$$

Example 406.

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}} \quad \text{Multiply by conjugate, } 4+\sqrt{5x^3}$$

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\frac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{8\sqrt{3x}+2\sqrt{15x^4}}{16-5x^3} \quad \text{Simplify radicals where possible}$$

$$\frac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3} \quad \text{Our Solution}$$

The same process can be used when there is a binomial in the numerator and denominator. We just need to remember to FOIL out the numerator.

Example 407.

$$\frac{3-\sqrt{5}}{2-\sqrt{3}} \quad \text{Multiply by conjugate, } 2+\sqrt{3}$$

$$\frac{3-\sqrt{5}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) \quad \text{FOIL in numerator, denominator is difference of squares}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4-3} \quad \text{Simplify denominator}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1} \quad \text{Divide each term by 1}$$

$$6+3\sqrt{3}-2\sqrt{5}-\sqrt{15} \quad \text{Our Solution}$$

Example 408.

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}} \quad \text{Multiply by the conjugate, } 5\sqrt{6}-4\sqrt{2}$$

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}}\left(\frac{5\sqrt{6}-4\sqrt{2}}{5\sqrt{6}-4\sqrt{2}}\right) \quad \text{FOIL numerator, denominator is difference of squares}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{25\cdot 6-16\cdot 2} \quad \text{Multiply in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{150-32} \quad \text{Subtract in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{118} \quad \text{Our Solution}$$

The same process is used when we have variables

Example 409.

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}} \quad \text{Multiply by the conjugate, } 5x+\sqrt{3x}$$

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}}\left(\frac{5x+\sqrt{3x}}{5x+\sqrt{3x}}\right) \quad \text{FOIL in numerator, denominator is difference of squares}$$

$$\frac{15x^2\sqrt{2x}+3x\sqrt{6x^2}+5x\sqrt{4x^3}+\sqrt{12x^4}}{25x^2-3x} \quad \text{Simplify radicals}$$

$$\frac{15x^2\sqrt{2x}+3x^2\sqrt{6}+10x^2\sqrt{x}+2x^2\sqrt{3}}{25x^2-3x} \quad \text{Divide each term by } x$$

$$\frac{15x\sqrt{2x}+3x\sqrt{6}+10x\sqrt{x}+2x\sqrt{3}}{25x-3} \quad \text{Our Solution}$$

World View Note: During the 5th century BC in India, Aryabhata published a treatise on astronomy. His work included a method for finding the square root of numbers that have many digits.

8.5 Practice - Rationalize Denominators

Simplify.

1) $\frac{4+2\sqrt{3}}{\sqrt{9}}$

2) $\frac{-4+\sqrt{3}}{4\sqrt{9}}$

3) $\frac{4+2\sqrt{3}}{5\sqrt{4}}$

4) $\frac{2\sqrt{3}-2}{2\sqrt{16}}$

5) $\frac{2-5\sqrt{5}}{4\sqrt{13}}$

6) $\frac{\sqrt{5}+4}{4\sqrt{17}}$

7) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

8) $\frac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

9) $\frac{5}{3\sqrt{5}+\sqrt{2}}$

10) $\frac{5}{\sqrt{3}+4\sqrt{5}}$

11) $\frac{2}{5+\sqrt{2}}$

12) $\frac{5}{2\sqrt{3}-\sqrt{2}}$

13) $\frac{3}{4-3\sqrt{3}}$

14) $\frac{4}{\sqrt{2}-2}$

15) $\frac{4}{3+\sqrt{5}}$

16) $\frac{2}{2\sqrt{5}+2\sqrt{3}}$

17) $-\frac{4}{4-4\sqrt{2}}$

18) $\frac{4}{4\sqrt{3}-\sqrt{5}}$

19) $\frac{1}{1+\sqrt{2}}$

20) $\frac{3+\sqrt{3}}{\sqrt{3}-1}$

21) $\frac{\sqrt{14}-2}{\sqrt{7}-\sqrt{2}}$

22) $\frac{2+\sqrt{10}}{\sqrt{2}+\sqrt{5}}$

23) $\frac{\sqrt{\mathrm{ab}}-a}{\sqrt{b}-\sqrt{a}}$

24) $\frac{\sqrt{14}-\sqrt{7}}{\sqrt{14}+\sqrt{7}}$

25) $\frac{a+\sqrt{\mathrm{ab}}}{\sqrt{a}+\sqrt{b}}$

26) $\frac{a+\sqrt{\mathrm{ab}}}{\sqrt{a}+\sqrt{b}}$

27) $\frac{2+\sqrt{6}}{2+\sqrt{3}}$

28) $\frac{2\sqrt{5}+\sqrt{3}}{1-\sqrt{3}}$

29) $\frac{a-\sqrt{b}}{a+\sqrt{b}}$

30) $\frac{a-b}{\sqrt{a}+\sqrt{b}}$

31) $\frac{6}{3\sqrt{2}-2\sqrt{3}}$

32) $\frac{\mathrm{ab}}{a\sqrt{b}-b\sqrt{a}}$

33) $\frac{a-b}{a\sqrt{b}-b\sqrt{a}}$

34) $\frac{4\sqrt{2}+3}{3\sqrt{2}+\sqrt{3}}$

35) $\frac{2-\sqrt{5}}{-3+\sqrt{5}}$

36) $\frac{-1+\sqrt{5}}{2\sqrt{5}+5\sqrt{2}}$

37) $\frac{5\sqrt{2}+\sqrt{3}}{5+5\sqrt{2}}$

38) $\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

8.6

Radicals - Rational Exponents

Objective: Convert between radical notation and exponential notation and simplify expressions with rational exponents using the properties of exponents.

When we simplify radicals with exponents, we divide the exponent by the index. Another way to write division is with a fraction bar. This idea is how we will define rational exponents.

Definition of Rational Exponents: $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

The denominator of a rational exponent becomes the index on our radical, likewise the index on the radical becomes the denominator of the exponent. We can use this property to change any radical expression into an exponential expression.

Example 410.

$(\sqrt[5]{x})^3 = x^{\frac{3}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$	Index is denominator
$\dfrac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\dfrac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$	Negative exponents from reciprocals

We can also change any rational exponent into a radical expression by using the denominator as the index.

Example 411.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$	Index is denominator
$x^{-\frac{4}{5}} = \dfrac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \dfrac{1}{(\sqrt[9]{xy})^2}$	Negative exponent means reciprocals

World View Note: Nicole Oresme, a Mathematician born in Normandy was the first to use rational exponents. He used the notation $\frac{1}{3}\bullet 9^p$ to represent $9^{\frac{1}{3}}$. However his notation went largely unnoticed.

The ability to change between exponential expressions and radical expressions allows us to evaluate problems we had no means of evaluating before by changing to a radical.

Example 412.

$27^{-\frac{4}{3}}$	Change to radical, denominator is index, negative means reciprocal
$\dfrac{1}{(\sqrt[3]{27})^4}$	Evaluate radical

$\frac{1}{(3)^4}$ Evaluate exponent

$\frac{1}{81}$ Our solution

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

Properties of Exponents

$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \frac{1}{a^m}$
$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{1}{a^{-m}} = a^m$
$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$

When adding and subtracting with fractions we need to be sure to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 413.

$a^{\frac{2}{3}} b^{\frac{1}{2}} a^{\frac{1}{6}} b^{\frac{1}{5}}$ Need common denominator on $a's$ (6) and $b's$ (10)

$a^{\frac{4}{6}} b^{\frac{5}{10}} a^{\frac{1}{6}} b^{\frac{2}{10}}$ Add exponents on $a's$ and $b's$

$a^{\frac{5}{6}} b^{\frac{7}{10}}$ Our Solution

Example 414.

$\left(x^{\frac{1}{3}} y^{\frac{2}{5}}\right)^{\frac{3}{4}}$ Multiply $\frac{3}{4}$ by each exponent

$x^{\frac{1}{4}} y^{\frac{3}{10}}$ Our Solution

Example 415.

$\frac{x^2 y^{\frac{2}{3}} \cdot 2x^{\frac{1}{2}} y^{\frac{5}{6}}}{x^{\frac{7}{2}} y^0}$ In numerator, need common denominator to add exponents

$$\frac{x^{\frac{4}{2}}y^{\frac{4}{6}}\cdot 2x^{\frac{1}{2}}y^{\frac{5}{6}}}{x^{\frac{7}{2}}y^{0}}$$ Add exponents in numerator, in denominator, $y^0=1$

$$\frac{2x^{\frac{5}{2}}y^{\frac{9}{6}}}{x^{\frac{7}{2}}}$$ Subtract exponents on x, reduce exponent on y

$$2x^{-1}y^{\frac{3}{2}}$$ Negative exponent moves down to denominator

$$\frac{2y^{\frac{3}{2}}}{x}$$ Our Solution

Example 416.

$$\left(\frac{25x^{\frac{1}{3}}y^{\frac{2}{5}}}{9x^{\frac{4}{5}}y^{-\frac{3}{2}}}\right)^{-\frac{1}{2}}$$ Using order of operations, simplify inside parenthesis first
Need common denominators before we can subtract exponents

$$\left(\frac{25x^{\frac{5}{15}}y^{\frac{4}{10}}}{9x^{\frac{12}{15}}y^{-\frac{15}{10}}}\right)^{-\frac{1}{2}}$$ Subtract exponents, be careful of the negative:
$\frac{4}{10}-\left(-\frac{15}{10}\right)=\frac{4}{10}+\frac{15}{10}=\frac{19}{10}$

$$\left(\frac{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}{9}\right)^{-\frac{1}{2}}$$ The negative exponent will flip the fraction

$$\left(\frac{9}{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}\right)^{\frac{1}{2}}$$ The exponent $\frac{1}{2}$ goes on each factor

$$\frac{9^{\frac{1}{2}}}{25^{\frac{1}{2}}x^{-\frac{7}{30}}y^{\frac{19}{20}}}$$ Evaluate $9^{\frac{1}{2}}$ and $25^{\frac{1}{2}}$ and move negative exponent

$$\frac{3x^{\frac{7}{30}}}{5y^{\frac{19}{20}}}$$ Our Solution

It is important to remember that as we simplify with rational exponents we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure your comfortable with using the properties.

8.6 Practice - Rational Exponents

Write each expression in radical form.

1) $m^{\frac{3}{5}}$

2) $(10r)^{-\frac{3}{4}}$

3) $(7x)^{\frac{3}{2}}$

4) $(6b)^{-\frac{4}{3}}$

Write each expression in exponential form.

5) $\frac{1}{(\sqrt{6x})^3}$

6) $\sqrt{v}$

7) $\frac{1}{(\sqrt[4]{n})^7}$

8) $\sqrt{5a}$

Evaluate.

9) $8^{\frac{2}{3}}$

10) $16^{\frac{1}{4}}$

11) $4^{\frac{3}{2}}$

12) $100^{-\frac{3}{2}}$

Simplify. Your answer should contain only positive exponents.

13) $yx^{\frac{1}{3}}\cdot xy^{\frac{3}{2}}$

14) $4v^{\frac{2}{3}}\cdot v^{-1}$

15) $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{-1}$

16) $(x^{\frac{5}{3}}y^{-2})^0$

17) $\frac{a^2b^0}{3a^4}$

18) $\frac{2x^{\frac{1}{2}}y^{\frac{1}{3}}}{2x^{\frac{4}{3}}y^{-\frac{7}{4}}}$

19) $uv\cdot u\cdot (v^{\frac{3}{2}})^3$

20) $(x\cdot xy^2)^0$

21) $(x^0y^{\frac{1}{3}})^{\frac{3}{2}}x^0$

22) $u^{-\frac{5}{4}}v^2\cdot (u^{\frac{3}{2}})^{-\frac{3}{2}}$

23) $\frac{a^{\frac{3}{4}}b^{-1}\cdot b^{\frac{7}{4}}}{3b^{-1}}$

24) $\frac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}}\cdot xy^{\frac{1}{2}}}$

25) $\frac{3y^{-\frac{5}{4}}}{y^{-1}\cdot 2y^{-\frac{1}{3}}}$

26) $\frac{ab^{\frac{1}{3}}\cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$

27) $\left(\frac{m^{\frac{3}{2}}n^{-2}}{(mn^{\frac{4}{3}})^{-1}}\right)^{\frac{7}{4}}$

28) $\frac{(y^{-\frac{1}{2}})^{\frac{3}{2}}}{x^{\frac{3}{2}}y^{\frac{1}{2}}}$

29) $\frac{(m^2n^{\frac{1}{2}})^0}{n^{\frac{3}{4}}}$

30) $\frac{y^0}{(x^{\frac{3}{4}}y^{-1})^{\frac{1}{3}}}$

31) $\frac{(x^{-\frac{4}{3}}y^{-\frac{1}{3}}\cdot y)^{-1}}{x^{\frac{1}{3}}y^{-2}}$

32) $\frac{(x^{\frac{1}{2}}y^0)^{-\frac{4}{3}}}{y^4\cdot x^{-2}y^{-\frac{2}{3}}}$

33) $\frac{(uv^2)^{\frac{1}{2}}}{v^{-\frac{1}{4}}v^2}$

34) $\left(\frac{y^{\frac{1}{3}}y^{-2}}{(x^{\frac{5}{3}}y^3)^{-\frac{3}{2}}}\right)^{\frac{3}{2}}$

8.7

Radicals - Radicals of Mixed Index

Objective: Reduce the index on a radical and multiply or divide radicals of different index.

Knowing that a radical has the same properties as exponents (written as a ratio) allows us to manipulate radicals in new ways. One thing we are allowed to do is reduce, not just the radicand, but the index as well. This is shown in the following example.

Example 417.

$$\begin{array}{rl} \sqrt[8]{x^6y^2} & \text{Rewrite as rational exponent} \\ (x^6y^2)^{\frac{1}{5}} & \text{Multiply exponents} \\ x^{\frac{6}{8}}y^{\frac{2}{8}} & \text{Reduce each fraction} \\ x^{\frac{3}{4}}y^{\frac{1}{4}} & \text{All exponents have denominator of 4, this is our new index} \\ \sqrt[4]{x^3y} & \text{Our Solution} \end{array}$$

What we have done is reduced our index by dividing the index and all the exponents by the same number (2 in the previous example). If we notice a common factor in the index and all the exponnets on every factor we can reduce by dividing by that common factor. This is shown in the next example

Example 418.

$$\begin{array}{rl} \sqrt[24]{a^6b^9c^{15}} & \text{Index and all exponents are divisible by 3} \\ \sqrt[8]{a^2b^3c^5} & \text{Our Solution} \end{array}$$

We can use the same process when there are coefficients in the problem. We will first write the coefficient as an exponential expression so we can divide the exponet by the common factor as well.

Example 419.

$$\begin{array}{rl} \sqrt[9]{8m^6n^3} & \text{Write 8 as } 2^3 \\ \sqrt[9]{2^3m^6n^3} & \text{Index and all exponents are divisible by 3} \\ \sqrt[3]{2m^2n} & \text{Our Solution} \end{array}$$

We can use a very similar idea to also multiply radicals where the index does not match. First we will consider an example using rational exponents, then identify the pattern we can use.

Example 420.

$\sqrt[3]{ab^2}\,\sqrt[4]{a^2b}$	Rewrite as rational exponents
$(ab^2)^{\frac{1}{3}}(a^2b)^{\frac{1}{4}}$	Multiply exponents
$a^{\frac{1}{3}}b^{\frac{2}{3}}a^{\frac{2}{4}}b^{\frac{1}{4}}$	To have one radical need a common denominator, 12
$a^{\frac{4}{12}}b^{\frac{8}{12}}a^{\frac{6}{12}}b^{\frac{3}{12}}$	Write under a single radical with common index, 12
$\sqrt[12]{a^4b^8a^6b^3}$	Add exponents
$\sqrt[12]{a^{10}b^{11}}$	Our Solution

To combine the radicals we need a common index (just like the common denominator). We will get a common index by multiplying each index and exponent by an integer that will allow us to build up to that desired index. This process is shown in the next example.

Example 421.

$\sqrt[4]{a^2b^3}\,\sqrt[6]{a^2b}$	Common index is 12.
	Multiply first index and exponents by 3, second by 2
$\sqrt[12]{a^6b^9a^4b^2}$	Add exponents
$\sqrt[12]{a^{10}b^{11}}$	Our Solution

Often after combining radicals of mixed index we will need to simplify the resulting radical.

Example 422.

$\sqrt[5]{x^3y^4}\,\sqrt[3]{x^2y}$	Common index: 15.
	Multiply first index and exponents by 3, second by 5
$\sqrt[15]{x^9y^{12}x^{10}y^5}$	Add exponents
$\sqrt[15]{x^{19}y^{17}}$	Simplify by dividing exponents by index, remainder is left inside
$xy\,\sqrt[15]{x^4y^2}$	Our Solution

Just as with reducing the index, we will rewrite coefficients as exponential expressions. This will also allow us to use exponent properties to simplify.

Example 423.

$\sqrt[3]{4x^2y}\,\sqrt[4]{8xy^3}$	Rewrite 4 as 2^2 and 8 as 2^3
$\sqrt[3]{2^2x^2y}\,\sqrt[4]{2^3xy^3}$	Common index: 12.
	Multiply first index and exponents by 4, second by 3
$\sqrt[12]{2^8x^8y^42^9x^3y^9}$	Add exponents (even on the 2)
$\sqrt[12]{2^{17}x^{11}y^{13}}$	Simplify by dividing exponents by index, remainder is left inside
$2y\,\sqrt[12]{2^5x^{11}y}$	Simplify 2^5
$2y\,\sqrt[12]{32x^{11}y}$	Our Solution

If there is a binomial in the radical then we need to keep that binomial together through the entire problem.

Example 424.

$\sqrt{3x(y+z)}\sqrt[3]{9x(y+z)^2}$	Rewrite 9 as 3^2
$\sqrt{3x(y+z)}\sqrt[3]{3^2x(y+z)^2}$	Common index: 6. Multiply first group by 3, second by 2
$\sqrt[6]{3^3x^3(y+z)^3 3^4x^2(y+z)^4}$	Add exponents, keep $(y+z)$ as binomial
$\sqrt[6]{3^7x^5(y+z)^7}$	Simplify, dividing exponent by index, remainder inside
$3(y+z)\sqrt[6]{3x^5(y+z)}$	Our Solution

World View Note: Originally the radical was just a check mark with the rest of the radical expression in parenthesis. In 1637 Rene Descartes was the first to put a line over the entire radical expression.

The same process is used for dividing mixed index as with multiplying mixed index. The only difference is our final answer cannot have a radical over the denominator.

Example 425.

$\dfrac{\sqrt[6]{x^4y^3z^2}}{\sqrt[8]{x^7y^2z}}$	Common index is 24. Multiply first group by 4, second by 3
$\sqrt[24]{\dfrac{x^{16}y^{12}z^8}{x^{21}y^6z^3}}$	Subtract exponents
$\sqrt[24]{x^{-5}y^6z^5}$	Negative exponent moves to denominator
$\sqrt[24]{\dfrac{y^6z^5}{x^5}}$	Cannot have denominator in radical, need $12x's$, or 7 more
$\sqrt[24]{\dfrac{y^6z^5}{x^5}}\left(\sqrt[24]{\dfrac{x^{19}}{x^{19}}}\right)$	Multiply numerator and denominator by $\sqrt[12]{x^7}$
$\dfrac{\sqrt[24]{x^{19}y^6z^5}}{x}$	Our Solution

8.7 Practice - Radicals of Mixed Index

Reduce the following radicals.

1) $\sqrt[8]{16x^4y^6}$

2) $\sqrt[4]{9x^2y^6}$

3) $\sqrt[12]{64x^4y^6z^8}$

4) $\sqrt[4]{\frac{25x^3}{16x^5}}$

5) $\sqrt[6]{\frac{16x^2}{9y^4}}$

6) $\sqrt[15]{x^9y^{12}z^6}$

7) $\sqrt[12]{x^6y^9}$

8) $\sqrt[10]{64x^8y^4}$

9) $\sqrt[8]{x^6y^4z^2}$

10) $\sqrt[4]{25y^2}$

11) $\sqrt[9]{8x^3y^6}$

12) $\sqrt[16]{81x^8y^{12}}$

Combine the following radicals.

13) $\sqrt[3]{5}\sqrt{6}$

14) $\sqrt[3]{7}\sqrt[4]{5}$

15) $\sqrt{x}\sqrt[3]{7y}$

16) $\sqrt[3]{y}\ \sqrt[5]{3z}$

17) $\sqrt{x}\sqrt[3]{x-2}$

18) $\sqrt[4]{3x}\ \sqrt{y+4}$

19) $\sqrt[5]{x^2y}\sqrt{xy}$

20) $\sqrt{ab}\ \sqrt[5]{2a^2b^2}$

21) $\sqrt[4]{xy^2}\sqrt[3]{x^2y}$

22) $\sqrt[5]{a^2b^3}\ \sqrt[4]{a^2b}$

23) $\sqrt[4]{a^2bc^2}\sqrt[5]{a^2b^3c}$

24) $\sqrt[6]{x^2yz^3}\ \sqrt[5]{x^2yz^2}$

25) $\sqrt{a}\ \sqrt[4]{a^3}$

26) $\sqrt[3]{x^2}\ \sqrt[6]{x^5}$

27) $\sqrt[5]{b^2}\sqrt{b^3}$

28) $\sqrt[4]{a^3}\ \sqrt[3]{a^2}$

29) $\sqrt{xy^3}\ \sqrt[3]{x^2y}$

30) $\sqrt[5]{a^3b}\ \sqrt{ab}$

31) $\sqrt[4]{9ab^3}\ \sqrt{3a^4b}$

32) $\sqrt{2x^3y^3}\ \sqrt[3]{4xy^2}$

33) $\sqrt[3]{3xy^2z}\sqrt[4]{9x^3yz^2}$

34) $\sqrt{a^4b^3c^4}\ \sqrt[3]{ab^2c}$

35) $\sqrt{27a^5(b+1)}\ \sqrt[3]{81a(b+1)^4}$

36) $\sqrt{8x\,(y+z)^5}\ \sqrt[3]{4x^2(y+z)^2}$

37) $\frac{\sqrt[3]{a^2}}{\sqrt[4]{a}}$

38) $\frac{\sqrt[3]{x^2}}{\sqrt[5]{x}}$

39) $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

40) $\frac{\sqrt[5]{a^4b^2}}{\sqrt[3]{ab^2}}$

41) $\frac{\sqrt{ab^3c}}{\sqrt[5]{a^2b^3c^{-1}}}$

42) $\frac{\sqrt[5]{x^3y^4z^9}}{\sqrt{xy^{-2}z}}$

43) $\frac{\sqrt[4]{(3x-1)^3}}{\sqrt[5]{(3x-1)^3}}$

44) $\frac{\sqrt[3]{(2+5x)^2}}{\sqrt[4]{(2+5x)}}$

45) $\frac{\sqrt[3]{(2x+1)^2}}{\sqrt[5]{(2x+1)^2}}$

46) $\frac{\sqrt[4]{(5-3x)^3}}{\sqrt[3]{(5-3x)^2}}$

8.8

Radicals - Complex Numbers

Objective: Add, subtract, multiply, rationalize, and simplify expressions using complex numbers.

World View Note: When mathematics was first used, the primary purpose was for counting. Thus they did not originally use negatives, zero, fractions or irrational numbers. However, the ancient Egyptians quickly developed the need for "a part" and so they made up a new type of number, the ratio or fraction. The Ancient Greeks did not believe in irrational numbers (people were killed for believing otherwise). The Mayans of Central America later made up the number zero when they found use for it as a placeholder. Ancient Chinese Mathematicians made up negative numbers when they found use for them.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, we tend to make up new ways for dealing with the problem that can solve the problem. Throughout history this has been the case with the need for a number that is nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for the square roots of negative numbers. To work with the square root of negative numbers mathematicians have defined what are called imaginary and complex numbers.

Definition of Imaginary Numbers: $\boldsymbol{i^2 = -1}$ (thus $\boldsymbol{i = \sqrt{-1}}$)

Examples of imaginary numbers include $3i$, $-6i$, $\frac{3}{5}i$ and $3i\sqrt{5}$. A **complex number** is one that contains both a real and imaginary part, such as $2 + 5i$.

With this definition, the square root of a negative number is no longer undefined. We now are allowed to do basic operations with the square root of negatives. First we will consider exponents on imaginary numbers. We will do this by manipulating our definition of $i^2 = -1$. If we multiply both sides of the definition by i, the equation becomes $i^3 = -i$. Then if we multiply both sides of the equation again by i, the equation becomes $i^4 = -i^2 = -(-1) = 1$, or simply $i^4 = 1$. Multiplying again by i gives $i^5 = i$. One more time gives $i^6 = i^2 = -1$. And if this pattern continues we see a cycle forming, the exponents on i change we cycle through simplified answers of i, -1, $-i$, 1. As there are 4 different possible answers in this cycle, if we divide the exponent by 4 and consider the remainder, we can simplify any exponent on i by learning just the following four values:

Cyclic Property of Powers of $\boldsymbol{i}$

$$i^0 = 1$$
$$i = i$$
$$i^2 = -1$$
$$i^3 = -i$$

Example 426.

$$\begin{array}{rl} i^{35} & \text{Divide exponent by 4} \\ 8R3 & \text{Use remainder as exponent on } i \\ i^3 & \text{Simplify} \\ -i & \text{Our Solution} \end{array}$$

Example 427.

$$\begin{array}{rl} i^{124} & \text{Divide exponent by 4} \\ 31R0 & \text{Use remainder as exponent on } i \\ i^0 & \text{Simplify} \\ 1 & \text{Our Solution} \end{array}$$

When performing operations (add, subtract, multilpy, divide) we can handle i just like we handle any other variable. This means when adding and subtracting complex numbers we simply add or combine like terms.

Example 428.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms } 2+4 \text{ and } 5i-7i \\ 6-2i & \text{Our Solution} \end{array}$$

It is important to notice what operation we are doing. Students often see the parenthesis and think that means FOIL. We only use FOIL to multiply. This problem is an addition problem so we simply add the terms, or combine like terms.

For subtraction problems the idea is the same, we need to remember to first distribute the negative onto all the terms in the parentheses.

Example 429.

$$\begin{array}{rl} (4-8i)-(3-5i) & \text{Distribute the negative} \\ 4-8i-3+5i & \text{Combine like terms } 4-3 \text{ and } -8i+5i \\ 1-3i & \text{Our Solution} \end{array}$$

Addition and subtraction can be combined into one problem.

Example 430.

$$\begin{array}{rl} (5i)-(3+8i)+(-4+7i) & \text{Distribute the negative} \\ 5i-3-8i-4+7i & \text{Combine like terms } 5i-8i+7i \text{ and } -3-4 \\ -7+4i & \text{Our Solution} \end{array}$$

Multiplying with complex numbers is the same as multiplying with variables with one exception, we will want to simplify our final answer so there are no exponents on i.

Example 431.

$$\begin{aligned} (3i)(7i) &\quad \text{Multilpy coefficients and } i's \\ 21i^2 &\quad \text{Simplify } i^2=-1 \\ 21(-1) &\quad \text{Multiply} \\ -21 &\quad \text{Our Solution} \end{aligned}$$

Example 432.

$$\begin{aligned} 5i(3i-7) &\quad \text{Distribute} \\ 15i^2-35i &\quad \text{Simplify } i^2=-1 \\ 15(-1)-35i &\quad \text{Multiply} \\ -15-35i &\quad \text{Our Solution} \end{aligned}$$

Example 433.

$$\begin{aligned} (2-4i)(3+5i) &\quad \text{FOIL} \\ 6+10i-12i-20i^2 &\quad \text{Simplify } i^2=-1 \\ 6+10i-12i-20(-1) &\quad \text{Multiply} \\ 6+10i-12i+20 &\quad \text{Combine like terms } 6+20 \text{ and } 10i-12i \\ 26-2i &\quad \text{Our Solution} \end{aligned}$$

Example 434.

$$\begin{aligned} (3i)(6i)(2-3i) &\quad \text{Multiply first two monomials} \\ 18i^2(2-3i) &\quad \text{Distribute} \\ 36i^2-54i^3 &\quad \text{Simplify } i^2=-1 \text{ and } i^3=-i \\ 36(-1)-54(-i) &\quad \text{Multiply} \\ -36+54i &\quad \text{Our Solution} \end{aligned}$$

Remember when squaring a binomial we either have to FOIL or use our shortcut to square the first, twice the product and square the last. The next example uses the shortcut

Example 435.

$$\begin{aligned} (4-5i)^2 &\quad \text{Use perfect square shortcut} \\ 4^2=16 &\quad \text{Square the first} \\ 2(4)(-5i)=-40i &\quad \text{Twice the product} \\ (5i)^2=25i^2=25(-1)=-25 &\quad \text{Square the last, simplify } i^2=-1 \\ 16-40i-25 &\quad \text{Combine like terms} \\ -9-40i &\quad \text{Our Solution} \end{aligned}$$

Dividing with complex numbers also has one thing we need to be careful of. If i is $\sqrt{-1}$, and it is in the denominator of a fraction, then we have a radical in the denominator! This means we will want to rationalize our denominator so there are no i's. This is done the same way we rationalized denominators with square roots.

Example 436.

$$\frac{7+3i}{-5i} \quad \text{Just } a \text{ monomial in denominator, multiply by } i$$

$$\frac{7+3i}{-5i}\left(\frac{i}{i}\right) \quad \text{Distribute } i \text{ in numerator}$$

$$\frac{7i+3i^2}{-5i^2} \quad \text{Simplify } i^2 = -1$$

$$\frac{7i+3(-1)}{-5(-1)} \quad \text{Multiply}$$

$$\frac{7i-3}{5} \quad \text{Our Solution}$$

The solution for these problems can be written several different ways, for example $\frac{-3+7i}{5}$ or $\frac{-3}{5}+\frac{7}{5}i$, The author has elected to use the solution as written, but it is important to express your answer in the form your instructor prefers.

Example 437.

$$\frac{2-6i}{4+8i} \quad \text{Binomial in denominator, multiply by conjugate, } 4-8i$$

$$\frac{2-6i}{4+8i}\left(\frac{4-8i}{4-8i}\right) \quad \text{FOIL in numerator, denominator is } a \text{ difference of squares}$$

$$\frac{8-16i-24i+48i^2}{16-64i^2} \quad \text{Simplify } i^2 = -1$$

$$\frac{8-16i-24i+48(-1)}{16-64(-1)} \quad \text{Multiply}$$

$$\frac{8-16i-24i-48}{16+64} \quad \text{Combine like terms } 8-48 \text{ and } -16i-24i \text{ and } 16+64$$

$$\frac{-40-40i}{80} \quad \text{Reduce, divide each term by } 40$$

$$\frac{-1-i}{2} \quad \text{Our Solution}$$

Using i we can simplify radicals with negatives under the root. We will use the product rule and simplify the negative as a factor of negative one. This is shown in the following examples.

Example 438.

$$\begin{array}{rl}\sqrt{-16} & \text{Consider the negative as } a \text{ factor of } -1\\ \sqrt{-1\cdot 16} & \text{Take each root, square root of } -1 \text{ is } i\\ 4i & \text{Our Solution}\end{array}$$

Example 439.

$$\begin{array}{rl}\sqrt{-24} & \text{Find perfect square factors, including } -1\\ \sqrt{-1\cdot 4\cdot 6} & \text{Square root of } -1 \text{ is } i\text{, square root of 4 is 2}\\ 2i\sqrt{6} & \text{Our Solution}\end{array}$$

When simplifying complex radicals it is important that we take the -1 out of the radical (as an i) before we combine radicals.

Example 440.

$$\begin{array}{rl}\sqrt{-6}\sqrt{-3} & \text{Simplify the negatives, bringing } i \text{ out of radicals}\\ (i\sqrt{6})(i\sqrt{3}) & \text{Multiply } i \text{ by } i \text{ is } i^2=-1\text{, also multiply radicals}\\ -\sqrt{18} & \text{Simplify the radical}\\ -\sqrt{9\cdot 2} & \text{Take square root of 9}\\ -3\sqrt{2} & \text{Our Solution}\end{array}$$

If there are fractions, we need to make sure to reduce each term by the same number. This is shown in the following example.

Example 441.

$$\begin{array}{rl}\dfrac{-15-\sqrt{-200}}{20} & \text{Simplify the radical first}\\ \sqrt{-200} & \text{Find perfect square factors, including } -1\\ \sqrt{-1\cdot 100\cdot 2} & \text{Take square root of } -1 \text{ and } 100\\ 10i\sqrt{2} & \text{Put this back into the expression}\\ \dfrac{-15-10i\sqrt{2}}{20} & \text{All the factors are divisible by 5}\\ \dfrac{-3-2i\sqrt{2}}{4} & \text{Our Solution}\end{array}$$

By using $i=\sqrt{-1}$ we will be able to simplify and solve problems that we could not simplify and solve before. This will be explored in more detail in a later section.

8.8 Practice - Complex Numbers

Simplify.

1) $3-(-8+4i)$

2) $(3i)-(7i)$

3) $(7i)-(3-2i)$

4) $5+(-6-6i)$

5) $(-6i)-(3+7i)$

6) $(-8i)-(7i)-(5-3i)$

7) $(3-3i)+(-7-8i)$

8) $(-4-i)+(1-5i)$

9) $(i)-(2+3i)-6$

10) $(5-4i)+(8-4i)$

11) $(6i)(-8i)$

12) $(3i)(-8i)$

13) $(-5i)(8i)$

14) $(8i)(-4i)$

15) $(-7i)^2$

16) $(-i)(7i)(4-3i)$

17) $(6+5i)^2$

18) $(8i)(-2i)(-2-8i)$

19) $(-7-4i)(-8+6i)$

20) $(3i)(-3i)(4-4i)$

21) $(-4+5i)(2-7i)$

22) $-8(4-8i)-2(-2-6i)$

23) $(-8-6i)(-4+2i)$

24) $(-6i)(3-2i)-(7i)(4i)$

25) $(1+5i)(2+i)$

26) $(-2+i)(3-5i)$

27) $\frac{-9+5i}{i}$

28) $\frac{-3+2i}{-3i}$

29) $\frac{-10-9i}{6i}$

30) $\frac{-4+2i}{3i}$

31) $\frac{-3-6i}{4i}$

32) $\frac{-5+9i}{9i}$

33) $\frac{10-i}{-i}$

34) $\frac{10}{5i}$

35) $\frac{4i}{-10+i}$

36) $\frac{9i}{1-5i}$

37) $\frac{8}{7-6i}$

38) $\frac{4}{4+6i}$

39) $\frac{7}{10-7i}$

40) $\frac{9}{-8-6i}$

41) $\frac{5i}{-6-i}$

42) $\frac{8i}{6-7i}$

43) $\sqrt{-81}$

44) $\sqrt{-45}$

45) $\sqrt{-10}\sqrt{-2}$

46) $\sqrt{-12}\sqrt{-2}$

47 $\frac{3+\sqrt{-27}}{6}$

48) $\frac{-4-\sqrt{-8}}{-4}$

49) $\frac{8-\sqrt{-16}}{4}$

50) $\frac{6+\sqrt{-32}}{4}$

51) i^{73}

52) i^{251}

53) i^{48}

54) i^{68}

55) i^{62}

56) i^{181}

57) i^{154}

58) i^{51}

Chapter 9 : Quadratics

9.1

Quadratics - Solving with Radicals

Objective: Solve equations with radicals and check for extraneous solutions.

Here we look at equations that have roots in the problem. As you might expect, to clear a root we can raise both sides to an exponent. So to clear a square root we can rise both sides to the second power. To clear a cubed root we can raise both sides to a third power. There is one catch to solving a problem with roots in it, sometimes we end up with solutions that do not actually work in the equation. This will only happen if the index on the root is even, and it will not happen all the time. So for these problems it will be required that we check our answer in the original problem. If a value does not work it is called an extraneous solution and not included in the final solution.

When solving a radical problem with an even index: check answers!

Example 442.

$$
\begin{array}{rl}
\sqrt{7x+2}=4 & \text{Even index! We will have to check answers} \\
(\sqrt{7x+2})^2=4^2 & \text{Square both sides, simplify exponents} \\
7x+2=16 & \text{Solve} \\
\underline{-2\quad -2} & \text{Subtract 2 from both sides} \\
7x=14 & \text{Divide both sides by 7} \\
\overline{7}\quad \overline{7} & \\
x=2 & \text{Need to check answer in original problem} \\
\sqrt{7(2)+2}=4 & \text{Multiply} \\
\sqrt{14+2}=4 & \text{Add} \\
\sqrt{16}=4 & \text{Square root} \\
4=4 & \text{True! It works!} \\
x=2 & \text{Our Solution}
\end{array}
$$

Example 443.

$$
\begin{array}{rl}
\sqrt[3]{x-1}=-4 & \text{Odd index, we don}'t\text{ need to check answer} \\
(\sqrt[3]{x-1})^3=(-4)^3 & \text{Cube both sides, simplify exponents} \\
x-1=-64 & \text{Solve}
\end{array}
$$

$$\begin{array}{rl} \underline{+1 \quad +1} & \text{Add 1 to both sides} \\ x = -63 & \text{Our Solution} \end{array}$$

Example 444.

$$\begin{array}{rl} \sqrt[4]{3x+6} = -3 & \text{Even index! We will have to check answers} \\ (\sqrt[4]{3x+6}) = (-3)^4 & \text{Rise both sides to fourth power} \\ 3x+6 = 81 & \text{Solve} \\ \underline{-6 \quad -6} & \text{Subtract 6 from both sides} \\ 3x = 75 & \text{Divide both sides by 3} \\ \overline{3} \quad \overline{3} & \\ x = 25 & \text{Need to check answer in original problem} \\ \sqrt[4]{3(25)+6} = -3 & \text{Multiply} \\ \sqrt[4]{75+6} = -3 & \text{Add} \\ \sqrt[4]{81} = -3 & \text{Take root} \\ 3 = -3 & \text{False, extraneous solution} \\ \text{No Solution} & \text{Our Solution} \end{array}$$

If the radical is not alone on one side of the equation we will have to solve for the radical before we raise it to an exponent

Example 445.

$$\begin{array}{rl} x + \sqrt{4x+1} = 5 & \text{Even index! We will have to check solutions} \\ \underline{-x \qquad\qquad -x} & \text{Isolate radical by subtracting } x \text{ from both sides} \\ \sqrt{4x+1} = 5 - x & \text{Square both sides} \\ (\sqrt{4x+1})^2 = (5-x)^2 & \text{Evaluate exponents, recal } (a-b)^2 = a^2 - 2ab + b^2 \\ 4x+1 = 25 - 10x + x^2 & \text{Re} - \text{order terms} \\ 4x+1 = x^2 - 10x + 25 & \text{Make equation equal zero} \\ \underline{-4x-1 \qquad -4x \quad -1} & \text{Subtract } 4x \text{ and 1 from both sides} \\ 0 = x^2 - 14x + 24 & \text{Factor} \\ 0 = (x-12)(x-2) & \text{Set each factor equal to zero} \\ x - 12 = 0 \ \text{ or } \ x - 2 = 0 & \text{Solve each equation} \\ \underline{+12+12} \qquad \underline{+2+2} & \\ x = 12 \ \text{ or } \ x = 2 & \text{Need to check answers in original problem} \\ & \\ (12) + \sqrt{4(12)+1} = 5 & \text{Check } x = 5 \text{ first} \end{array}$$

$$
\begin{aligned}
12+\sqrt{48+1}&=5 && \text{Add}\\
12+\sqrt{49}&=5 && \text{Take root}\\
12+7&=5 && \text{Add}\\
19&=5 && \text{False, extraneous root}
\end{aligned}
$$

$$
\begin{aligned}
(2)+\sqrt{4(2)+1}&=5 && \text{Check } x=2\\
2+\sqrt{8+1}&=5 && \text{Add}\\
2+\sqrt{9}&=5 && \text{Take root}\\
2+3&=5 && \text{Add}\\
5&=5 && \text{True! It works}
\end{aligned}
$$

$$
x=2 \qquad \text{Our Solution}
$$

The above example illustrates that as we solve we could end up with an x^2 term or a quadratic. In this case we remember to set the equation to zero and solve by factoring. We will have to check both solutions if the index in the problem was even. Sometimes both values work, sometimes only one, and sometimes neither works.

World View Note: The babylonians were the first known culture to solve quadratics in radicals - as early as 2000 BC!

If there is more than one square root in a problem we will clear the roots one at a time. This means we must first isolate one of them before we square both sides.

Example 446.

$$
\begin{aligned}
\sqrt{3x-8}-\sqrt{x}&=0 && \text{Even index! We will have to check answers}\\
+\sqrt{x}\ \ &+\sqrt{x} && \text{Isolate first root by adding } \sqrt{x} \text{ to both sides}\\
\sqrt{3x-8}&=\sqrt{x} && \text{Square both sides}\\
(\sqrt{3x-8})^2&=(\sqrt{x})^2 && \text{Evaluate exponents}\\
3x-8&=x && \text{Solve}\\
\underline{-3x\qquad}&\underline{\ -3x} && \text{Subtract } 3x \text{ from both sides}\\
-8&=-2x && \text{Divide both sides by } -2\\
\overline{-2}\ \ &\ \ \overline{-2} && \\
4&=x && \text{Need to check answer in original}\\
\sqrt{3(4)-8}-\sqrt{4}&=0 && \text{Multiply}\\
\sqrt{12-8}-\sqrt{4}&=0 && \text{Subtract}\\
\sqrt{4}-\sqrt{4}&=0 && \text{Take roots}
\end{aligned}
$$

$$
\begin{array}{rl}
2-2=0 & \text{Subtract} \\
0=0 & \text{True! It works} \\
x=4 & \text{Our Solution}
\end{array}
$$

When there is more than one square root in the problem, after isolating one root and squaring both sides we may still have a root remaining in the problem. In this case we will again isolate the term with the second root and square both sides. When isolating, we will isolate the *term* with the square root. This means the square root can be multiplied by a number after isolating.

Example 447.

$$
\begin{array}{rl}
\sqrt{2x+1}-\sqrt{x}=1 & \text{Even index! We will have to check answers} \\
\underline{+\sqrt{x}+\sqrt{x}} & \text{Isolate first root by adding } \sqrt{x} \text{ to both sides} \\
\sqrt{2x+1}=\sqrt{x}+1 & \text{Square both sides} \\
(\sqrt{2x+1})^2=(\sqrt{x}+1)^2 & \text{Evaluate exponents, recall } (a+b)^2=a^2+2ab+b^2 \\
2x+1=x+2\sqrt{x}+1 & \text{Isolate the term with the root} \\
\underline{-x-1-x\qquad\quad -1} & \text{Subtract } x \text{ and } 1 \text{ from both sides} \\
x=2\sqrt{x} & \text{Square both sides} \\
(x)^2=(2\sqrt{x})^2 & \text{Evaluate exponents} \\
x^2=4x & \text{Make equation equal zero} \\
-4x-4x & \text{Subtract } x \text{ from both sides} \\
x^2-4x=0 & \text{Factor} \\
x(x-4)=0 & \text{Set each factor equal to zero} \\
x=0 \ \text{ or } \ x-4=0 & \text{Solve} \\
\underline{+4+4} & \text{Add 4 to both sides of second equation} \\
x=0 \ \text{ or } \ x=4 & \text{Need to check answers in original}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt{2(0)+1}-\sqrt{(0)}=1 & \text{Check } x=0 \text{ first} \\
\sqrt{1}-\sqrt{0}=1 & \text{Take roots} \\
1-0=1 & \text{Subtract} \\
1=1 & \text{True! It works}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt{2(4)+1}-\sqrt{(4)}=1 & \text{Check } x=4 \\
\sqrt{8+1}-\sqrt{4}=1 & \text{Add} \\
\sqrt{9}-\sqrt{4}=1 & \text{Take roots} \\
3-2=1 & \text{Subtract} \\
1=1 & \text{True! It works}
\end{array}
$$

$x=0$ or 4 Our Solution

Example 448.

$$
\begin{array}{rl}
\sqrt{3x+9}-\sqrt{x+4}=-1 & \text{Even index! We will have to check answers} \\
\underline{+\sqrt{x+4}+\sqrt{x+4}} & \text{Isolate the first root by adding } \sqrt{x+4} \\
\sqrt{3x+9}=\sqrt{x+4}-1 & \text{Square both sides} \\
(\sqrt{3x+9})^2=(\sqrt{x+4}-1)^2 & \text{Evaluate exponents} \\
3x+9=x+4-2\sqrt{x+4}+1 & \text{Combine like terms} \\
3x+9=x+5-2\sqrt{x+4} & \text{Isolate the term with radical} \\
\underline{-x-5-x-5} & \text{Subtract } x \text{ and 5 from both sides} \\
2x+4=-2\sqrt{x+4} & \text{Square both sides} \\
(2x+4)^2=(-2\sqrt{x+4})^2 & \text{Evaluate exponents} \\
4x^2+16x+16=4(x+4) & \text{Distribute} \\
4x^2+16x+16=4x+16 & \text{Make equation equal zero} \\
\underline{-4x-16-4x-16} & \text{Subtract } 4x \text{ and 16 from both sides} \\
4x^2+12x=0 & \text{Factor} \\
4x(x+3)=0 & \text{Set each factor equal to zero} \\
4x=0 \text{ or } x+3=0 & \text{Solve} \\
\overline{4}\quad\overline{4}\qquad\underline{-3-3} & \\
x=0 \text{ or } x=-3 & \text{Check solutions in original}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt{3(0)+9}-\sqrt{(0)+4}=-1 & \text{Check } x=0 \text{ first} \\
\sqrt{9}-\sqrt{4}=-1 & \text{Take roots} \\
3-2=-1 & \text{Subtract} \\
1=-1 & \text{False, extraneous solution}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt{3(-3)+9}-\sqrt{(-3)+4}=-1 & \text{Check } x=-3 \\
\sqrt{-9+9}-\sqrt{(-3)+4}=-1 & \text{Add} \\
\sqrt{0}-\sqrt{1}=-1 & \text{Take roots} \\
0-1=-1 & \text{Subtract} \\
-1=-1 & \text{True! It works}
\end{array}
$$

$x=-3$ Our Solution

9.1 Practice - Solving with Radicals

Solve.

1) $\sqrt{2x+3}-3=0$

2) $\sqrt{5x+1}-4=0$

3) $\sqrt{6x-5}-x=0$

4) $\sqrt{x+2}-\sqrt{x}=2$

5) $3+x=\sqrt{6x+13}$

6) $x-1=\sqrt{7-x}$

7) $\sqrt{3-3x}-1=2x$

8) $\sqrt{2x+2}=3+\sqrt{2x-1}$

9) $\sqrt{4x+5}-\sqrt{x+4}=2$

10) $\sqrt{3x+4}-\sqrt{x+2}=2$

11) $\sqrt{2x+4}-\sqrt{x+3}=1$

12) $\sqrt{7x+2}-\sqrt{3x+6}=6$

13) $\sqrt{2x+6}-\sqrt{x+4}=1$

14) $\sqrt{4x-3}-\sqrt{3x+1}=1$

15) $\sqrt{6-2x}-\sqrt{2x+3}=3$

16) $\sqrt{2-3x}-\sqrt{3x+7}=3$

9.2

Quadratics - Solving with Exponents

Objective: Solve equations with exponents using the odd root property and the even root property.

Another type of equation we can solve is one with exponents. As you might expect we can clear exponents by using roots. This is done with very few unexpected results when the exponent is odd. We solve these problems very straight forward using the odd root property

$$\textbf{Odd Root Property: if } \boldsymbol{a^n = b}\textbf{, then } \boldsymbol{a = \sqrt[n]{b}} \textbf{ when } \boldsymbol{n} \textbf{ is odd}$$

Example 449.

$$\begin{aligned} x^5 &= 32 && \text{Use odd root property} \\ \sqrt[5]{x^5} &= \sqrt[5]{32} && \text{Simplify roots} \\ x &= 2 && \text{Our Solution} \end{aligned}$$

However, when the exponent is even we will have two results from taking an even root of both sides. One will be positive and one will be negative. This is because both $3^2 = 9$ and $(-3)^2 = 9$. so when solving $x^2 = 9$ we will have two solutions, one positive and one negative: $x = 3$ and -3

$$\textbf{Even Root Property: if } \boldsymbol{a^n = b}\textbf{, then } \boldsymbol{a = \pm\sqrt[n]{b}} \textbf{ when } \boldsymbol{n} \textbf{ is even}$$

Example 450.

$$x^4 = 16 \qquad \text{Use even root property } (\pm)$$

$$\sqrt[4]{x^4} = \pm\sqrt[4]{16} \quad \text{Simplify roots}$$
$$x = \pm 2 \quad \text{Our Solution}$$

World View Note: In 1545, French Mathematicain Gerolamo Cardano published his book *The Great Art, or the Rules of Algebra* which included the solution of an equation with a fourth power, but it was considered absurd by many to take a quantity to the fourth power because there are only three dimensions!

Example 451.

$$(2x+4)^2 = 36 \quad \text{Use even root property } (\pm)$$
$$\sqrt{(2x+4)^2} = \pm\sqrt{36} \quad \text{Simplify roots}$$
$$2x+4 = \pm 6 \quad \text{To avoid sign errors we need two equations}$$
$$2x+4 = 6 \text{ or } 2x+4 = -6 \quad \text{One equation for } +\text{, one equation for } -$$
$$\underline{-4\ -4} \qquad \underline{-4\quad -4} \quad \text{Subtract 4 from both sides}$$
$$2x = 2 \text{ or } 2x = -10 \quad \text{Divide both sides by 2}$$
$$\overline{2}\ \ \overline{2} \qquad \overline{2}\quad \overline{2}$$
$$x = 1 \text{ or } x = -5 \quad \text{Our Solutions}$$

In the previous example we needed two equations to simplify because when we took the root, our solutions were two rational numbers, 6 and -6. If the roots did not simplify to rational numbers we can keep the $\pm$ in the equation.

Example 452.

$$(6x-9)^2 = 45 \quad \text{Use even root property } (\pm)$$
$$\sqrt{(6x-9)^2} = \pm\sqrt{45} \quad \text{Simplify roots}$$
$$6x-9 = \pm 3\sqrt{5} \quad \text{Use one equation because root did not simplify to rational}$$
$$\underline{+9\ +9} \quad \text{Add 9 to both sides}$$
$$6x = 9 \pm 3\sqrt{5} \quad \text{Divide both sides by 6}$$
$$\overline{6} \qquad \overline{\quad 6 \quad}$$
$$x = \frac{9 \pm 3\sqrt{5}}{6} \quad \text{Simplify, divide each term by 3}$$
$$x = \frac{3 \pm \sqrt{5}}{2} \quad \text{Our Solution}$$

When solving with exponents, it is important to first isolate the part with the exponent before taking any roots.

Example 453.

$$
\begin{array}{rl}
(x+4)^3 - 6 = 119 & \text{Isolate part with exponent} \\
\underline{+6 \quad +6} & \\
(x+4)^3 = 125 & \text{Use odd root property} \\
\sqrt[3]{(x+4)^3} = \sqrt{125} & \text{Simplify roots} \\
x+4 = 5 & \text{Solve} \\
\underline{-4 - 4} & \text{Subtract 4 from both sides} \\
x = 1 & \text{Our Solution}
\end{array}
$$

Example 454.

$$
\begin{array}{rl}
(6x+1)^2 + 6 = 10 & \text{Isolate part with exponent} \\
\underline{-6 - 6} & \text{Subtract 6 from both sides} \\
(6x+1)^2 = 4 & \text{Use even root property } (\pm) \\
\sqrt{(6x+1)^2} = \pm\sqrt{4} & \text{Simplify roots} \\
6x+1 = \pm 2 & \text{To avoid sign errors, we need two equations} \\
6x+1 = 2 \ \text{ or } \ 6x+1 = -2 & \text{Solve each equation} \\
\underline{-1 - 1} \qquad \underline{-1 \quad -1} & \text{Subtract 1 from both sides} \\
\frac{6x}{6} = \frac{1}{6} \ \text{ or } \ \frac{6x}{6} = \frac{-3}{6} & \text{Divide both sides by 6} \\
x = \frac{1}{6} \ \text{ or } \ x = -\frac{1}{2} & \text{Our Solution}
\end{array}
$$

When our exponents are a fraction we will need to first convert the fractional exponent into a radical expression to solve. Recall that $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. Once we have done this we can clear the exponent using either the even ($\pm$) or odd root property. Then we can clear the radical by raising both sides to an exponent (remember to check answers if the index is even).

Example 455.

$$
\begin{array}{rl}
(4x+1)^{\frac{2}{5}} = 9 & \text{Rewrite as } a \text{ radical expression} \\
(\sqrt[5]{4x+1})^2 = 9 & \text{Clear exponent first with even root property } (\pm) \\
\sqrt{(\sqrt[5]{4x+1})^2} = \pm\sqrt{9} & \text{Simplify roots}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt[5]{4x+1}=\pm 3 & \text{Clear radical by raising both sides to 5th power}\\
(\sqrt[5]{4x+1})^5=(\pm 3)^5 & \text{Simplify exponents}\\
4x+1=\pm 243 & \text{Solve, need 2 equations!}\\
4x+1=243 \text{ or } 4x+1=-243 & \\
\underline{-1\quad -1} \qquad \underline{-1\quad -1} & \text{Subtract 1 from both sides}\\
4x=242 \text{ or } 4x=-244 & \text{Divide both sides by 4}\\
\overline{4}\quad \overline{4} \qquad \overline{4}\quad \overline{4} & \\
x=\dfrac{121}{2},-61 & \text{Our Solution}
\end{array}
$$

Example 456.

$$
\begin{array}{rl}
(3x-2)^{\frac{3}{4}}=64 & \text{Rewrite as radical expression}\\
(\sqrt[4]{3x-2})^3=64 & \text{Clear exponent first with odd root property}\\
\sqrt[3]{(\sqrt[4]{3x-2})^3}=\sqrt[3]{64} & \text{Simplify roots}\\
\sqrt[4]{3x-2}=4 & \text{Even Index! Check answers.}\\
(\sqrt[4]{3x-2})^4=4^4 & \text{Raise both sides to 4th power}\\
3x-2=256 & \text{Solve}\\
\underline{+2\quad +2} & \text{Add 2 to both sides}\\
3x=258 & \text{Divide both sides by 3}\\
\overline{3}\quad \overline{3} & \\
x=86 & \text{Need to check answer in radical form of problem}\\
(\sqrt[4]{3(86)-2})^3=64 & \text{Multiply}\\
(\sqrt[4]{258-2})^3=64 & \text{Subtract}\\
(\sqrt[4]{256})^3=64 & \text{Evaluate root}\\
4^3=64 & \text{Evaluate exponent}\\
64=64 & \text{True! It works}\\
x=86 & \text{Our Solution}
\end{array}
$$

With rational exponents it is very helpful to convert to radical form to be able to see if we need a $\pm$ because we used the even root property, or to see if we need to check our answer because there was an even root in the problem. When checking we will usually want to check in the radical form as it will be easier to evaluate.

9.2 Practice - Solving with Exponents

Solve.

1) $x^2 = 75$

2) $x^3 = -8$

3) $x^2 + 5 = 13$

4) $4x^3 - 2 = 106$

5) $3x^2 + 1 = 73$

6) $(x - 4)^2 = 49$

7) $(x + 2)^5 = -243$

8) $(5x + 1)^4 = 16$

9) $(2x + 5)^3 - 6 = 21$

10) $(2x + 1)^2 + 3 = 21$

11) $(x - 1)^{\frac{2}{3}} = 16$

12) $(x - 1)^{\frac{3}{2}} = 8$

13) $(2 - x)^{\frac{3}{2}} = 27$

14) $(2x + 3)^{\frac{4}{3}} = 16$

15) $(2x - 3)^{\frac{2}{3}} = 4$

16) $(x + 3)^{-\frac{1}{3}} = 4$

17) $(x + \frac{1}{2})^{-\frac{2}{3}} = 4$

18) $(x - 1)^{-\frac{5}{3}} = 32$

19) $(x - 1)^{-\frac{5}{2}} = 32$

20) $(x + 3)^{\frac{3}{2}} = -8$

21) $(3x - 2)^{\frac{4}{5}} = 16$

22) $(2x + 3)^{\frac{3}{2}} = 27$

23) $(4x + 2)^{\frac{3}{5}} = -8$

24) $(3 - 2x)^{\frac{4}{3}} = -81$

9.3

Quadratics - Complete the Square

Objective: Solve quadratic equations by completing the square.

When solving quadratic equations in the past we have used factoring to solve for our variable. This is exactly what is done in the next example.

Example 457.

$$\begin{array}{rcrl} x^2+5x+6=0 & & & \text{Factor} \\ (x+3)(x+2)=0 & & & \text{Set each factor equal to zero} \\ x+3=0 & \text{or} & x+2=0 & \text{Solve each equation} \\ \underline{-3-3} & & \underline{-2-2} & \\ x=-3 & \text{or} & x=-2 & \text{Our Solutions} \end{array}$$

However, the problem with factoring is all equations cannot be factored. Consider the following equation: $x^2-2x-7=0$. The equation cannot be factored, however there are two solutions to this equation, $1+2\sqrt{2}$ and $1-2\sqrt{2}$. To find these two solutions we will use a method known as completing the square. When completing the square we will change the quadratic into a perfect square which can easily be solved with the square root property. The next example reviews the square root property.

Example 458.

$$\begin{array}{rl} (x+5)^2=18 & \text{Square root of both sides} \\ \sqrt{(x+5)^2}=\pm\sqrt{18} & \text{Simplify each radical} \\ x+5=\pm 3\sqrt{2} & \text{Subtract 5 from both sides} \\ \underline{-5\quad -5} & \\ x=-5\pm 3\sqrt{2} & \text{Our Solution} \end{array}$$

To complete the square, or make our problem into the form of the previous example, we will be searching for the third term in a trinomial. If a quadratic is of the form $x^2 + bx + c$, and a perfect square, the third term, c, can be easily found by the formula $\left(\frac{1}{2} \cdot b\right)^2$. This is shown in the following examples, where we find the number that completes the square and then factor the perfect square.

Example 459.

$$\begin{array}{rl} x^2 + 8x + c & c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 8 \\ \left(\frac{1}{2} \cdot 8\right)^2 = 4^2 = 16 & \text{The third term to complete the square is } 16 \\ x^2 + 8x + 16 & \text{Our equation as } a \text{ perfect square, factor} \\ (x+4)^2 & \text{Our Solution} \end{array}$$

Example 460.

$$\begin{array}{rl} x^2 - 7x + c & c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 7 \\ \left(\frac{1}{2} \cdot 7\right)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4} & \text{The third term to complete the square is } \frac{49}{4} \\ x^2 - 11x + \frac{49}{4} & \text{Our equation as } a \text{ perfect square, factor} \\ \left(x - \frac{7}{2}\right)^2 & \text{Our Solution} \end{array}$$

Example 461.

$$\begin{array}{rl} x^2 + \frac{5}{3}x + c & c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 8 \\ \left(\frac{1}{2} \cdot \frac{5}{3}\right)^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36} & \text{The third term to complete the square is } \frac{25}{36} \end{array}$$

$$x^2+\frac{5}{3}x+\frac{25}{36} \quad \text{Our equation as } a \text{ perfect square, factor}$$

$$\left(x+\frac{5}{6}\right)^2 \quad \text{Our Solution}$$

The process in the previous examples, combined with the even root property, is used to solve quadratic equations by completing the square. The following five steps describe the process used to complete the square, along with an example to demonstrate each step.

Problem	$3x^2+18x-6=0$
1. Separate constant term from variables	$\underline{+6+6}$ $3x^2+18x \quad =6$
2. Divide each term by a	$\frac{3}{3}x^2+\frac{18}{3}x \quad =\frac{6}{3}$ $x^2+6x \quad =2$
3. Find value to complete the square: $\left(\frac{1}{2}\cdot b\right)^2$	$\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$
4. Add to both sides of equation	$x^2+6x \quad =2$ $\underline{+9\ +9}$ $x^2+6x+9=11$
5. Factor	$(x+3)^2=11$
Solve by even root property	$\sqrt{(x+3)^2}=\pm\sqrt{11}$ $x+3=\pm\sqrt{11}$ $\underline{-3\quad -3}$ $x=-3\pm\sqrt{11}$

World View Note: The Chinese in 200 BC were the first known culture group to use a method similar to completing the square, but their method was only used to calculate positive roots.

The advantage of this method is it can be used to solve any quadratic equation. The following examples show how completing the square can give us rational solutions, irrational solutions, and even complex solutions.

Example 462.

$$2x^2+20x+48=0 \quad \text{Separate constant term from varaibles}$$

$$\begin{array}{rl}
\underline{-48-48} & \text{Subtract 24} \\
2x^2+20x \quad =-48 & \text{Divide by } a \text{ or } 2 \\
\overline{\ 2\ } \quad \overline{\ 2\ } \quad\quad \overline{\ 2\ } & \\
x^2+10x \quad =-24 & \text{Find number to complete the square: } \left(\frac{1}{2}\cdot b\right)^2 \\
\left(\frac{1}{2}\cdot 10\right)^2 = 5^2 = 25 & \text{Add 25 to both sides of the equation} \\
x^2+10x \quad =-24 & \\
\underline{+25 \quad +25} & \\
x^2+10x+25=1 & \text{Factor} \\
(x+5)^2=1 & \text{Solve with even root property} \\
\sqrt{(x+5)^2} = \pm\sqrt{1} & \text{Simplify roots} \\
x+5=\pm 1 & \text{Subtract 5 from both sides} \\
\underline{-5-5} & \\
x=-5\pm 1 & \text{Evaluate} \\
x=-4 \text{ or } -6 & \text{Our Solution}
\end{array}$$

Example 463.

$$\begin{array}{rl}
x^2-3x-2=0 & \text{Separate constant from variables} \\
\underline{+2+2} & \text{Add 2 to both sides} \\
x^2-3x \quad =2 & \text{No } a\text{, find number to complete the square } \left(\frac{1}{2}\cdot b\right)^2 \\
\left(\frac{1}{2}\cdot 3\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} & \text{Add } \frac{9}{4} \text{ to both sides,} \\
\frac{2}{1}\left(\frac{4}{4}\right)+\frac{9}{4} = \frac{8}{4}+\frac{9}{4} = \frac{17}{4} & \text{Need common denominator (4) on right} \\
x^2-3x+\frac{9}{4} = \frac{8}{4}+\frac{9}{4} = \frac{17}{4} & \text{Factor} \\
\left(x-\frac{3}{2}\right)^2 = \frac{17}{4} & \text{Solve using the even root property} \\
\sqrt{\left(x-\frac{3}{2}\right)^2} = \pm\sqrt{\frac{17}{4}} & \text{Simplify roots} \\
x-\frac{3}{2} = \frac{\pm\sqrt{17}}{2} & \text{Add } \frac{3}{2} \text{ to both sides,}
\end{array}$$

$$\underline{+\frac{3}{2}+\frac{3}{2}} \quad \text{we already have } a \text{ common denominator}$$

$$x=\frac{3\pm\sqrt{17}}{2} \quad \text{Our Solution}$$

Example 464.

$$3x^2=2x-7 \quad \text{Separate the constant from the variables}$$

$$\underline{-2x-2x} \quad \text{Subtract } 2x \text{ from both sides}$$

$$\frac{3x^2}{3}-\frac{2x}{3}=\frac{-7}{3} \quad \text{Divide each term by } a \text{ or } 3$$

$$x^2-\frac{2}{3}x=-\frac{7}{3} \quad \text{Find the number to complete the square } \left(\frac{1}{2}\cdot b\right)^2$$

$$\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9} \quad \text{Add to both sides,}$$

$$-\frac{7}{3}\left(\frac{3}{3}\right)+\frac{1}{9}=\frac{-21}{3}+\frac{1}{9}=\frac{-20}{9} \quad \text{get common denominator on right}$$

$$x^2-\frac{2}{3}x+\frac{1}{3}=-\frac{20}{9} \quad \text{Factor}$$

$$\left(x-\frac{1}{3}\right)^2=-\frac{20}{9} \quad \text{Solve using the even root property}$$

$$\sqrt{\left(x-\frac{1}{3}\right)^2}=\pm\sqrt{\frac{-20}{9}} \quad \text{Simplify roots}$$

$$x-\frac{1}{3}=\frac{\pm 2i\sqrt{5}}{3} \quad \text{Add } \frac{1}{3} \text{ to both sides,}$$

$$\underline{+\frac{1}{3}+\frac{1}{3}} \quad \text{Already have common denominator}$$

$$x=\frac{1\pm 2i\sqrt{5}}{3} \quad \text{Our Solution}$$

As several of the examples have shown, when solving by completing the square we will often need to use fractions and be comfortable finding common denominators and adding fractions together. Once we get comfortable solving by completing the square and using the five steps, any quadratic equation can be easily solved.

9.3 Practice - Complete the Square

Find the value that completes the square and then rewrite as a perfect square.

1) $x^2 - 30x + \_\_$
2) $a^2 - 24a + \_\_$
3) $m^2 - 36m + \_\_$
4) $x^2 - 34x + \_\_$
5) $x^2 - 15x + \_\_$
6) $r^2 - \frac{1}{9}r + \_\_$
7) $y^2 - y + \_\_$
8) $p^2 - 17p + \_\_$

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 12 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $6x^2 + 12x + 63 = 0$
14) $3x^2 - 6x + 47 = 0$
15) $5k^2 - 10k + 48 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 3 = 6$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$
27) $n^2 = -21 + 10n$
28) $a^2 - 56 = -10a$
29) $3k^2 + 9 = 6k$
30) $5x^2 = -26 + 10x$
31) $2x^2 + 63 = 8x$
32) $5n^2 = -10n + 15$
33) $p^2 - 8p = -55$
34) $x^2 + 8x + 15 = 8$
35) $7n^2 - n + 7 = 7n + 6n^2$
36) $n^2 + 4n = 12$
37) $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$
38) $-3r^2 + 12r + 49 = -6r^2$
39) $5x^2 + 5x = -31 - 5x$
40) $8n^2 + 16n = 64$
41) $v^2 + 5v + 28 = 0$
42) $b^2 + 7b - 33 = 0$
43) $7x^2 - 6x + 40 = 0$
44) $4x^2 + 4x + 25 = 0$
45) $k^2 - 7k + 50 = 3$
46) $a^2 - 5a + 25 = 3$
47) $5x^2 + 8x - 40 = 8$
48) $2p^2 - p + 56 = -8$
49) $m^2 = -15 + 9m$
50) $n^2 - n = -41$
51) $8r^2 + 10r = -55$
52) $3x^2 - 11x = -18$
53) $5n^2 - 8n + 60 = -3n + 6 + 4n^2$
54) $4b^2 - 15b + 56 = 3b^2$
55) $-2x^2 + 3x - 5 = -4x^2$
56) $10v^2 - 15v = 27 + 4v^2 - 6v$

9.4

Quadratics - Quadratic Formula

Objective: Solve quadratic equations by using the quadratic formula.

The general from of a quadratic is $ax^2 + bx + c = 0$. We will now solve this formula for x by completing the square

Example 465.

$$\begin{array}{rl}
ax^2 + bc + c = 0 & \text{Separate constant from variables} \\
\underline{-c - c} & \text{Subtract } c \text{ from both sides} \\
\dfrac{ax^2}{a} + \dfrac{bx}{a} \quad = \dfrac{-c}{a} & \text{Divide each term by } a \\
x^2 + \dfrac{b}{a}x \quad = \dfrac{-c}{a} & \text{Find the number that completes the square} \\
\left(\dfrac{1}{2} \cdot \dfrac{b}{a}\right)^2 = \left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} & \text{Add to both sides,} \\
\dfrac{b^2}{4a^2} - \dfrac{c}{a}\left(\dfrac{4a}{4a}\right) = \dfrac{b^2}{4a^2} - \dfrac{4ac}{4a^2} = \dfrac{b^2 - 4ac}{4a^2} & \text{Get common denominator on right} \\
x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{b^2}{4a^2} - \dfrac{4\mathrm{ac}}{4a^2} = \dfrac{b^2 - 4ac}{4a^2} & \text{Factor} \\
\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2} & \text{Solve using the even root property} \\
\sqrt{\left(x + \dfrac{b}{2a}\right)^2} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}} & \text{Simplify roots} \\
x + \dfrac{b}{2a} = \dfrac{\pm\sqrt{b^2 - 4ac}}{2a} & \text{Subtract } \dfrac{b}{2a} \text{ from both sides} \\
x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} & \text{Our Solution}
\end{array}$$

This solution is a very important one to us. As we solved a general equation by completing the square, we can use this formula to solve any quadratic equation. Once we identify what a, b, and c are in the quadratic, we can substitute those values into $x = \frac{-b \pm \sqrt{b^2 - 4\text{ac}}}{2a}$ and we will get our two solutions. This formula is known as the quadratic fromula

$$\textbf{Quadratic Formula: if } \boldsymbol{ax^2 + bx + c = 0} \textbf{ then } \boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

World View Note: Indian mathematician Brahmagupta gave the first explicit formula for solving quadratics in 628. However, at that time mathematics was not done with variables and symbols, so the formula he gave was, "To the absolute number multiplied by four times the square, add the square of the middle term; the square root of the same, less the middle term, being divided by twice the square is the value." This would translate to $\frac{\sqrt{4\text{ac} + b^2} - b}{2a}$ as the solution to the equation $ax^2 + bx = c$.

We can use the quadratic formula to solve any quadratic, this is shown in the following examples.

Example 466.

$$\begin{aligned}
x^2 + 3x + 2 &= 0 && a = 1, b = 3, c = 2, \text{use quadratic formula} \\
x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} && \text{Evaluate exponent and multiplication} \\
x &= \frac{-3 \pm \sqrt{9 - 8}}{2} && \text{Evaluate subtraction under root} \\
x &= \frac{-3 \pm \sqrt{1}}{2} && \text{Evaluate root} \\
x &= \frac{-3 \pm 1}{2} && \text{Evaluate} \pm \text{to get two answers} \\
x &= \frac{-2}{2} \text{ or } \frac{-4}{2} && \text{Simplify fractions} \\
x &= -1 \text{ or } -2 && \text{Our Solution}
\end{aligned}$$

As we are solving using the quadratic formula, it is important to remember the equation must fist be equal to zero.

Example 467.

$$\begin{aligned}
25x^2 &= 30x + 11 && \text{First set equal to zero} \\
\underline{-30x - 11} &\ \ \underline{-30x - 11} && \text{Subtract } 30x \text{ and } 11 \text{ from both sides} \\
25x^2 - 30x - 11 &= 0 && a = 25, b = -30, c = -11, \text{use quadratic formula} \\
x &= \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \text{Evaluate exponent and multiplication}
\end{aligned}$$

$$x = \frac{30 \pm \sqrt{900 + 1100}}{50} \quad \text{Evaluate addition inside root}$$
$$x = \frac{30 \pm \sqrt{2000}}{50} \quad \text{Simplify root}$$
$$x = \frac{30 \pm 20\sqrt{5}}{50} \quad \text{Reduce fraction by dividing each term by 10}$$
$$x = \frac{3 \pm 2\sqrt{5}}{5} \quad \text{Our Solution}$$

Example 468.

$$\begin{array}{rl}
3x^2 + 4x + 8 = 2x^2 + 6x - 5 & \text{First set equation equal to zero} \\
\underline{-2x^2 - 6x + 5 \quad -2x^2 - 6x + 5} & \text{Subtract } 2x^2 \text{ and } 6x \text{ and add } 5 \\
x^2 - 2x + 13 = 0 & a = 1, b = -2, c = 13, \text{ use quadratic formula} \\
x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)} & \text{Evaluate exponent and multiplication} \\
x = \dfrac{2 \pm \sqrt{4 - 52}}{2} & \text{Evaluate subtraction inside root} \\
x = \dfrac{2 \pm \sqrt{-48}}{2} & \text{Simplify root} \\
x = \dfrac{2 \pm 4i\sqrt{3}}{2} & \text{Reduce fraction by dividing each term by 2} \\
x = 1 \pm 2i\sqrt{3} & \text{Our Solution}
\end{array}$$

When we use the quadratic formula we don't necessarily get two unique answers. We can end up with only one solution if the square root simplifies to zero.

Example 469.

$$\begin{array}{rl}
4x^2 - 12x + 9 = 0 & a = 4, b = -12, c = 9, \text{ use quadratic formula} \\
x = \dfrac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} & \text{Evaluate exponents and multiplication} \\
x = \dfrac{12 \pm \sqrt{144 - 144}}{8} & \text{Evaluate subtraction inside root} \\
x = \dfrac{12 \pm \sqrt{0}}{8} & \text{Evaluate root} \\
x = \dfrac{12 \pm 0}{8} & \text{Evaluate } \pm \\
x = \dfrac{12}{8} & \text{Reduce fraction} \\
x = \dfrac{3}{2} & \text{Our Solution}
\end{array}$$

If a term is missing from the quadratic, we can still solve with the quadratic formula, we simply use zero for that term. The order is important, so if the term with x is missing, we have $b=0$, if the constant term is missing, we have $c=0$.

Example 470.

$$3x^2+7=0 \quad a=3, b=0\,(\text{missing term}), c=7$$

$$x=\frac{-0\pm\sqrt{0^2-4(3)(7)}}{2(3)} \quad \text{Evaluate exponnets and multiplication, zeros not needed}$$

$$x=\frac{\pm\sqrt{-84}}{6} \quad \text{Simplify root}$$

$$x=\frac{\pm 2i\sqrt{21}}{6} \quad \text{Reduce, dividing by 2}$$

$$x=\frac{\pm i\sqrt{21}}{3} \quad \text{Our Solution}$$

We have covered three different methods to use to solve a quadratic: factoring, complete the square, and the quadratic formula. It is important to be familiar with all three as each has its advantage to solving quadratics. The following table walks through a suggested process to decide which method would be best to use for solving a problem.

1. If it can easily factor, solve by factoring	$x^2-5x+6=0$ $(x-2)(x-3)=0$ $x=2$ or $x=3$
2. If $a=1$ and b is even, complete the square	$x^2+2x=4$ $\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$ $x^2+2x+1=5$ $(x+1)^2=5$ $x+1=\pm\sqrt{5}$ $x=-1\pm\sqrt{5}$
3. Otherwise, solve by the quadratic formula	$x^2-3x+4=0$ $x=\frac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$ $x=\frac{3\pm i\sqrt{7}}{2}$

The above table is mearly a suggestion for deciding how to solve a quadtratic. Remember completing the square and quadratic formula will always work to solve any quadratic. Factoring only woks if the equation can be factored.

9.4 Practice - Quadratic Formula

Solve each equation with the quadratic formula.

1) $4a^2+6=0$

2) $3k^2+2=0$

3) $2x^2-8x-2=0$

4) $6n^2-1=0$

5) $2m^2-3=0$

6) $5p^2+2p+6=0$

7) $3r^2-2r-1=0$

8) $2x^2-2x-15=0$

9) $4n^2-36=0$

10) $3b^2+6=0$

11) $v^2-4v-5=-8$

12) $2x^2+4x+12=8$

13) $2a^2+3a+14=6$

14) $6n^2-3n+3=-4$

15) $3k^2+3k-4=7$

16) $4x^2-14=-2$

17) $7x^2+3x-16=-2$

18) $4n^2+5n=7$

19) $2p^2+6p-16=4$

20) $m^2+4m-48=-3$

21) $3n^2+3n=-3$

22) $3b^2-3=8b$

23) $2x^2=-7x+49$

24) $3r^2+4=-6r$

25) $5x^2=7x+7$

26) $6a^2=-5a+13$

27) $8n^2=-3n-8$

28) $6v^2=4+6v$

29) $2x^2+5x=-3$

30) $x^2=8$

31) $4a^2-64=0$

32) $2k^2+6k-16=2k$

33) $4p^2+5p-36=3p^2$

34) $12x^2+x+7=5x^2+5x$

35) $-5n^2-3n-52=2-7n^2$

36) $7m^2-6m+6=-m$

37) $7r^2-12=-3r$

38) $3x^2-3=x^2$

39) $2n^2-9=4$

40) $6b^2=b^2+7-b$

9.5

Quadratics - Build Quadratics From Roots

Objective: Find a quadratic equation that has given roots using reverse factoring and reverse completing the square.

Up to this point we have found the solutions to quadratics by a method such as factoring or completing the square. Here we will take our solutions and work backwards to find what quadratic goes with the solutions.

We will start with rational solutions. If we have rational solutions we can use factoring in reverse, we will set each solution equal to x and then make the equation equal to zero by adding or subtracting. Once we have done this our expressions will become the factors of the quadratic.

Example 471.

$$\begin{array}{rl} \text{The solutions are } 4 \text{ and } -2 & \text{Set each solution equal to } x \\ x = 4 \ \text{ or } \ x = -2 & \text{Make each equation equal zero} \\ \underline{-4-4} \quad \underline{+2 \quad +2} & \text{Subtract 4 from first, add 2 to second} \\ x - 4 = 0 \ \text{ or } \ x + 2 = 0 & \text{These expressions are the factors} \\ (x-4)(x+2) = 0 & \text{FOIL} \\ x^2 + 2x - 4x - 8 & \text{Combine like terms} \\ x^2 - 2x - 8 = 0 & \text{Our Solution} \end{array}$$

If one or both of the solutions are fractions we will clear the fractions by multiplying by the denominators.

Example 472.

$$\begin{array}{rl} \text{The solution are } \dfrac{2}{3} \text{ and } \dfrac{3}{4} & \text{Set each solution equal to } x \\ x = \dfrac{2}{3} \ \text{ or } \ x = \dfrac{3}{4} & \text{Clear fractions by multiplying by denominators} \\ 3x = 2 \ \text{ or } \ 4x = 3 & \text{Make each equation equal zero} \\ \underline{-2-2} \qquad \underline{-3-3} & \text{Subtract 2 from the first, subtract 3 from the second} \\ 3x - 2 = 0 \ \text{ or } \ 4x - 3 = 0 & \text{These expressions are the factors} \\ (3x-2)(4x-3) = 0 & \text{FOIL} \\ 12x^2 - 9x - 8x + 6 = 0 & \text{Combine like terms} \end{array}$$

$$12x^2 - 17x + 6 = 0 \quad \text{Our Solution}$$

If the solutions have radicals (or complex numbers) then we cannot use reverse factoring. In these cases we will use reverse completing the square. When there are radicals the solutions will always come in pairs, one with a plus, one with a minus, that can be combined into "one" solution using $\pm$. We will then set this solution equal to x and square both sides. This will clear the radical from our problem.

Example 473.

$$\begin{array}{rl}
\text{The solutions are } \sqrt{3} \text{ and } -\sqrt{3} & \text{Write as "one" expression equal to } x \\
x = \pm\sqrt{3} & \text{Square both sides} \\
x^2 = 3 & \text{Make equal to zero} \\
\underline{-3 - 3} & \text{Subtract 3 from both sides} \\
x^2 - 3 = 0 & \text{Our Solution}
\end{array}$$

We may have to isolate the term with the square root (with plus or minus) by adding or subtracting. With these problems, remember to square a binomial we use the formula $(a+b)^2 = a^2 + 2ab + b^2$

Example 474.

$$\begin{array}{rl}
\text{The solutions are } 2 - 5\sqrt{2} \text{ and } 2 + 5\sqrt{2} & \text{Write as "one" expression equal to } x \\
x = 2 \pm 5\sqrt{2} & \text{Isolate the square root term} \\
\underline{-2 - 2} \quad & \text{Subtract 2 from both sides} \\
x - 2 = \pm 5\sqrt{2} & \text{Square both sides} \\
x^2 - 4x + 4 = 25 \cdot 2 & \\
x^2 - 4x + 4 = 50 & \text{Make equal to zero} \\
\underline{-50 - 50} & \text{Subtract 50} \\
x^2 - 4x - 46 = 0 & \text{Our Solution}
\end{array}$$

World View Note: Before the quadratic formula, before completing the square, before factoring, quadratics were solved geometrically by the Greeks as early as 300 BC! In 1079 Omar Khayyam, a Persian mathematician solved cubic equations geometrically!

If the solution is a fraction we will clear it just as before by multiplying by the denominator.

Example 475.

$$\begin{array}{rl}
\text{The solutions are } \dfrac{2+\sqrt{3}}{4} \text{ and } \dfrac{2-\sqrt{3}}{4} & \text{Write as ''one'' expresion equal to } x \\
x = \dfrac{2 \pm \sqrt{3}}{4} & \text{Clear fraction by multiplying by 4} \\
4x = 2 \pm \sqrt{3} & \text{Isolate the square root term} \\
\underline{-2 - 2} \quad & \text{Subtract 2 from both sides} \\
4x - 2 = \pm \sqrt{3} & \text{Square both sides} \\
16x^2 - 16x + 4 = 3 & \text{Make equal to zero} \\
\underline{-3 - 3} & \text{Subtract 3} \\
16x^2 - 16x + 1 = 0 & \text{Our Solution}
\end{array}$$

The process used for complex solutions is identical to the process used for radicals.

Example 476.

$$\begin{array}{rl}
\text{The solutions are } 4 - 5i \text{ and } 4 + 5i & \text{Write as ''one'' expression equal to } x \\
x = 4 \pm 5i & \text{Isolate the } i \text{ term} \\
\underline{-4 - 4} \quad & \text{Subtract 4 from both sides} \\
x - 4 = \pm 5i & \text{Square both sides} \\
x^2 - 8x + 16 = 25i^2 & i^2 = -1 \\
x^2 - 8x + 16 = -25 & \text{Make equal to zero} \\
\underline{+25 \quad +25} & \text{Add 25 to both sides} \\
x^2 - 8x + 41 = 0 & \text{Our Solution}
\end{array}$$

Example 477.

$$\begin{array}{rl}
\text{The solutions are } \dfrac{3-5i}{2} \text{ and } \dfrac{3+5i}{2} & \text{Write as ''one'' expression equal to } x \\
x = \dfrac{3 \pm 5i}{2} & \text{Clear fraction by multiplying by denominator} \\
2x = 3 \pm 5i & \text{Isolate the } i \text{ term} \\
\underline{-3 - 3} \quad & \text{Subtract 3 from both sides} \\
2x - 3 = \pm 5i & \text{Square both sides} \\
4x^2 - 12x + 9 = 5i^2 & i^2 = -1 \\
4x^2 - 12x + 9 = -25 & \text{Make equal to zero} \\
\underline{+25 \quad +25} & \text{Add 25 to both sides} \\
4x^2 - 12x + 34 = 0 & \text{Our Solution}
\end{array}$$

9.5 Practice - Build Quadratics from Roots

From each problem, find a quadratic equation with those numbers as its solutions.

1) $2, 5$

2) $3, 6$

3) $20, 2$

4) $13, 1$

5) $4, 4$

6) $0, 9$

7) $0, 0$

8) $-2, -5$

9) $-4, 11$

10) $3, \ -1$

11) $\frac{3}{4}, \frac{1}{4}$

12) $\frac{5}{8}, \frac{5}{7}$

13) $\frac{1}{2}, \frac{1}{3}$

14) $\frac{1}{2}, \frac{2}{3}$

15) $\frac{3}{7}, 4$

16) $2, \frac{2}{9}$

17) $-\frac{1}{3}, \frac{5}{6}$

18) $\frac{5}{3}, -\frac{1}{2}$

19) $-6, \frac{1}{9}$

20) $-\frac{2}{5}, 0$

21) ± 5

22) ± 1

23) $\pm\frac{1}{5}$

24) $\pm\sqrt{7}$

25) $\pm\sqrt{11}$

26) $\pm 2\sqrt{3}$

27) $\pm\frac{\sqrt{3}}{4}$

28) $\pm 11i$

29) $\pm i\sqrt{13}$

30) $\pm 5i\sqrt{2}$

31) $2\pm\sqrt{6}$

32) $-3\pm\sqrt{2}$

33) $1\pm 3i$

34) $-2\pm 4i$

35) $6\pm i\sqrt{3}$

36) $-9\pm i\sqrt{5}$

37) $\frac{-1\pm\sqrt{6}}{2}$

38) $\frac{2\pm 5i}{3}$

39) $\frac{6\pm i\sqrt{2}}{8}$

40) $\frac{-2\pm i\sqrt{15}}{2}$

9.6

Quadratics - Quadratic in Form

Objective: Solve equations that are quadratic in form by substitution to create a quadratic equation.

We have seen three different ways to solve quadratics: factoring, completing the square, and the quadratic formula. A quadratic is any equation of the form $0 = ax^2 + bx + c$, however, we can use the skills learned to solve quadratics to solve problems with higher (or sometimes lower) powers if the equation is in what is called quadratic form.

$$\textbf{Quadratic Form: } \boldsymbol{0 = ax^m + bx^n + c} \textbf{ where } \boldsymbol{m = 2n}$$

An equation is in quadratic form if one of the exponents on a variable is double the exponent on the same variable somewhere else in the equation. If this is the case we can create a new variable, set it equal to the variable with smallest exponent. When we substitute this into the equation we will have a quadratic equation we can solve.

World View Note: Arab mathematicians around the year 1000 were the first to use this method!

Example 478.

$x^4 - 13x^2 + 36 = 0$	Quadratic form, one exponent, 4, double the other, 2
$y = x^2$	New variable equal to the variable with smaller exponent
$y^2 = x^4$	Square both sides
$y^2 - 13y + 36 = 0$	Substitute y for x^2 and y^2 for x^4
$(y - 9)(y - 4) = 0$	Solve. We can solve this equation by factoring
$y - 9 = 0 \text{ or } y - 4 = 0$	Set each factor equal to zero
$\underline{+9+9 \qquad +4+4}$	Solve each equation
$y = 9 \text{ or } y = 4$	Solutions for y, need x. We will use $y = x^2$ equation
$9 = x^2 \text{ or } 4 = x^2$	Substitute values for y
$\pm\sqrt{9} = \sqrt{x^2} \text{ or } \pm\sqrt{4} = \sqrt{x^2}$	Solve using the even root property, simplify roots
$x = \pm 3, \pm 2$	Our Solutions

When we have higher powers of our variable, we could end up with many more solutions. The previous equation had four unique solutions.

Example 479.

$$\begin{array}{rl}
a^{-2}-a^{-1}-6=0 & \text{Quadratic form, one exponent, } -2\text{, is double the other, } -1\\
b=a^{-1} & \text{Make } a \text{ new variable equal to the variable with lowest exponent}\\
b^2=a^{-2} & \text{Square both sides}\\
b^2-b-6=0 & \text{Substitute } b^2 \text{ for } a^{-2} \text{ and } b \text{ for } a^{-1}\\
(b-3)(b+2)=0 & \text{Solve. We will solve by factoring}\\
b-3=0 \ \text{ or } \ b+2=0 & \text{Set each factor equal to zero}\\
\underline{+3+3 \qquad\quad -2-2} & \text{Solve each equation}\\
b=3 \ \text{ or } \ b=-2 & \text{Solutions for } b\text{, still need } a\text{, substitute into } b=a^{-1}\\
3=a^{-1} \ \text{ or } \ -2=a^{-1} & \text{Raise both sides to } -1 \text{ power}\\
3^{-1}=a \ \text{ or } \ (-2)^{-1}=a & \text{Simplify negative exponents}\\
a=\dfrac{1}{3},-\dfrac{1}{2} & \text{Our Solution}
\end{array}$$

Just as with regular quadratics, these problems will not always have rational solutions. We also can have irrational or complex solutions to our equations.

Example 480.

$$\begin{array}{rl}
2x^4+x^2=6 & \text{Make equation equal to zero}\\
\underline{-6-6} & \text{Subtract 6 from both sides}\\
2x^4+x^2-6=0 & \text{Quadratic form, one exponent, 4, double the other, 2}\\
y=x^2 & \text{New variable equal variable with smallest exponent}\\
y^2=x^4 & \text{Square both sides}\\
2y^2+y-6=0 & \text{Solve. We will factor this equation}\\
(2y-3)(y+2)=0 & \text{Set each factor equal to zero}\\
2y-3=0 \ \text{ or } \ y+2=0 & \text{Solve each equation}\\
\underline{+3+3} \qquad\quad \underline{-2-2} & \\
2y=3 \ \text{ or } \ y=-2 & \\
\overline{2} \quad \overline{2} \qquad\qquad\quad & \\
y=\dfrac{3}{2} \ \text{ or } \ y=-2 & \text{We have } y\text{, still need } x\text{. Substitute into } y=x^2\\
\dfrac{3}{2}=x^2 \ \text{ or } \ -2=x^2 & \text{Square root of each side}\\
\pm\sqrt{\dfrac{3}{2}}=\sqrt{x^2} \ \text{ or } \ \pm\sqrt{-2}=\sqrt{x^2} & \text{Simplify each root, rationalize denominator}\\
x=\dfrac{\pm\sqrt{6}}{2},\pm i\sqrt{2} & \text{Our Solution}
\end{array}$$

When we create a new variable for our substitution, it won't always be equal to just another variable. We can make our substitution variable equal to an expression as shown in the next example.

Example 481.

$$
\begin{array}{rl}
3(x-7)^2-2(x-7)+5=0 & \text{Quadratic form} \\
y=x-7 & \text{Define new variable} \\
y^2=(x-7)^2 & \text{Square both sides} \\
3y^2-2y+5=0 & \text{Substitute values into original} \\
(3y-5)(y+1)=0 & \text{Factor} \\
3y-5=0 \ \text{ or } \ y+1=0 & \text{Set each factor equal to zero} \\
\underline{+5+5} \qquad \underline{-1-1} & \text{Solve each equation} \\
3y=5 \ \text{ or } \ y=-1 & \\
\overline{3} \ \ \overline{3} \qquad\qquad & \\
y=\dfrac{5}{3} \ \text{ or } \ y=-1 & \text{We have } y\text{, we still need } x. \\
\dfrac{5}{3}=x-7 \ \text{ or } \ -1=x-7 & \text{Substitute into } y=x-7 \\
\underline{+\dfrac{21}{3} \quad +7} \quad \underline{+7 \quad +7} & \text{Add 7. Use common denominator as needed} \\
x=\dfrac{26}{3},6 & \text{Our Solution}
\end{array}
$$

Example 482.

$$
\begin{array}{rl}
(x^2-6x)^2=7(x^2-6x)-12 & \text{Make equation equal zero} \\
-7(x^2-6x)+12-7(x^2-6x)+12 & \text{Move all terms to left} \\
(x^2-6x)^2-7(x^2-6x)+12=0 & \text{Quadratic form} \\
y=x^2-6x & \text{Make new variable} \\
y^2=(x^2-6x)^2 & \text{Square both sides} \\
y^2-7y+12=0 & \text{Substitute into original equation} \\
(y-3)(y-4)=0 & \text{Solve by factoring} \\
y-3=0 \ \text{ or } \ y-4=0 & \text{Set each factor equal to zero} \\
\underline{+3+3} \qquad \underline{+4+4} & \text{Solve each equation} \\
y=3 \ \text{ or } \ y=4 & \text{We have } y\text{, still need } x. \\
3=x^2-6x \ \text{ or } \ 4=x^3-6x & \text{Solve each equation, complete the square} \\
\left(\dfrac{1}{2}\cdot 6\right)^2=3^2=9 & \text{Add 9 to both sides of each equation} \\
12=x^2-6x+9 \ \text{ or } \ 13=x^2-6x+9 & \text{Factor}
\end{array}
$$

$$
\begin{array}{rl}
12 = (x-3)^2 \text{ or } 13 = (x-3)^2 & \text{Use even root property} \\
\pm\sqrt{12} = \sqrt{(x-3)^2} \text{ or } \pm\sqrt{13} = \sqrt{(x-3)^2} & \text{Simplify roots} \\
\pm 2\sqrt{3} = x-3 \text{ or } \pm\sqrt{13} = x-3 & \text{Add 3 to both sides} \\
\underline{+3 \quad +3} \qquad \underline{+3 \quad +3} & \\
x = 3 \pm 2\sqrt{3}, 3 \pm \sqrt{13} & \text{Our Solution}
\end{array}
$$

The higher the exponent, the more solution we could have. This is illustrated in the following example, one with six solutions.

Example 483.

$$
\begin{array}{rl}
x^6 - 9x^3 + 8 = 0 & \text{Quadratic form, one exponent, 6, double the other, 3} \\
y = x^3 & \text{New variable equal to variable with lowest exponent} \\
y^2 = x^6 & \text{Square both sides} \\
y^2 - 9y + 8 = 0 & \text{Substitute } y^2 \text{ for } x^6 \text{ and } y \text{ for } x^3 \\
(y-1)(y-8) = 0 & \text{Solve. We will solve by factoring.} \\
y - 1 = 0 \text{ or } y - 8 = 0 & \text{Set each factor equal to zero} \\
\underline{+1+1} \qquad \underline{+8+8} & \text{Solve each equation} \\
y = 1 \text{ or } y = 8 & \text{Solutions for } y \text{, we need } x \text{. Substitute into } y = x^3 \\
x^3 = 1 \text{ or } x^3 = 8 & \text{Set each equation equal to zero} \\
\underline{-1-1} \qquad \underline{-8-8} & \\
x^3 - 1 = 0 \text{ or } x^3 - 8 = 0 & \text{Factor each equation, difference of cubes} \\
(x-1)(x^2+x+1) = 0 & \text{First equation factored. Set each factor equal to zero} \\
x - 1 = 0 \text{ or } x^2 + x + 1 = 0 & \text{First equation is easy to solve} \\
\underline{+1+1} & \\
x = 1 & \text{First solution} \\
\dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm i\sqrt{3}}{2} & \text{Quadratic formula on second factor} \\
(x-2)(x^2+2x+4) = 0 & \text{Factor the second difference of cubes} \\
x - 2 = 0 \text{ or } x^2 + 2x + 4 = 0 & \text{Set each factor equal to zero.} \\
\underline{+2+2} & \text{First equation is easy to solve} \\
x = 2 & \text{Our fourth solution} \\
\dfrac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = -1 \pm i\sqrt{3} & \text{Quadratic formula on second factor} \\
x = 1, 2, \dfrac{1 \pm i\sqrt{3}}{2}, -1 \pm i\sqrt{3} & \text{Our final six solutions}
\end{array}
$$

9.6 Practice - Quadratic in Form

Solve each of the following equations. Some equations will have complex roots.

1) $x^4 - 5x^2 + 4 = 0$

2) $y^4 - 9y^2 + 20 = 0$

3) $m^4 - 7m^2 - 8 = 0$

4) $y^4 - 29y^2 + 100 = 0$

5) $a^4 - 50a^2 + 49 = 0$

6) $b^4 - 10b^2 + 9 = 0$

7) $x^4 - 25x^2 + 144 = 0$

8) $y^4 - 40y^2 + 144 = 0$

9) $m^4 - 20m^2 + 64 = 0$

10) $x^6 - 35x^3 + 216 = 0$

11) $z^6 - 216 = 19z^3$

12) $y^4 - 2y^2 = 24$

13) $6z^4 - z^2 = 12$

14) $x^{-2} - x^{-1} - 12 = 0$

15) $x^{\frac{2}{3}} - 35 = 2x^{\frac{1}{3}}$

16) $5y^{-2} - 20 = 21y^{-1}$

17) $y^{-6} + 7y^{-3} = 8$

18) $x^4 - 7x^2 + 12 = 0$

19) $x^4 - 2x^2 - 3 = 0$

20) $x^4 + 7x^2 + 10 = 0$

21) $2x^4 - 5x^2 + 2 = 0$

22) $2x^4 - x^2 - 3 = 0$

23) $x^4 - 9x^2 + 8 = 0$

24) $x^6 - 10x^3 + 16 = 0$

25) $8x^6 - 9x^3 + 1 = 0$

26) $8x^6 + 7x^3 - 1 = 0$

27) $x^8 - 17x^4 + 16 = 0$

28) $(x-1)^2 - 4(x-1) = 5$

29) $(y+b)^2 - 4(y+b) = 21$

30) $(x+1)^2 + 6(x+1) + 9 = 0$

31) $(y+2)^2 - 6(y+2) = 16$

32) $(m-1)^2 - 5(m-1) = 14$

33) $(x-3)^2 - 2(x-3) = 35$

34) $(a+1)^2 + 2(a-1) = 15$

35) $(r-1)^2 - 8(r-1) = 20$

36) $2(x-1)^2 - (x-1) = 3$

37) $3(y+1)^2 - 14(y+1) = 5$

38) $(x^2-3)^2 - 2(x^2-3) = 3$

39) $(3x^2-2x)^2 + 5 = 6(3x^2-2x)$

40) $(x^2+x+3)^2 + 15 = 8(x^2+x+3)$

41) $2(3x+1)^{\frac{2}{3}} - 5(3x+1)^{\frac{1}{3}} = 88$

42) $(x^2+x)^2 - 8(x^2+x) + 12 = 0$

43) $(x^2+2x)^2 - 2(x^2+2x) = 3$

44) $(2x^2+3x)^2 = 8(2x^2+3x) + 9$

45) $(2x^2-x)^2 - 4(2x^2-x) + 3 = 0$

46) $(3x^2-4x)^2 = 3(3x^2-4x) + 4$

9.7

Quadratics - Rectangles

Objective: Solve applications of quadratic equations using rectangles.

An application of solving quadratic equations comes from the formula for the area of a rectangle. The area of a rectangle can be calculated by multiplying the width by the length. To solve problems with rectangles we will first draw a picture to represent the problem and use the picture to set up our equation.

Example 484.

The length of a rectangle is 3 more than the width. If the area is 40 square inches, what are the dimensions?

$\boxed{\quad 40 \quad}\, x$	We do not know the width, x.
$x+3$	Length is 4 more, or $x+4$, and area is 40.

$$
\begin{array}{rl}
x(x+3)=40 & \text{Multiply length by width to get area}\\
x^2+3x=40 & \text{Distribute}\\
\underline{-40\;-40} & \text{Make equation equal zero}\\
x^2+3x-40=0 & \text{Factor}\\
(x-5)(x+8)=0 & \text{Set each factor equal to zero}\\
x-5=0 \;\text{ or }\; x+8=0 & \text{Solve each equation}\\
\underline{+5+5} \qquad\quad \underline{-8-8} & \\
x=5 \;\text{ or }\; x=-8 & \text{Our } x \text{ is } a \text{ width, cannot be negative.}\\
(5)+3=8 & \text{Length is } x+3\text{, substitute 5 for } x \text{ to find length}\\
5\text{ in by }8\text{in} & \text{Our Solution}
\end{array}
$$

The above rectangle problem is very simple as there is only one rectangle involved. When we compare two rectangles, we may have to get a bit more creative.

Example 485.

If each side of a square is increased by 6, the area is multiplied by 16. Find the side of the original square.

$$
\begin{array}{rl}
\boxed{x^2}\,x & \text{Square has all sides the same length}\\
x & \text{Area is found by multiplying length by width}\\
\boxed{\quad 16x^2 \quad}\;x+6 & \text{Each side is increased by 6,}\\
x+6 & \text{Area is 16 times original area}\\
(x+6)(x+6)=16x^2 & \text{Multiply length by width to get area}\\
x^2+12x+36=16x^2 & \text{FOIL}\\
\underline{-16x^2 \qquad\qquad -16x^2} & \text{Make equation equal zero}\\
-15x^2+12x+36=0 & \text{Divide each term by } -1\text{, changes the signs}\\
15x^2-12x-36=0 & \text{Solve using the quadratic formula}\\
x=\dfrac{12\pm\sqrt{(-12)^2-4(15)(-36)}}{2(15)} & \text{Evaluate}\\
x=\dfrac{16\pm\sqrt{2304}}{30} & \\
x=\dfrac{16\pm48}{30} & \text{Can}'t \text{ have } a \text{ negative solution, we will only add}\\
x=\dfrac{60}{30}=2 & \text{Our } x \text{ is the original square}
\end{array}
$$

2 Our Solution

Example 486.

The length of a rectangle is 4 ft greater than the width. If each dimension is increased by 3, the new area will be 33 square feet larger. Find the dimensions of the original rectangle.

$\boxed{x(x+4)}\ x$	We don't know width, x, length is 4 more, $x+4$
$x+4$	Area is found by multiplying length by width
$\boxed{x(x+4)+33}\ x+3$	Increase each side by 3. width becomes $x+3$, length $x+4+3=x+7$
$x+7$	Area is 33 more than original, $x(x+4)+33$
$(x+3)(x+7)=x(x+4)+33$	Set up equation, length times width is area
$x^2+10x+21=x^2+4x+33$	Subtract x^2 from both sides
$\underline{-x^2 \qquad -x^2}$	
$10x+21=4x+33$	Move variables to one side
$\underline{-4x \qquad -4x}$	Subtract $4x$ from each side
$6x+21=33$	Subtract 21 from both sides
$\underline{-21-21}$	
$\frac{6x}{6}=\frac{12}{6}$	Divide both sides by 6
$x=2$	x is the width of the original
$(2)+4=6$	$x+4$ is the length. Substitute 2 to find
2 ft by 6 ft	Our Solution

From one rectangle we can find two equations. Perimeter is found by adding all the sides of a polygon together. A rectangle has two widths and two lengths, both the same size. So we can use the equation $P=2l+2w$ (twice the length plus twice the width).

Example 487.

The area of a rectangle is 168 cm^2. The perimeter of the same rectangle is 52 cm. What are the dimensions of the rectangle?

$\boxed{}\ x$	We don't know anything about length or width
y	Use two variables, x and y
$xy=168$	Length times width gives the area.
$2x+2y=52$	Also use perimeter formula.
$\underline{-2x \qquad -2x}$	Solve by substitution, isolate y
$2y=-2x+52$	Divide each term by 2

$\overline{2}\quad\overline{2}\quad\overline{2}$

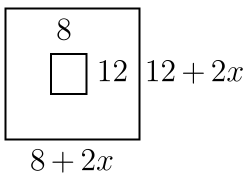
$y = -x + 26$	Substitute into area equation
$x(-x+26) = 168$	Distribute
$-x^2 + 26x = 168$	Divide each term by -1, changing all the signs
$x^2 - 26x = -168$	Solve by completing the square.
$\left(\frac{1}{2}\cdot 26\right)^2 = 13^2 = 169$	Find number to complete the square: $\left(\frac{1}{2}\cdot b\right)^2$
$x^2 - 26x + 324 = 1$	Add 169 to both sides
$(x-13)^2 = 1$	Factor
$x - 13 = \pm 1$	Square root both sides
$\underline{+13 \quad +13}$	
$x = 13 \pm 1$	Evaluate
$x = 14 \text{ or } 12$	Two options for first side.
$y = -(14) + 26 = 12$	Substitute 14 into $y = -x + 26$
$y = -(12) + 26 = 14$	Substitute 12 into $y = -x + 26$
	Both are the same rectangle, variables switched!
12 cm by 14 cm	Our Solution

World View Note: Indian mathematical records from the 9th century demonstrate that their civilization had worked extensivly in geometry creating religious alters of various shapes including rectangles.

Another type of rectangle problem is what we will call a "frame problem". The idea behind a frame problem is that a rectangle, such as a photograph, is centered inside another rectangle, such as a frame. In these cases it will be important to rememember that the frame extends on all sides of the rectangle. This is shown in the following example.

Example 488.

An 8 in by 12 in picture has a frame of uniform width around it. The area of the frame is equal to the area of the picture. What is the width of the frame?



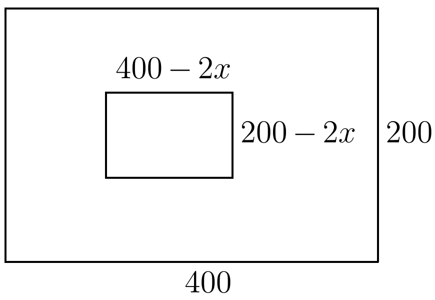
Draw picture, picture if 8 by 10
If frame has width x, on both sides, we add $2x$

$8\cdot 12 = 96$	Area of the picture, length times width
$2\cdot 96 = 192$	Frame is the same as the picture. Total area is double this.
$(12+2x)(8+2x) = 192$	Area of everything, length times width
$96 + 24x + 16x + 4x^2 = 192$	FOIL
$4x^2 + 40x + 96 = 192$	Combine like terms
$\underline{-192 \quad -192}$	Make equation equal to zero by subtracting 192
$4x^2 + 40x - 96 = 0$	Factor out GCF of 4

$$
\begin{array}{rl}
4(x^2+10x-24)=0 & \text{Factor trinomial}\\
4(x-2)(x+12)=0 & \text{Set each factor equal to zero}\\
x-2=0 \quad \text{or} \quad x+12=0 & \text{Solve each equation}\\
\underline{+2+2} \qquad \underline{-12-12} & \\
x=2 \;\; \text{or} \;\; -12 & \text{Can}'t \text{ have negative frame width.}\\
2\text{ inches} & \text{Our Solution}
\end{array}
$$

Example 489.

A farmer has a field that is 400 rods by 200 rods. He is mowing the field in a spiral pattern, starting from the outside and working in towards the center. After an hour of work, 72% of the field is left uncut. What is the size of the ring cut around the outside?



Draw picture, outside is 200 by 400
If frame has width x on both sides,
subtract $2x$ from each side to get center

$$
\begin{array}{rl}
400\cdot 200=80000 & \text{Area of entire field, length times width}\\
80000\cdot(0.72)=57600 & \text{Area of center, multiply by 28\% as decimal}\\
(400-2x)(200-2x)=57600 & \text{Area of center, length times width}\\
80000-800x-400x+4x^2=57600 & \text{FOIL}\\
4x^2-1200x+80000=57600 & \text{Combine like terms}\\
\underline{-57600-57600} & \text{Make equation equal zero}\\
4x^2-1200x+22400=0 & \text{Factor out GCF of 4}\\
4(x^2-300x+5600)=0 & \text{Factor trinomial}\\
4(x-280)(x-20)=0 & \text{Set each factor equal to zero}\\
x-280=0 \quad \text{or} \quad x-20=0 & \text{Solve each equation}\\
\underline{+280+280} \qquad \underline{+20+20} & \\
x=280 \;\; \text{or} \;\; 20 & \text{The field is only 200 rods wide,}\\
 & \text{Can}'t \text{ cut 280 off two sides!}\\
20\text{ rods} & \text{Our Solution}
\end{array}
$$

For each of the frame problems above we could have also completed the square or use the quadratic formula to solve the trinomials. Remember that completing the square or the quadratic formula always will work when solving, however, factoring only works if we can factor the trinomial.

9.7 Practice - Rectangles

1) In a landscape plan, a rectangular flowerbed is designed to be 4 meters longer than it is wide. If 60 square meters are needed for the plants in the bed, what should the dimensions of the rectangular bed be?

2) If the side of a square is increased by 5 the area is multiplied by 4. Find the side of the original square.

3) A rectangular lot is 20 yards longer than it is wide and its area is 2400 square yards. Find the dimensions of the lot.

4) The length of a room is 8 ft greater than it is width. If each dimension is increased by 2 ft, the area will be increased by 60 sq. ft. Find the dimensions of the rooms.

5) The length of a rectangular lot is 4 rods greater than its width, and its area is 60 square rods. Find the dimensions of the lot.

6) The length of a rectangle is 15 ft greater than its width. If each dimension is decreased by 2 ft, the area will be decreased by 106 ft^2. Find the dimensions.

7) A rectangular piece of paper is twice as long as a square piece and 3 inches wider. The area of the rectangular piece is 108 in^2. Find the dimensions of the square piece.

8) A room is one yard longer than it is wide. At 75¢ per sq. yd. a covering for the floor costs $31.50. Find the dimensions of the floor.

9) The area of a rectangle is 48 ft^2 and its perimeter is 32 ft. Find its length and width.

10) The dimensions of a picture inside a frame of uniform width are 12 by 16 inches. If the whole area (picture and frame) is 288 in^2, what is the width of the frame?

11) A mirror 14 inches by 15 inches has a frame of uniform width. If the area of the frame equals that of the mirror, what is the width of the frame.

12) A lawn is 60 ft by 80 ft. How wide a strip must be cut around it when mowing the grass to have cut half of it.

13) A grass plot 9 yards long and 6 yards wide has a path of uniform width around it. If the area of the path is equal to the area of the plot, determine the width of the path.

14) A landscape architect is designing a rectangular flowerbed to be border with 28 plants that are placed 1 meter apart. He needs an inner rectangular space in the center for plants that must be 1 meter from the border of the bed and

that require 24 square meters for planting. What should the overall dimensions of the flowerbed be?

15) A page is to have a margin of 1 inch, and is to contain 35 in^2 of painting. How large must the page be if the length is to exceed the width by 2 inches?

16) A picture 10 inches long by 8 inches wide has a frame whose area is one half the area of the picture. What are the outside dimensions of the frame?

17) A rectangular wheat field is 80 rods long by 60 rods wide. A strip of uniform width is cut around the field, so that half the grain is left standing in the form of a rectangular plot. How wide is the strip that is cut?

18) A picture 8 inches by 12 inches is placed in a frame of uniform width. If the area of the frame equals the area of the picture find the width of the frame.

19) A rectangular field 225 ft by 120 ft has a ring of uniform width cut around the outside edge. The ring leaves 65% of the field uncut in the center. What is the width of the ring?

20) One Saturday morning George goes out to cut his lot that is 100 ft by 120 ft. He starts cutting around the outside boundary spiraling around towards the center. By noon he has cut 60% of the lawn. What is the width of the ring that he has cut?

21) A frame is 15 in by 25 in and is of uniform width. The inside of the frame leaves 75% of the total area available for the picture. What is the width of the frame?

22) A farmer has a field 180 ft by 240 ft. He wants to increase the area of the field by 50% by cultivating a band of uniform width around the outside. How wide a band should he cultivate?

23) The farmer in the previous problem has a neighber who has a field 325 ft by 420 ft. His neighbor wants to increase the size of his field by 20% by cultivating a band of uniform width around the outside of his lot. How wide a band should his neighbor cultivate?

24) A third farmer has a field that is 500 ft by 550 ft. He wants to increase his field by 20%. How wide a ring should he cultivate around the outside of his field?

25) Donna has a garden that is 30 ft by 36 ft. She wants to increase the size of the garden by 40%. How wide a ring around the outside should she cultivate?

26) A picture is 12 in by 25 in and is surrounded by a frame of uniform width. The area of the frame is 30% of the area of the picture. How wide is the frame?

9.8

Quadratics - Teamwork

Objective: Solve teamwork problems by creating a rational equation to model the problem.

If it takes one person 4 hours to paint a room and another person 12 hours to paint the same room, working together they could paint the room even quicker, it turns out they would paint the room in 3 hours together. This can be reasoned by the following logic, if the first person paints the room in 4 hours, she paints $\frac{1}{4}$ of the room each hour. If the second person takes 12 hours to paint the room, he paints $\frac{1}{12}$ of the room each hour. So together, each hour they paint $\frac{1}{4}+\frac{1}{12}$ of the room. Using a common denominator of 12 gives: $\frac{3}{12}+\frac{1}{12}=\frac{4}{12}=\frac{1}{3}$. This means each hour, working together they complete $\frac{1}{3}$ of the room. If $\frac{1}{3}$ is completed each hour, it follows that it will take 3 hours to complete the entire room.

This pattern is used to solve teamwork problems. If the first person does a job in A, a second person does a job in B, and together they can do a job in T (total). We can use the team work equation.

$$\textbf{Teamwork Equation: } \boldsymbol{\frac{1}{A}+\frac{1}{B}=\frac{1}{T}}$$

Often these problems will involve fractions. Rather than thinking of the first fraction as $\frac{1}{A}$, it may be better to think of it as the reciprocal of A's time.

World View Note: When the Egyptians, who were the first to work with fractions, wrote fractions, they were all unit fractions (numerator of one). They only used these type of fractions for about 2000 years! Some believe that this cumbersome style of using fractions was used for so long out of tradition, others believe the Egyptians had a way of thinking about and working with fractions that has been completely lost in history.

Example 490.

Adam can clean a room in 3 hours. If his sister Maria helps, they can clean it in $2\frac{2}{5}$ hours. How long will it take Maria to do the job alone?

$$2\frac{2}{5}=\frac{12}{5} \qquad \text{Together time, } 2\frac{2}{5}\text{, needs to be converted to fraction}$$

$$\text{Adan: } 3, \text{Maria: } x, \text{Total: } \frac{5}{12} \qquad \text{Clearly state times for each and total, using } x \text{ for Maria}$$

$$\frac{1}{3}+\frac{1}{x}=\frac{5}{12} \quad \text{Using reciprocals, add the individual times gives total}$$

$$\frac{1(12x)}{3}+\frac{1(12x)}{x}=\frac{5(12x)}{12} \quad \text{Multiply each term by LCD of } 12x$$

$$\begin{array}{rl} 4x+12=5x & \text{Reduce each fraction} \\ \underline{-4x \qquad -4x} & \text{Move variables to one side, subtracting } 4x \\ 12=x & \text{Our solution for } x \\ \text{It takes Maria 12 hours} & \text{Our Solution} \end{array}$$

Somtimes we only know how two people's times are related to eachother as in the next example.

Example 491.

Mike takes twice as long as Rachel to complete a project. Together they can complete the project in 10 hours. How long will it take each of them to complete the project alone?

$$\begin{array}{rl} \text{Mike: } 2x, \text{Rachel: } x, \text{Total: } 10 & \text{Clearly define variables. If Rachel is } x, \text{Mike is } 2x \\ \dfrac{1}{2x}+\dfrac{1}{x}=\dfrac{1}{10} & \text{Using reciprocals, add individal times equaling total} \\ \dfrac{1(10x)}{2x}+\dfrac{1(10x)}{x}=\dfrac{1(10x)}{10} & \text{Multiply each term by LCD, } 10x \\ 5+10=x & \text{Combine like terms} \\ 15=x & \text{We have our } x, \text{we said } x \text{ was Rachel}'s \text{ time} \\ 2(15)=30 & \text{Mike is double Rachel, this gives Mike}'s \text{ time.} \\ \text{Mike: 30 hr, Rachel: 15 hr} & \text{Our Solution} \end{array}$$

With problems such as these we will often end up with a quadratic to solve.

Example 492.

Brittney can build a large shed in 10 days less than Cosmo can. If they built it together it would take them 12 days. How long would it take each of them working alone?

$$\begin{array}{rl} \text{Britney: } x-10, \text{Cosmo: } x, \text{Total: } 12 & \text{If Cosmo is } x, \text{Britney is } x-10 \\ \dfrac{1}{x-10}+\dfrac{1}{x}=\dfrac{1}{12} & \text{Using reciprocals, make equation} \end{array}$$

$$\frac{1(12x(x-10))}{x-10}+\frac{1(12x(x-10))}{x}=\frac{1(12x(x-10))}{12} \quad \text{Multiply by LCD: } 12x(x-10)$$

$$12x+12(x-10)=x(x-10) \quad \text{Reduce fraction}$$
$$12x+12x-120=x^2-10x \quad \text{Distribute}$$
$$24x-120=x^2-10x \quad \text{Combine like terms}$$
$$\underline{-24x+120 \quad -24x+120} \quad \text{Move all terms to one side}$$
$$0=x^2-34x+120 \quad \text{Factor}$$
$$0=(x-30)(x-4) \quad \text{Set each factor equal to zero}$$
$$x-30=0 \text{ or } x-4=0 \quad \text{Solve each equation}$$
$$\underline{+30+30} \qquad \underline{+4+4}$$
$$x=30 \text{ or } x=4 \quad \text{This, } x\text{, was defined as Cosmo.}$$
$$30-10=20 \text{ or } 4-10=-6 \quad \text{Find Britney, can't have negative time}$$
$$\text{Britney: 20 days, Cosmo: 30 days} \quad \text{Our Solution}$$

In the previous example, when solving, one of the possible times ended up negative. We can't have a negative amount of time to build a shed, so this possibility is ignored for this problem. Also, as we were solving, we had to factor $x^2-34x+120$. This may have been difficult to factor. We could have also chosen to complete the square or use the quadratic formula to find our solutions.

It is important that units match as we solve problems. This means we may have to convert minutes into hours to match the other units given in the problem.

Example 493.

An electrician can complete a job in one hour less than his apprentice. Together they do the job in 1 hour and 12 minutes. How long would it take each of them working alone?

$$1\text{ hr }12\text{ min}=1\frac{12}{60}\text{ hr} \quad \text{Change 1 hour 12 minutes to mixed number}$$

$$1\frac{12}{60}=1\frac{1}{5}=\frac{6}{5} \quad \text{Reduce and convert to fraction}$$

$$\text{Electrician: } x-1\text{, Apprentice: } x\text{, Total: }\frac{6}{5} \quad \text{Clearly define variables}$$

$$\frac{1}{x-1}+\frac{1}{x}=\frac{5}{6} \quad \text{Using reciprocals, make equation}$$

$$\frac{1(6x(x-1))}{x-1}+\frac{1(6x(x-1))}{x}=\frac{5(6x(x-1)}{6} \quad \text{Multiply each term by LCD } 6x(x-1)$$

$$
\begin{array}{rl}
6x+6(x-1)=5x(x-1) & \text{Reduce each fraction}\\
6x+6x-6=5x^2-5x & \text{Distribute}\\
12x-6=5x^2-5x & \text{Combine like terms}\\
\underline{-12x+6 \quad -12x+6} & \text{Move all terms to one side of equation}\\
0=5x^2-17x+6 & \text{Factor}\\
0=(5x-2)(x-3) & \text{Set each factor equal to zero}\\
5x-2=0 \ \text{ or } \ x-3=0 & \text{Solve each equation}\\
\underline{+2+2} \qquad \underline{+3+3} & \\
5x=2 \ \text{ or } \ x=3 & \\
\overline{5} \quad \overline{5} \qquad\qquad & \\
x=\dfrac{2}{5} \ \text{ or } \ x=3 & \text{Subtract 1 from each to find electrician}\\
\dfrac{2}{5}-1=\dfrac{-3}{5} \ \text{ or } \ 3-1=2 & \text{Ignore negative.}\\
\text{Electrician: 2 hr, Apprentice: 3 hours} & \text{Our Solution}
\end{array}
$$

Very similar to a teamwork problem is when the two involved parts are working against each other. A common example of this is a sink that is filled by a pipe and emptied by a drain. If they are working against eachother we need to make one of the values negative to show they oppose eachother. This is shown in the next example..

Example 494.

A sink can be filled by a pipe in 5 minutes but it takes 7 minutes to drain a full sink. If both the pipe and the drain are open, how long will it take to fill the sink?

$$
\begin{array}{rl}
\text{Sink: 5, Drain: 7, Total: } x & \text{Define variables, drain is negative}\\
\dfrac{1}{5}-\dfrac{1}{7}=\dfrac{1}{x} & \text{Using reciprocals to make equation,}\\
 & \text{Subtract because they are opposite}\\
\dfrac{1(35x)}{5}-\dfrac{1(35x)}{7}=\dfrac{1(35x)}{x} & \text{Multiply each term by LCD: } 35x\\
7x-5x=35 & \text{Reduce fractions}\\
2x=35 & \text{Combine like terms}\\
\overline{2} \quad \overline{2} & \text{Divide each term by 2}\\
x=17.5 & \text{Our answer for } x\\
\text{17.5 min or 17 min 30 sec} & \text{Our Solution}
\end{array}
$$

9.8 Practice - Teamwork

1) Bills father can paint a room in two hours less than Bill can paint it. Working together they can complete the job in two hours and 24 minutes. How much time would each require working alone?

2) Of two inlet pipes, the smaller pipe takes four hours longer than the larger pipe to fill a pool. When both pipes are open, the pool is filled in three hours and forty-five minutes. If only the larger pipe is open, how many hours are required to fill the pool?

3) Jack can wash and wax the family car in one hour less than Bob can. The two working together can complete the job in $1\ \frac{1}{5}$ hours. How much time would each require if they worked alone?

4) If A can do a piece of work alone in 6 days and B can do it alone in 4 days, how long will it take the two working together to complete the job?

5) Working alone it takes John 8 hours longer than Carlos to do a job. Working together they can do the job in 3 hours. How long will it take each to do the job working alone?

6) A can do a piece of work in 3 days, B in 4 days, and C in 5 days each working alone. How long will it take them to do it working together?

7) A can do a piece of work in 4 days and B can do it in half the time. How long will it take them to do the work together?

8) A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take both pipes together to fill the tank?

9) If A can do a piece of work in 24 days and A and B together can do it in 6 days, how long would it take B to do the work alone?

10) A carpenter and his assistant can do a piece of work in $3\frac{3}{4}$ days. If the carpenter himself could do the work alone in 5 days, how long would the assistant take to do the work alone?

11) If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

12) Tim can finish a certain job in 10 hours. It take his wife JoAnn only 8 hours to do the same job. If they work together, how long will it take them to complete the job?

13) Two people working together can complete a job in 6 hours. If one of them works twice as fast as the other, how long would it take the faster person, working alone, to do the job?

14) If two people working together can do a job in 3 hours, how long will it take the slower person to do the same job if one of them is 3 times as fast as the other?

15) A water tank can be filled by an inlet pipe in 8 hours. It takes twice that long for the outlet pipe to empty the tank. How long will it take to fill the tank if both pipes are open?

16) A sink can be filled from the faucet in 5 minutes. It takes only 3 minutes to empty the sink when the drain is open. If the sink is full and both the faucet and the drain are open, how long will it take to empty the sink?

17) It takes 10 hours to fill a pool with the inlet pipe. It can be emptied in 15 hrs with the outlet pipe. If the pool is half full to begin with, how long will it take to fill it from there if both pipes are open?

18) A sink is $\frac{1}{4}$ full when both the faucet and the drain are opened. The faucet alone can fill the sink in 6 minutes, while it takes 8 minutes to empty it with the drain. How long will it take to fill the remaining $\frac{3}{4}$ of the sink?

19) A sink has two faucets, one for hot water and one for cold water. The sink can be filled by a cold-water faucet in 3.5 minutes. If both faucets are open, the sink is filled in 2.1 minutes. How long does it take to fill the sink with just the hot-water faucet open?

20) A water tank is being filled by two inlet pipes. Pipe A can fill the tank in $4\frac{1}{2}$ hrs, while both pipes together can fill the tank in 2 hours. How long does it take to fill the tank using only pipe B?

21) A tank can be emptied by any one of three caps. The first can empty the tank in 20 minutes while the second takes 32 minutes. If all three working together could empty the tank in $8\frac{8}{59}$ minutes, how long would the third take to empty the tank?

22) One pipe can fill a cistern in $1\frac{1}{2}$ hours while a second pipe can fill it in $2\frac{1}{3}$ hrs. Three pipes working together fill the cistern in 42 minutes. How long would it take the third pipe alone to fill the tank?

23) Sam takes 6 hours longer than Susan to wax a floor. Working together they can wax the floor in 4 hours. How long will it take each of them working alone to wax the floor?

24) It takes Robert 9 hours longer than Paul to rapair a transmission. If it takes them $2\frac{2}{5}$ hours to do the job if they work together, how long will it take each of them working alone?

25) It takes Sally $10\frac{1}{2}$ minutes longer than Patricia to clean up their dorm room. If they work together they can clean it in 5 minutes. How long will it take each of them if they work alone?

26) A takes $7\frac{1}{2}$ minutes longer than B to do a job. Working together they can do the job in 9 minutes. How long does it take each working alone?

27) Secretary A takes 6 minutes longer than Secretary B to type 10 pages of manuscript. If they divide the job and work together it will take them $8\frac{3}{4}$ minutes to type 10 pages. How long will it take each working alone to type the 10 pages?

28) It takes John 24 minutes longer than Sally to mow the lawn. If they work together they can mow the lawn in 9 minutes. How long will it take each to mow the lawn if they work alone?

9.9

Quadratics - Simultaneous Products

Objective: Solve simultaneous product equations using substitution to create a rational equation.

When solving a system of equations where the variables are multiplied together we can use the same idea of substitution that we used with linear equations. When we do so we may end up with a quadratic equation to solve. When we used substitution we solved for a variable and substitute this expression into the other equation. If we have two products we will choose a variable to solve for first and divide both sides of the equations by that variable or the factor containing the variable. This will create a situation where substitution can easily be done.

Example 495.

$$xy=48 \quad (x+3)(y-2)=54 \qquad \text{To solve for } x\text{, divide first equation by } x\text{, second by } x+3$$

$$y=\frac{48}{x} \text{ and } y-2=\frac{54}{x+3} \qquad \text{Substitute } \frac{48}{x} \text{ for } y \text{ in the second equation}$$

$$\frac{48}{x}-2=\frac{54}{x+3} \qquad \text{Multiply each term by LCD: } x(x+3)$$

$$\frac{48x(x+3)}{x}-2x(x+3)=\frac{54x(x+3)}{x+3} \qquad \text{Reduce each fraction}$$

$$48(x+3)-2x(x+3)=54x \qquad \text{Distribute}$$

$$48x+144-2x^2-6x=54x \qquad \text{Combine like terms}$$

$$-2x^2+42x+144=54x \qquad \text{Make equation equal zero}$$

$$\underline{-54x \qquad -54x} \qquad \text{Subtract } 54x \text{ from both sides}$$

$$-2x^2-12x+144=0 \qquad \text{Divide each term by GCF of } -2$$

$$x^2+6x-72=0 \qquad \text{Factor}$$

$$(x-6)(x+12)=0 \qquad \text{Set each factor equal to zero}$$

$$x-6=0 \text{ or } x+12=0 \qquad \text{Solve each equation}$$

$$\underline{+6+6} \qquad \underline{-12-12}$$

$$x=6 \text{ or } x=-12 \qquad \text{Substitute each solution into } xy=48$$

$$\frac{6y}{6}=\frac{48}{6} \text{ or } \frac{-12y}{-12}=\frac{48}{-12} \qquad \text{Solve each equation}$$

$$y=8 \text{ or } y=-4 \qquad \text{Our solutions for } y,$$

$$(6,8) \text{ or } (-12,-4) \qquad \text{Our Solutions as ordered pairs}$$

These simultaneous product equations will also solve by the exact same pattern. We pick a variable to solve for, divide each side by that variable, or factor containing the variable. This will allow us to use substitution to create a rational expression we can use to solve. Quite often these problems will have two solutions.

Example 496.

$$\begin{array}{rl}
xy = -35 & \text{To solve for } x\text{, divide the first} \\
(x+6)(y-2) = 5 & \text{equation by } x\text{, second by } x+6 \\
y = \dfrac{-35}{x} \text{ and } y - 2 = \dfrac{5}{x+6} & \text{Substitute } \dfrac{-35}{x} \text{ for } y \text{ in the second equation} \\
\dfrac{-35}{x} - 2 = \dfrac{5}{x+6} & \text{Multiply each term by LCD: } x(x+6) \\
\dfrac{-35x(x+6)}{x} - 2x(x+6) = \dfrac{5x(x+6)}{x+6} & \text{Reduce fractions} \\
-35(x+6) - 2x(x+6) = 5x & \text{Distribute} \\
-35x - 210 - 2x^2 - 12x = 5x & \text{Combine like terms} \\
-2x^2 - 47x - 210 = 5x & \text{Make equation equal zero} \\
\underline{-5x \qquad\quad -5x} & \\
-2x^2 - 52x - 210 = 0 & \text{Divide each term by } -2 \\
x^2 + 26x + 105 = 0 & \text{Factor} \\
(x+5)(x+21) = 0 & \text{Set each factor equal to zero} \\
x + 5 = 0 \text{ or } x + 21 = 0 & \text{Solve each equation} \\
\underline{-5 - 5} \qquad \underline{-21 - 21} & \\
x = -5 \text{ or } x = -21 & \text{Substitute each solution into } xy = -35 \\
\dfrac{-5y}{-5} = \dfrac{-35}{-5} \text{ or } \dfrac{-21y}{-21} = \dfrac{-35}{-21} & \text{Solve each equation} \\
y = 7 \text{ or } y = \dfrac{5}{3} & \text{Our solutions for } y \\
(-5, 7) \text{ or } \left(-21, \dfrac{5}{3}\right) & \text{Our Solutions as ordered pairs}
\end{array}$$

The processes used here will be used as we solve applications of quadratics including distance problems and revenue problems. These will be covered in another section.

World View Note: William Horner, a British mathematician from the late 18th century/early 19th century is credited with a method for solving simultaneous equations, however, Chinese mathematician Chu Shih-chieh in 1303 solved these equations with exponents as high as 14!

9.9 Practice - Simultaneous Product

Solve.

1) $\begin{aligned} xy &= 72 \\ (x+2)(y-4) &= 128 \end{aligned}$

2) $\begin{aligned} xy &= 180 \\ (x-1)(y-\tfrac{1}{2}) &= 205 \end{aligned}$

3) $\begin{aligned} xy &= 150 \\ (x-6)(y+1) &= 64 \end{aligned}$

4) $\begin{aligned} xy &= 120 \\ (x+2)(y-3) &= 120 \end{aligned}$

5) $\begin{aligned} xy &= 45 \\ (x+2)(y+1) &= 70 \end{aligned}$

6) $\begin{aligned} xy &= 65 \\ (x-8)(y+2) &= 35 \end{aligned}$

7) $\begin{aligned} xy &= 90 \\ (x-5)(y+1) &= 120 \end{aligned}$

8) $\begin{aligned} xy &= 48 \\ (x-6)(y+3) &= 60 \end{aligned}$

9) $\begin{aligned} xy &= 12 \\ (x+1)(y-4) &= 16 \end{aligned}$

10) $\begin{aligned} xy &= 60 \\ (x+5)(y+3) &= 150 \end{aligned}$

11) $\begin{aligned} xy &= 45 \\ (x-5)(y+3) &= 160 \end{aligned}$

12) $\begin{aligned} xy &= 80 \\ (x-5)(y+5) &= 45 \end{aligned}$

9.10

Quadratics - Revenue and Distance

Objective: Solve revenue and distance applications of quadratic equations.

A common application of quadratics comes from revenue and distance problems. Both are set up almost identical to each other so they are both included together. Once they are set up, we will solve them in exactly the same way we solved the simultaneous product equations.

Revenue problems are problems where a person buys a certain number of items for a certain price per item. If we multiply the number of items by the price per item we will get the total paid. To help us organize our information we will use the following table for revenue problems

	Number	Price	Total
First			
Second			

The price column will be used for the individual prices, the total column is used for the total paid, which is calculated by multiplying the number by the price. Once we have the table filled out we will have our equations which we can solve. This is shown in the following examples.

Example 497.

A man buys several fish for \$56. After three fish die, he decides to sell the rest at a profit of \$5 per fish. His total profit was \$4. How many fish did he buy to begin with?

	Number	Price	Total
Buy	n	p	56
Sell			

Using our table, we don$'t$ know the number he bought, or at what price, so we use varibles n and p. Total price was \$56.

	Number	Price	Total
Buy	n	p	56
Sell	$n-3$	$p+5$	60

When he sold, he sold 3 less $(n-3)$, for \$5 more $(p+5)$. Total profit was \$4, combined with \$56 spent is \$60

$$np = 56$$ Find equatinos by multiplying number by price

$$(n-3)(p+5) = 60$$ These are a simultaneous product

$$p = \frac{56}{n} \quad \text{and} \quad p+5 = \frac{60}{n-3}$$ Solving for number, divide by n or $(n-3)$

$$\frac{56}{n} + 5 = \frac{60}{n-3} \quad \text{Substitute } \frac{56}{n} \text{ for } p \text{ in second equation}$$

$$\frac{56n(n-3)}{n} + 5n(n-3) = \frac{60n(n-3)}{n-3} \quad \text{Multiply each term by LCD: } n(n-3)$$

$$56(n-3) + 5n(n-3) = 60n \quad \text{Reduce fractions}$$

$$56n - 168 + 5n^2 - 15n = 60n \quad \text{Combine like terms}$$

$$5n^2 + 41n - 168 = 60n \quad \text{Move all terms to one side}$$

$$\underline{-60n \qquad -60n}$$

$$5n^2 - 19n - 168 = 0 \quad \text{Solve with quadratic formula}$$

$$n = \frac{19 \pm \sqrt{(-19)^2 - 4(5)(-168)}}{2(5)} \quad \text{Simplify}$$

$$n = \frac{19 \pm \sqrt{3721}}{10} = \frac{19 \pm 61}{10} \quad \text{We don't want negative solutions, only do } +$$

$$n = \frac{80}{10} = 8 \quad \text{This is our } n$$

8 fish — Our Solution

Example 498.

A group of students together bought a couch for their dorm that cost \$96. However, 2 students failed to pay their share, so each student had to pay \$4 more. How many students were in the original group?

	Number	Price	Total
Deal	n	p	96
Paid			

\$96 was paid, but we don't know the number or the price agreed upon by each student.

	Number	Price	Total
Deal	n	p	96
Paid	$n-2$	$p+4$	96

There were 2 less that actually paid $(n-2)$ and they had to pay \$4 more $(p+4)$. The total here is still \$96.

$$np = 96 \quad \text{Equations are product of number and price}$$

$$(n-2)(p+4) = 96 \quad \text{This is } a \text{ simultaneous product}$$

$$p = \frac{96}{n} \text{ and } p + 4 = \frac{96}{n-2} \quad \text{Solving for number, divide by } n \text{ and } n-2$$

$$\frac{96}{n} + 4 = \frac{96}{n-2} \quad \text{Substitute } \frac{96}{n} \text{ for } p \text{ in the second equation}$$

$$\frac{96n(n-2)}{n} + 4n(n-2) = \frac{96n(n-2)}{n-2} \quad \text{Multiply each term by LCD: } n(n-2)$$

$$\begin{array}{rl}
96(n-2)+4n(n-2)=96n & \text{Reduce fractions}\\
96n-192+4n^2-8n=96n & \text{Distribute}\\
4n^2+88n-192=96n & \text{Combine like terms}\\
\underline{-96n \qquad -96n} & \text{Set equation equal to zero}\\
4n^2-8n-192=0 & \text{Solve by completing the square,}\\
\underline{+192+192} & \text{Separate variables and constant}\\
4n^2-8n=192 & \text{Divide each term by } a \text{ or } 4\\
\overline{\ 4\ } \quad \overline{\ 4\ } \quad \overline{\ 4\ } & \\
n^2-2n=48 & \text{Complete the square: } \left(b\cdot\frac{1}{2}\right)^2\\
\left(2\cdot\frac{1}{2}\right)^2=1^2=1 & \text{Add to both sides of equation}\\
n^2-2n+1=49 & \text{Factor}\\
(n-1)^2=49 & \text{Square root of both sides}\\
n-1=\pm 7 & \text{Add 1 to both sides}\\
n=1\pm 7 & \text{We don't want a negative solution}\\
n=1+7=8 & \\
8 \text{ students} & \text{Our Solution}
\end{array}$$

The above examples were solved by the quadratic formula and completing the square. For either of these we could have used either method or even factoring. Remember we have several options for solving quadratics. Use the one that seems easiest for the problem.

Distance problems work with the same ideas that the revenue problems work. The only difference is the variables are r and t (for rate and time), instead of n and p (for number and price). We already know that distance is calculated by multiplying rate by time. So for our distance problems our table becomes the following:

	rate	time	distance
First			
Second			

Using this table and the exact same patterns as the revenue problems is shown in the following example.

Example 499.

Greg went to a conference in a city 120 miles away. On the way back, due to road construction he had to drive 10 mph slower which resulted in the return trip taking 2 hours longer. How fast did he drive on the way to the conference?

	rate	time	distance
There	r	t	120
Back			

We do not know rate, r, or time, t he traveled on the way to the conference. But we do know the distance was 120 miles.

	rate	time	distance
There	r	t	120
Back	$r-10$	$t+2$	120

Coming back he drove 10 mph slower $(r-10)$ and took 2 hours longer $(t+2)$. The distance was still 120 miles.

$$rt = 120 \quad \text{Equations are product of rate and time}$$
$$(r-10)(t+2) = 120 \quad \text{We have simultaneous product equations}$$

$$t = \frac{120}{r} \text{ and } t+2 = \frac{120}{r-10} \quad \text{Solving for rate, divide by } r \text{ and } r-10$$

$$\frac{120}{r} + 2 = \frac{120}{r-10} \quad \text{Substitute } \frac{120}{r} \text{ for } t \text{ in the second equation}$$

$$\frac{120r(r-10)}{r} + 2r(r-10) = \frac{120r(r-10)}{r-10} \quad \text{Multiply each term by LCD: } r(r-10)$$

$$120(r-10) + 2r^2 - 20r = 120r \quad \text{Reduce each fraction}$$
$$120r - 1200 + 2r^2 - 20r = 120r \quad \text{Distribute}$$
$$2r^2 + 100r - 1200 = 120r \quad \text{Combine like terms}$$
$$\underline{-120r \qquad\qquad -120r} \quad \text{Make equation equal to zero}$$
$$2r^2 - 20r - 1200 = 0 \quad \text{Divide each term by 2}$$
$$r^2 - 10r - 600 = 0 \quad \text{Factor}$$
$$(r-30)(r+20) = 0 \quad \text{Set each factor equal to zero}$$
$$r - 30 = 0 \text{ and } r + 20 = 0 \quad \text{Solve each equation}$$
$$\underline{+30 + 30} \qquad \underline{-20 - 20}$$
$$r = 30 \text{ and } r = -20 \quad \text{Can't have a negative rate}$$
$$30 \text{ mph} \quad \text{Our Solution}$$

World View Note: The world's fastest man (at the time of printing) is Jamaican Usain Bolt who set the record of running 100 m in 9.58 seconds on August 16, 2009 in Berlin. That is a speed of over 23 miles per hour!

Another type of simultaneous product distance problem is where a boat is traveling in a river with the current or against the current (or an airplane flying with the wind or against the wind). If a boat is traveling downstream, the current will push it or increase the rate by the speed of the current. If a boat is traveling upstream, the current will pull against it or decrease the rate by the speed of the current. This is demonstrated in the following example.

Example 500.

A man rows down stream for 30 miles then turns around and returns to his original location, the total trip took 8 hours. If the current flows at 2 miles per hour, how fast would the man row in still water?

8

	rate	time	distance
down		t	30
up		$8-t$	30

Write total time above time column
We know the distance up and down is 30.
Put t for time downstream. Subtracting $8-t$ becomes time upstream

	rate	time	distance
down	$r+2$	t	30
up	$r-2$	$8-t$	30

Downstream the current of 2 mph pushes the boat $(r+2)$ and upstream the current pulls the boat $(r-2)$

$$(r+2)t=30$$
$$(r-2)(8-t)=30$$

Multiply rate by time to get equations
We have a simultaneous product

$$t=\frac{30}{r+2} \text{ and } 8-t=\frac{30}{r-2}$$

Solving for rate, divide by $r+2$ or $r-2$

$$8-\frac{30}{r+2}=\frac{30}{r-2}$$

Substitute $\frac{30}{r+2}$ for t in second equation

$$8(r+2)(r-2)-\frac{30(r+2)(r-2)}{r+2}=\frac{30(r+2)(r-2)}{r-2}$$

Multiply each term by LCD: $(r+2)(r-2)$

$$8(r+2)(r-2)-30(r-2)=30(r+2)$$

Reduce fractions

$$8r^2-32-30r+60=30r+60$$

Multiply and distribute

$$8r^2-30r+28=30r+60$$
$$\underline{-30r-60-30r-60}$$

Make equation equal zero

$$8r^2-60r-32=0$$

Divide each term by 4

$$2r^2-15r-8=0$$

Factor

$$(2r+1)(r-8)=0$$

Set each factor equal to zero

$$2r+1=0 \text{ or } r-8=0$$
$$\underline{-1-1} \qquad \underline{+8+8}$$
$$\frac{2r}{2}=\frac{-1}{2} \text{ or } r=8$$

Solve each equation

$$r=-\frac{1}{2} \text{ or } r=8$$

Can't have a negative rate

8 mph

Our Solution

9.10 Practice - Revenue and Distance

1) A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money, he would have paid \$15 less for each piece. Find the number of pieces purchased.

2) A number of men subscribed a certain amount to make up a deficit of \$100 but 5 men failed to pay and thus increased the share of the others by \$1 each. Find the amount that each man paid.

3) A merchant bought a number of barrels of apples for \$120. He kept two barrels and sold the remainder at a profit of \$2 per barrel making a total profit of \$34. How many barrels did he originally buy?

4) A dealer bought a number of sheep for \$440. After 5 had died he sold the remainder at a profit of \$2 each making a profit of \$60 for the sheep. How many sheep did he originally purchase?

5) A man bought a number of articles at equal cost for \$500. He sold all but two for \$540 at a profit of \$5 for each item. How many articles did he buy?

6) A clothier bought a lot of suits for \$750. He sold all but 3 of them for \$864 making a profit of \$7 on each suit sold. How many suits did he buy?

7) A group of boys bought a boat for \$450. Five boys failed to pay their share, hence each remaining boys were compelled to pay \$4.50 more. How many boys were in the original group and how much had each agreed to pay?

8) The total expenses of a camping party were \$72. If there had been 3 fewer persons in the party, it would have cost each person \$2 more than it did. How many people were in the party and how much did it cost each one?

9) A factory tests the road performance of new model cars by driving them at two different rates of speed for at least 100 kilometers at each rate. The speed rates range from 50 to 70 km/hr in the lower range and from 70 to 90 km/hr in the higher range. A driver plans to test a car on an available speedway by driving it for 120 kilometers at a speed in the lower range and then driving 120 kilometers at a rate that is 20 km/hr faster. At what rates should he drive if he plans to complete the test in $3\frac{1}{2}$ hours?

10) A train traveled 240 kilometers at a certain speed. When the engine was replaced by an improved model, the speed was increased by 20 km/hr and the travel time for the trip was decreased by 1 hour. What was the rate of each engine?

11) The rate of the current in a stream is 3 km/hr. A man rowed upstream for 3 kilometers and then returned. The round trip required 1 hour and 20 minutes. How fast was he rowing?

12) A pilot flying at a constant rate against a headwind of 50 km/hr flew for 750 kilometers, then reversed direction and returned to his starting point. He completed the round trip in 8 hours. What was the speed of the plane?

13) Two drivers are testing the same model car at speeds that differ by 20 km/hr. The one driving at the slower rate drives 70 kilometers down a speedway and returns by the same route. The one driving at the faster rate drives 76 kilometers down the speedway and returns by the same route. Both drivers leave at the same time, and the faster car returns $\frac{1}{2}$ hour earlier than the slower car. At what rates were the cars driven?

14) An athlete plans to row upstream a distance of 2 kilometers and then return to his starting point in a total time of 2 hours and 20 minutes. If the rate of the current is 2 km/hr, how fast should he row?

15) An automobile goes to a place 72 miles away and then returns, the round trip occupying 9 hours. His speed in returning is 12 miles per hour faster than his speed in going. Find the rate of speed in both going and returning.

16) An automobile made a trip of 120 miles and then returned, the round trip occupying 7 hours. Returning, the rate was increased 10 miles an hour. Find the rate of each.

17) The rate of a stream is 3 miles an hour. If a crew rows downstream for a distance of 8 miles and then back again, the round trip occupying 5 hours, what is the rate of the crew in still water?

18) The railroad distance between two towns is 240 miles. If the speed of a train were increased 4 miles an hour, the trip would take 40 minutes less. What is the usual rate of the train?

19) By going 15 miles per hour faster, a train would have required 1 hour less to travel 180 miles. How fast did it travel?

20) Mr. Jones visits his grandmother who lives 100 miles away on a regular basis. Recently a new freeway has opend up and, although the freeway route is 120 miles, he can drive 20 mph faster on average and takes 30 minutes less time to make the trip. What is Mr. Jones rate on both the old route and on the freeway?

21) If a train had traveled 5 miles an hour faster, it would have needed $1\ \frac{1}{2}$ hours less time to travel 150 miles. Find the rate of the train.

22) A traveler having 18 miles to go, calculates that his usual rate would make him one-half hour late for an appointment; he finds that in order to arrive on time he must travel at a rate one-half mile an hour faster. What is his usual rate?

9.11

Quadratics - Graphs of Quadratics

Objective: Graph quadratic equations using the vertex, x-intercepts, and y-intercept.

Just as we drew pictures of the solutions for lines or linear equations, we can draw a picture of solution to quadratics as well. One way we can do that is to make a table of values.

Example 501.

$y = x^2 - 4x + 3$ Make a table of values

x	y
0	
1	
2	
3	
4	

We will test 5 values to get an idea of shape

$y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 = 3$ Plug 0 in for x and evaluate

$y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 = 0$ Plug 1 in for x and evaluate

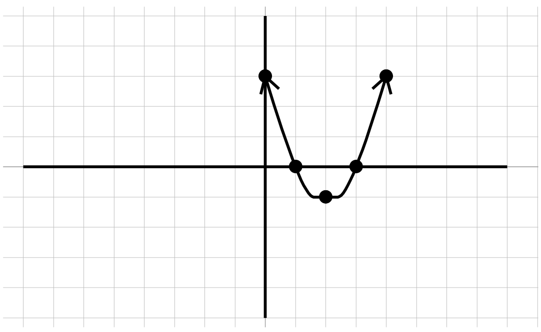
$y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$ Plug 2 in for x and evaluate

$y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 = 0$ Plug 3 in for x and evaluate

$y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 = 3$ Plug 4 in for x and evaluate

x	y
0	3
1	0
2	-1
3	0
4	3

Our completed table. Plot points on graph



Plot the points $(0, 3)$, $(1, 0)$, $(2, -1)$, $(3,0)$, and $(4,3)$.

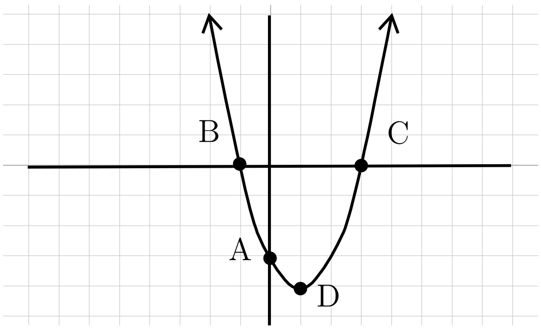
Connect the dots with a smooth curve.

Our Solution

When we have x^2 in our equations, the graph will no longer be a straight line. Quadratics have a graph that looks like a U shape that is called a parabola.

World View Note: The first major female mathematician was Hypatia of Egypt who was born around 370 AD. She studied conic sections. The parabola is one type of conic section.

The above method to graph a parabola works for any equation, however, it can be very tedious to find all the correct points to get the correct bend and shape. For this reason we identify several key points on a graph and in the equation to help us graph parabolas more efficiently. These key points are described below.



Point A: y-intercept: Where the graph crosses the vertical y-axis.

Points B and C: x-intercepts: Where the graph crosses the horizontal x-axis

Point D: Vertex: The point where the graph curves and changes directions.

We will use the following method to find each of the points on our parabola.

To graph the equation $\boldsymbol{y = ax^2 + bx + c}$, find the following points

1. y-intercept: Found by making $x = 0$, this simplifies down to $y = c$
2. x-intercepts: Found by making $y = 0$, this means solving $0 = ax^2 + bx + c$
3. Vertex: Let $x = \frac{-b}{2a}$ to find x. Then plug this value into the equation to find y.

After finding these points we can connect the dots with a smooth curve to find our graph!

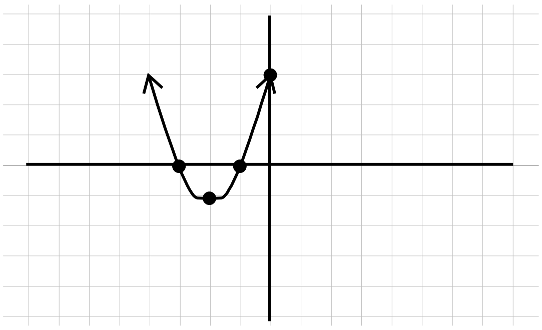
Example 502.

$$y = x^2 + 4x + 3 \quad \text{Find the key points}$$

$$y = 3 \quad y = c\,\text{is the}\, y - \text{intercept}$$

$$\begin{array}{rl}
0 = x^2 + 4x + 3 & \text{To find}\, x - \text{intercept we solve the equation} \\
0 = (x+3)(x+1) & \text{Factor} \\
x + 3 = 0 \ \text{ and } \ x + 1 = 0 & \text{Set each factor equal to zero} \\
\underline{-3 - 3} \qquad\qquad \underline{-1 - 1} & \text{Solve each equation} \\
x = -3 \ \text{ and } \ x = -1 & \text{Our}\, x - \text{intercepts}
\end{array}$$

$$\begin{array}{rl}
x = \dfrac{-4}{2(1)} = \dfrac{-4}{2} = -2 & \text{To find the vertex, first use}\, x = \dfrac{-b}{2a} \\
y = (-2)^2 + 4(-2) + 3 & \text{Plug this answer into equation to find}\, y - \text{coordinate} \\
y = 4 - 8 + 3 & \text{Evaluate} \\
y = -1 & \text{The}\, y - \text{coordinate} \\
(-2, -1) & \text{Vertex as}\, a\, \text{point}
\end{array}$$



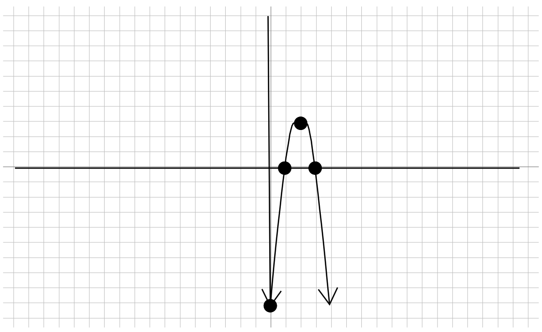
Graph the y-intercept at 3, the x-intercepts at -3 and -1, and the vertex at $(-2,-1)$. Connect the dots with a smooth curve in a U shape to get our parabola.

Our Solution

If the a in $y=ax^2+bx+c$ is a negative value, the parabola will end up being an upside-down U. The process to graph it is identical, we just need to be very careful of how our signs operate. Remember, if a is negative, then ax^2 will also be negative because we only square the x, not the a.

Example 503.

$$
\begin{array}{rl}
y=-3x^2+12x-9 & \text{Find key points}\\
\\
y=-9 & y-\text{intercept is } y=c\\
\\
0=-3x^2+12x-9 & \text{To find } x-\text{intercept solve this equation}\\
0=-3(x^2-4x+3) & \text{Factor out GCF first, then factor rest}\\
0=-3(x-3)(x-1) & \text{Set each factor with } a \text{ varaible equal to zero}\\
x-3=0 \text{ and } x-1=0 & \text{Solve each equation}\\
\underline{+3+3} \qquad \underline{+1+1} & \\
x=3 \text{ and } x=1 & \text{Our } x-\text{intercepts}\\
\\
x=\dfrac{-12}{2(-3)}=\dfrac{-12}{-6}=2 & \text{To find the vertex, first use } x=\dfrac{-b}{2a}\\
y=-3(2)^2+12(2)-9 & \text{Plug this value into equation to find } y-\text{coordinate}\\
y=-3(4)+24-9 & \text{Evaluate}\\
y=-12+24-9 & \\
y=3 & y-\text{value of vertex}\\
(2,3) & \text{Vertex as } a \text{ point}
\end{array}
$$



Graph the y-intercept at -9, the x-intercepts at 3 and 1, and the vertex at (2, 3). Connect the dots with smooth curve in an upside-down U shape to get our parabola.

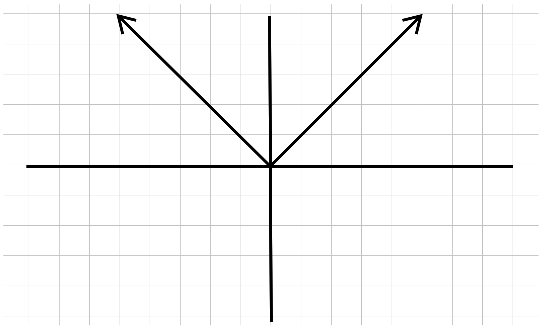
Our Solution

It is important to remember the graph of all quadratics is a parabola with the same U shape (they could be upside-down). If you plot your points and we cannot connect them in the correct U shape then one of your points must be wrong. Go back and check your work to be sure they are correct!

Just as all quadratics (equation with $y = x^2$) all have the same U-shape to them and all linear equations (equations such as $y = x$) have the same line shape when graphed, different equations have different shapes to them. Below are some common equations (some we have yet to cover!) with their graph shape drawn.

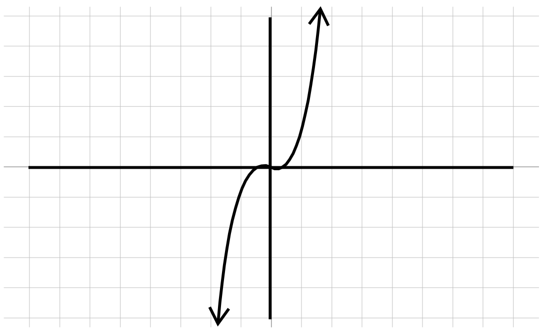
Absolute Value

$y = |x|$



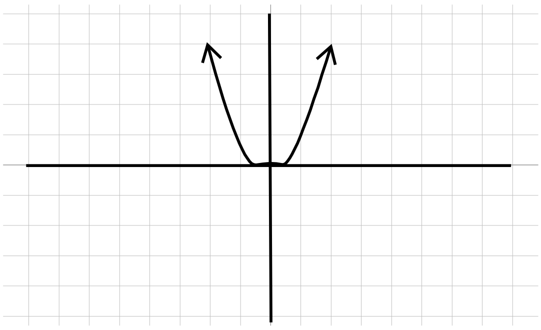
Cubic

$y = x^3$



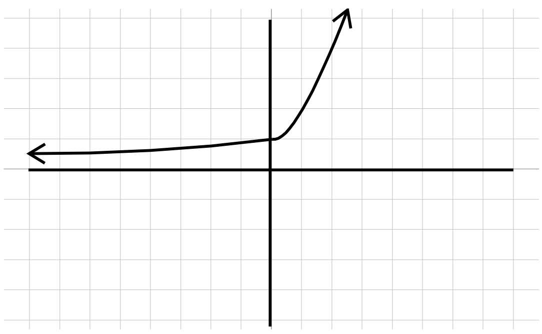
Quadratic

$y = x^2$



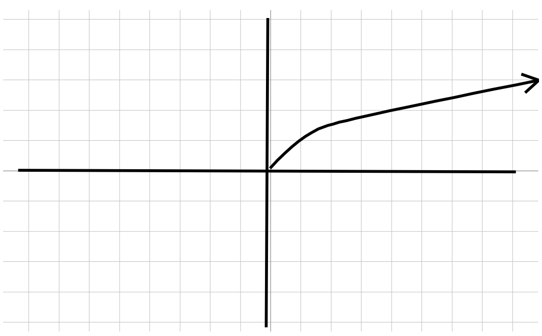
Exponential

$y = a^x$



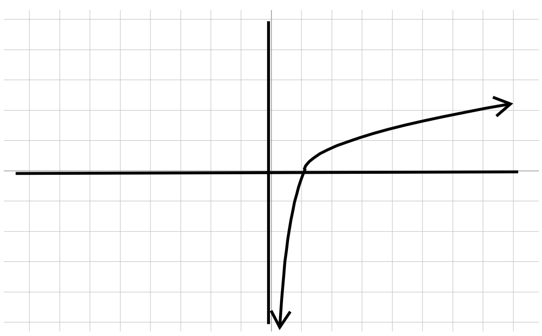
Square Root

$y = \sqrt{x}$



Logarithmic

$y = \log_a x$



9.11 Practice - Graphs of Quadratics

Find the vertex and intercepts of the following quadratics. Use this information to graph the quadratic.

1) $y = x^2 - 2x - 8$

2) $y = x^2 - 2x - 3$

3) $y = 2x^2 - 12x + 10$

4) $y = 2x^2 - 12x + 16$

5) $y = -2x^2 + 12x - 18$

6) $y = -2x^2 + 12x - 10$

7) $y = -3x^2 + 24x - 45$

8) $y = -3x^2 + 12x - 9$

9) $y = -x^2 + 4x + 5$

10) $y = -x^2 + 4x - 3$

11) $y = -x^2 + 6x - 5$

12) $y = -2x^2 + 16x - 30$

13) $y = -2x^2 + 16x - 24$

14) $y = 2x^2 + 4x - 6$

15) $y = 3x^2 + 12x + 9$

16) $y = 5x^2 + 30x + 45$

17) $y = 5x^2 - 40x + 75$

18) $y = 5x^2 + 20x + 15$

19) $y = -5x^2 - 60x - 175$

20) $y = -5x^2 + 20x - 15$

Beginning and Intermediate Algebra

Student Solutions Manual

Complete worked solutions to odd problems

Solutions manual has not been cross checked for accuracy.
If you disagree with this solutions manual you should check with your instructor.
Should you find an error, please E-mail tylerw@bigbend.edu so it can be corrected.
Thank you!

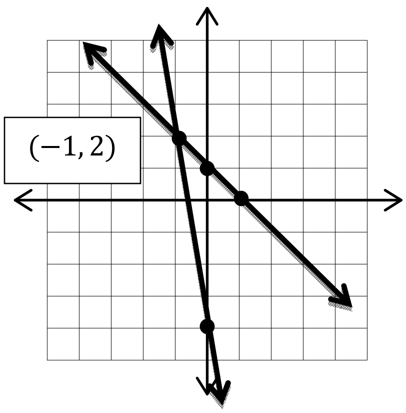
Beginning and Intermediate Algebra

Student Solutions Manual Table of Contents

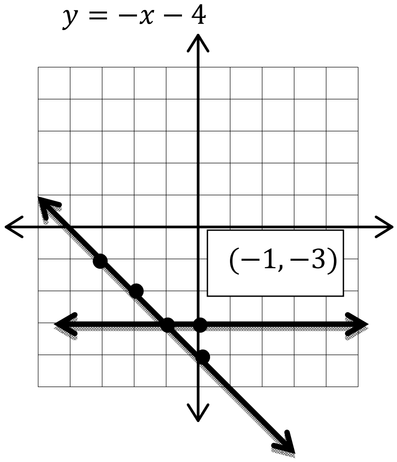
Chapter 4: Systems of Equations

4.1

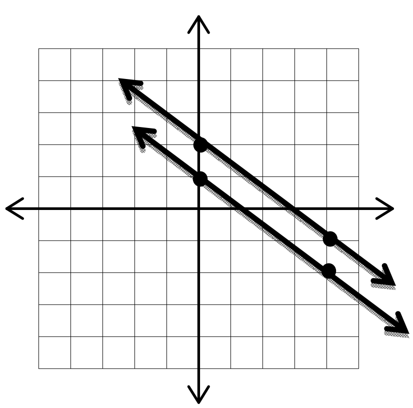
1) $y = -x + 1$
$y = -5x - 3$



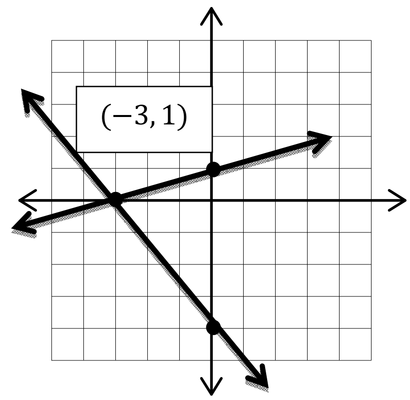
3) $y = -3$
$y = -x - 4$



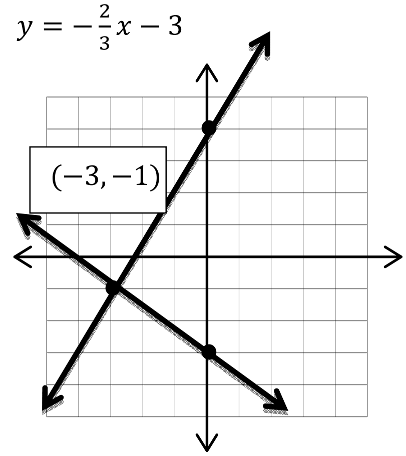
5) $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$ No Solution



7) $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$



9) $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

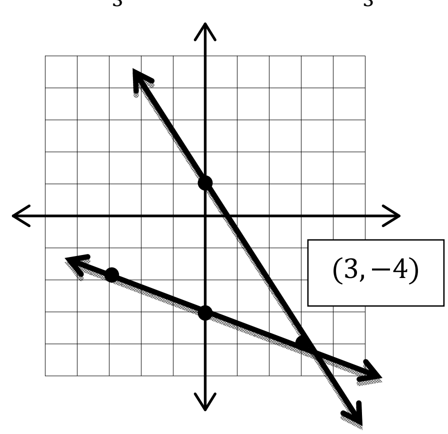


11) $x + 3y = -9$ $\quad$ $5x + 3y = 3$

$$\underline{-x \qquad -x} \qquad \underline{-5x \qquad -5x}$$

$$\frac{3y}{3} = -\frac{x}{3} - \frac{9}{3} \qquad \frac{3y}{3} = -\frac{5x}{3} + \frac{3}{3}$$

$$y = -\frac{1}{3}x - 3 \qquad y = -\frac{5}{3}x + 1$$

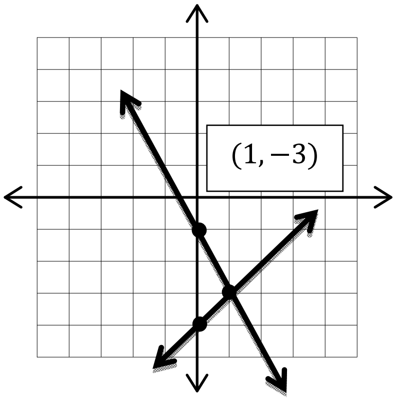


13) $x - y = 4 \qquad\qquad 2x + y = -1$

$\underline{-x \quad\quad -x \qquad\quad -2x \qquad -2x}$

$-\frac{y}{-1} = -\frac{x}{-1} + \frac{4}{-1} \qquad\qquad y = -2x - 1$

$y = x - 4$

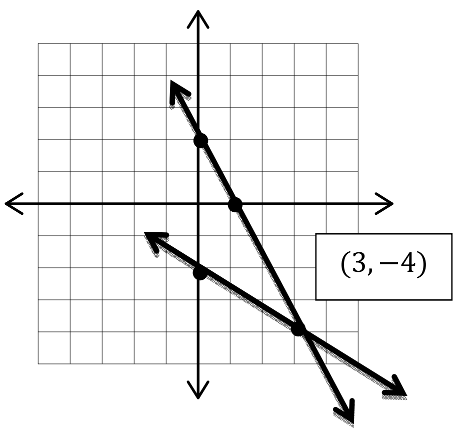


15) $2x + 3y = -6 \qquad\qquad 2x + y = 2$

$\underline{-2x \qquad -2x \quad\; -2x \qquad -2x}$

$\frac{3y}{3} = -\frac{2x}{3} - \frac{6}{3} \qquad\qquad y = -2x + 2$

$y = -\frac{2}{3}x - 2$

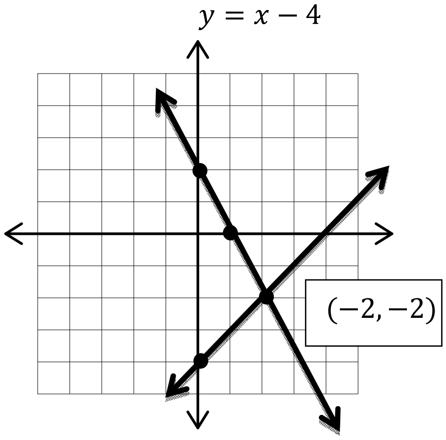


17) $2x + y = 2 \qquad x - y = 4$

$\underline{-2x \quad -2x \;\; -x \qquad -x}$

$y = -2x + 2 \qquad -\frac{y}{-1} = -\frac{x}{-1} + \frac{4}{-1}$

$y = x - 4$

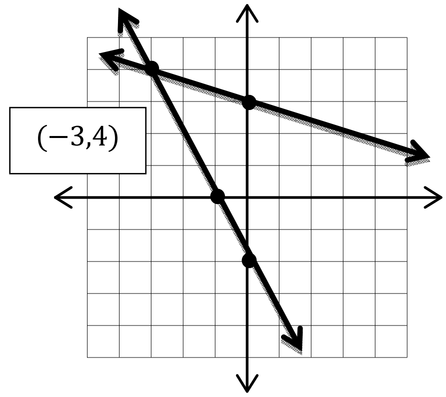


19) $2x + y = 2 \qquad x + 3y = 9$

$\underline{-2x \;\; -2x \;\; -x \qquad -x}$

$y = -2x + 2 \qquad \frac{3y}{3} = -\frac{x}{3} + \frac{9}{3}$

$y = -\frac{1}{3}x + 3$

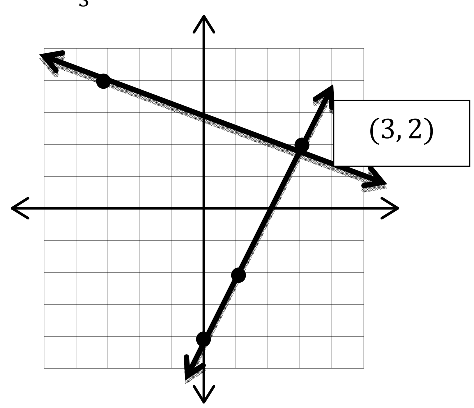


21) $0 = -6x - 9y + 36 \qquad 12 = 6x - 3y$

$\underline{+9y \qquad +9y \qquad\qquad -6x - 6x}$

$\frac{9y}{9} = -\frac{6x}{9} + \frac{36}{9} \qquad\qquad -\frac{6x}{-3} + \frac{12}{-3} = \frac{-3y}{-3}$

$y = -\frac{2}{3}x + 4 \qquad\qquad 2x - 4 = y$

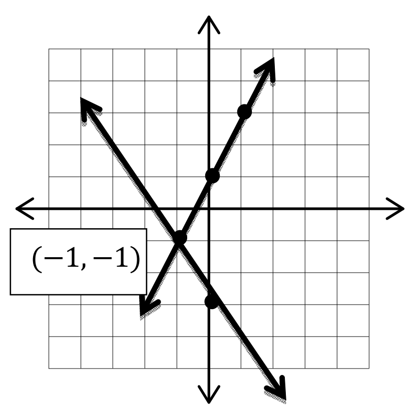


23) $2x - y = -1 \qquad\qquad 0 = -2x - y - 3$

$\underline{-2x \qquad -2x \qquad\quad +y \qquad +y}$

$-\frac{y}{-1} = -\frac{2x}{-1} - \frac{1}{-1} \quad y = -2x - 3$

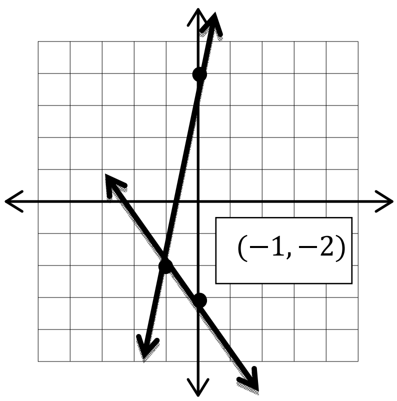
$y = 2x + 1$



25) $3 + y = -x \qquad -\frac{4}{-1} - \frac{6x}{-1} = -\frac{y}{-1}$

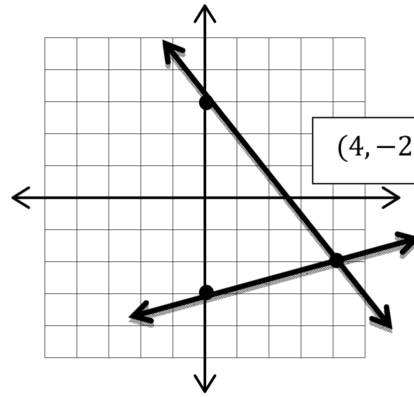
$\underline{-3 \qquad -3} \qquad 4 + 6x = y$

$y = -x - 3$



29) $-\frac{12}{4} + \frac{x}{4} = \frac{4y}{4} \qquad \frac{12}{4} - \frac{5x}{4} = \frac{4y}{4}$

$-3 + \frac{1}{4}x = y \qquad 3 - \frac{5}{4}x = y$

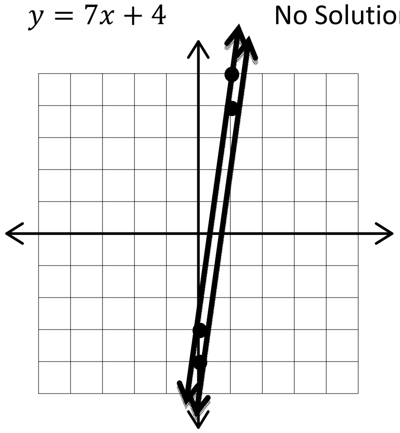


27) $-y + 7x = 4 \qquad -y - 3 + 7x = 0$

$\underline{\quad -7x \quad -7x \qquad +y \qquad\qquad +y}$

$-\frac{y}{-1} = -\frac{7x}{-1} + \frac{4}{-1} \qquad -3 + 7x = y$

$y = 7x + 4$ No Solution



4.2

1) $y = -3x$

$y = 6x - 9$

$-3x = 6x - 9$

$\underline{-6x \quad -6x}$

$-\frac{9x}{-9} = -\frac{9}{-9}$

$x = 1$

$y = -3(1) = -3$

$(1, -3)$

3) $y = -2x - 9$

$y = 2x - 1$

$-2x - 9 = 2x - 1$

$\underline{+2x \qquad +2x}$

$-9 = 4x - 1$

$\underline{+1 \qquad +1}$

$-\frac{8}{4} = \frac{4x}{4}$

$-2 = x$

$y = -2(-2) - 9$

$y = 4 - 9$

$y = -5$

$(-2, -5)$

5) $y = 6x + 4$

$y = -3x - 5$

$6x + 4 = -3x - 5$

$\underline{+3x \qquad +3x}$

$9x + 4 = -5$

$\underline{\quad -4 \quad -4}$

$\frac{9x}{9} = \frac{-9}{9}$

$x = -1$

$y = 6(-1) + 4$

$y = -6 + 4$

$y = -2$

$(-1, -2)$

7) $y = 3x + 2$
$y = -3x + 8$
$3x + 2 = -3x + 8$
$\underline{+3x \quad\quad + 3x}$
$6x + 2 = 8$
$\underline{\quad -2 \;\; -2}$
$\frac{6x}{6} = \frac{6}{6}$
$x = 1$
$y = 3(1) + 2$
$y = 3 + 2$
$y = 5$
$(1, 5)$

9) $y = 2x - 3$
$y = -2x + 9$
$2x - 3 = -2x + 9$
$\underline{+2x \quad\quad + 2x}$
$4x - 3 = 9$
$\underline{\quad +3 \;\; +3}$
$\frac{4x}{4} = \frac{12}{4}$
$x = 3$
$y = 2(3) - 3$
$y = 6 - 3$
$y = 3$
$(3, 3)$

11) $y = 6x - 6$
$-3x - 3y = -24$
$-3x - 3(6x - 6) =$
-24
$-3x - 18x + 18 = -24$
$-21x + 18 = -24$
$\underline{\quad -18 \quad -18}$
$\frac{-21x}{-21} = \frac{-42}{-21}$
$x = 2$
$y = 6(2) - 6$
$y = 12 - 6$
$y = 6$
$(2, 6)$

13) $y = -6$
$3x - 6y = 30$
$3x - 6(-6) = 30$
$3x + 36 = 30$
$\underline{\quad -36 \;\; -36}$
$\frac{3x}{3} = \frac{-6}{3}$
$x = -2$
$(-2, -6)$

15) $y = -5$
$3x + 4y = -17$
$3x + 4(-5) = -17$
$3x - 20 = -17$
$\underline{\quad +20 \quad +20}$
$\frac{3x}{3} = \frac{3}{3}$
$x = 1$
$(1, -5)$

17) $-2x + 2y = 18$
$y = 7x + 15$
$-2x + 2(7x + 15) = 18$
$-2x + 14x + 30 = 18$
$12x + 30 = 18$
$\underline{\quad -30 \;\; -30}$
$\frac{12x}{12} = \frac{-12}{12}$
$x = -1$
$y = 7(-1) + 15$
$y = -7 + 15$
$y = 8$
$(-1, 8)$

19) $y = -8x + 19$
$-x + 6y = 16$
$-x + 6(-8x + 19) =$
16
$-x - 48x + 114 = 16$
$-49x + 114 = 16$
$\underline{\quad -114} \;\; -114$
$\frac{-49x}{-49} = \frac{-98}{-49}$
$x = 2$
$y = -8(2) + 19$
$y = -16 + 19$
$y = 3$
$(2, 3)$

21) $7x - 2y = -7$
$y = 7$
$7x - 2(7) = -7$
$7x - 14 = -7$
$\underline{\quad +14 \quad +14}$
$\frac{7x}{7} = \frac{7}{7}$
$x = 1$
$(1, 7)$

23) $x - 5y = 7$
$2x + 7y = -20$
$x - 5y = 7$
$\underline{\quad +5y \;\; +5y}$
$x = 5y + 7$
$2(5y + 7) + 7y = -20$
$10y + 14 + 7y = -20$
$17y + 14 = -20$
$\underline{\quad -14 \quad -14}$
$\frac{17y}{17} = \frac{-34}{17}$
$y = -2$
$x - 5(-2) = 7$
$x + 10 = 7$
$\underline{\quad -10 \;\; -10}$
$x = -3$
$(-3, -2)$

25) $-2x - y = -5$

$x - 8y = -23$

$\underline{\quad +8y \quad + 8y}$

$x = 8y - 23$

$-2(8y - 23) - y = -5$

$-16y + 46 - y = -5$

$-17y + 46 = -5$

$\underline{\quad -46 \quad -46}$

$\frac{-17y}{-17} = \frac{-51}{-17}$

$y = 3$

$x - 8(3) = -23$

$x - 24 = -23$

$\underline{\quad +24 \quad +24}$

$x = 1$

$(1, 3)$

27) $-6x + y = 20$

$-3x - 3y = -18$

$-6x + y = 20$

$\underline{+6x \qquad + 6x}$

$y = 6x + 20$

$-3x - 3(6x + 20) = -18$

$-3x - 18x - 60 = -18$

$-21x - 60 = -18$

$\underline{\quad +60 \quad +60}$

$\frac{-21x}{-21} = \frac{42}{-21}$

$x = -2$

$y = 20 + 6(-2)$

$y = 20 - 12$

$y = 8$

$(-2, 8)$

29) $3x + y = 9$

$2x + 8y = -16$

$3x + y = 9$

$\underline{\quad -3x \quad - 3x}$

$y = -3x + 9$

$2x + 8(-3x + 9) =$
-16

$2x - 24x + 72 = -16$

$-22x + 72 = -16$

$\underline{\quad -72 \quad -72}$

$\frac{-22x}{-22} = \frac{-88}{-22}$

$x = 4$

$y = -3(4) + 9$

$y = -12 + 9$

$y = -3$

$(4, -3)$

32) $2x + y = 2$

$3x + 7y = 14$

$2x + y = 2$

$\underline{-2x \qquad - 2x}$

$y = 2 - 2x$

$3x + 7(2 - 2x) = 14$

$3x + 14 - 14x = 14$

$-11x + 14 = 14$

$\underline{\quad -14 \quad -14}$

$\frac{-11x}{-11} = \frac{0}{-11}$

$x = 0$

$y = 2 - 2(0)$

$y = 2 - 0$

$y = 2$

$(0, 2)$

34) $x + 5y = 15$

$-3x + 2y = 6$

$x + 5y = 15$

$\underline{\quad -5y \quad -5y}$

$x = 15 - 5y$

$-3(15 - 5y) + 2y = 6$

$-45 + 15y + 2y = 6$

$-45 + 17y = 6$

$\underline{+45 \qquad + 45}$

$\frac{17y}{17} = \frac{51}{17}$

$y = 3$

$x = 15 - 5(3)$

$x = 15 - 15$

$x = 0$

$(0, 3)$

36) $-2x + 4y = -16$

$y = -2$

$-2x + 4(-2) = -16$

$-2x - 8 = -16$

$\underline{\quad +8 \quad +8}$

$\frac{-2x}{-2} = \frac{-8}{-2}$

$x = 4$

$(4, -2)$

38) $-6x + 6y = -12$

$8x - 3y = 16$

$-6x + 6y = -12$

$\underline{+6x \qquad + 6x}$

$\frac{6y}{6} = \frac{6x}{6} - \frac{12}{6}$

$y = x - 2$

$8x - 3(x - 2) = 16$

$8x - 3x + 6 = 16$

$5x + 6 = 16$

$\underline{\quad -6 \quad -6}$

$\frac{5x}{5} = \frac{10}{5}$

$x = 2$

$y = (2) - 2$

$y = 0$

$(2, 0)$

39) $2x + 3y = 16$

$-7x - y = 20$

$\underline{+7x \qquad + 7x}$

$-\frac{y}{-1} = \frac{7x}{-1} + 20/-1$

$y = -7x - 20$

$2x + 3(-7x - 20) = 16$

$2x - 21x - 60 = 16$

$-19x - 60 = 16$

$\underline{+60 \quad + 60}$

$\frac{-19x}{-19} = \frac{76}{-19}$

$x = -4$

$y = -7(-4) - 20$

$y = 28 - 20$

$y = 8$

$(-4, 8)$

4.3

1) $4x + 2y = 0$

$-4x - 9y = -28$

$-\frac{7y}{-7} = -\frac{28}{-7}$

$y = 4$

$4x + 2(4) = 0$

$4x + 8 = 0$

$\underline{-8 \quad -8}$

$\frac{4x}{4} = -\frac{8}{4}$

$x = -2$

$(-2, 4)$

3) $-9x + 5y = -22$

$9x - 5y = 13$

$0 = -9$

$false$

$No\ Solution\ \emptyset$

5) $-6x + 9y = 3$

$6x - 9y = -9$

$0 = -6$

$false$

$No\ Solution\ \emptyset$

7) $-1(4x - 6y) = (-10) - 1$

$4x - 6y = -14$

$-4x + 6y = 10$

$0 = 10$

$false$

$No\ Solution\ \emptyset$

9) $-1(-x - 5y) = 28(-1)$

$-x + 4y = -17$

$\underline{x + 5y = -28}$

$\frac{9y}{9} = -\frac{45}{9}$

$y = -5$

$-x - 5(-5) = 28$

$-x + 25 = 28$

$\underline{-25 \quad -25}$

$\frac{-x}{-1} = \frac{3}{-1}$

$x = -3$

$(-3, -5)$

11) $2(2x - y) = (5)2$

$5x + 2y = -28$

$\underline{4x - 2y = 10}$

$\frac{9x}{9} = \frac{-18}{9}$

$x = -2$

$2(-2) - y = 5$

$-4 - y = 5$

$\underline{+4 \qquad +4}$

$\frac{-y}{-1} = \frac{9}{-1}$

$y = -9$

$(-2, -9)$

13) $10x + 6y = 24$

$-6(-6x + y) = (4)(-6)$

$10x + 6y = 24$

$\underline{36x - 6y = -24}$

$\frac{46x}{46} = \frac{0}{46}$

$x = 0$

$10(0) + 6y = 24$

$\frac{6y}{6} = \frac{24}{6}$

$y = 4 \quad (0, 4)$

15) $3(2x+4y)=(24)3$
$4x-12y=8$
$\underline{6x+12y=72}$
$\frac{10x}{10}=\frac{80}{10}$
$x=8$
$2(8)+4y=24$
$16+4y=24$
$\underline{-16 \quad -16}$
$\frac{4y}{4}=\frac{8}{4}$
$y=2$
$(8,2)$

17) $2(-7x+4y)=(-4)2$
$10x-8y=-8$
$\underline{-14x+8y=8}$
$\left(-\frac{4x}{4}\right)=-\frac{16}{-4}$
$x=4$
$-7(4)+4y=-4$
$-28+4y=-4$
$\underline{+28 \quad +28}$
$\frac{4y}{4}=\frac{24}{4}$
$y=6$
$(4,6)$

19) $5x+10y=20$
$2(-6x-5y)=(-3)2$
$5x+10y=20$
$\underline{-12x-10y=-6}$
$\left(-\frac{7x}{7}\right)=14/-7$
$x=-2$
$5(-2)+10y=20$
$-10+10y=20$
$\underline{+10 \quad +10}$
$\frac{10y}{10}=\frac{30}{10}$
$y=3$
$(-2,3)$

21) $5(-7x-3y)=12(5)$
$-3(-6x-5y)=20(-3)$
$-35x-15y=60$
$\underline{18x+15y=-60}$
$-\frac{17x}{-17}=\frac{0}{-17}$
$x=0$
$-7(0)-3y=12$
$\frac{-3y}{-3}=\frac{12}{-3}$
$y=-4$
$(0,-4)$

23) $7(9x-2y)=(-18)7$
$-2(5x-7y)=(-10)(-2)$
$63x-14y=-126$
$\underline{-10x+14y=20}$
$\frac{53x}{53}=\frac{-106}{53}$
$x=-2$
$9(-2)-2y=-18$
$-18-2y=-18$
$\underline{+18 \quad +18}$
$\frac{-2y}{-2}=\frac{0}{-2}$
$y=0$
$(-2,0)$

25) $3(9x+6y)=(-21)3$
$2(-10x-9y)=28(2)$
$27x+18y=-63$
$\underline{-20x-18y=56}$
$\frac{7x}{7}=\frac{-7}{7}$
$x=-1$
$9(-1)+6y=-21$
$-9+6y=-21$
$\underline{+9 \quad +9}$
$\frac{6y}{6}=-\frac{12}{6}$
$y=-2$
$(-1,-2)$

27) $3(-7x+5y)=(-8)3$
$5(-3x-3y)=12(5)$
$-21x+15=-24$
$\underline{-15x-15=60}$
$-\frac{36x}{36}=\frac{36}{-36}$
$x=-1$
$-7(-1)+5y=-8$
$7+5y=-8$
$\underline{-7 \quad -7}$
$\frac{5y}{5}=-\frac{15}{5}$
$y=-3$
$(-1,-3)$

29) $5(-8x-8y)=(-8)5$
$4(10x+9y)=(1)4$
$-40x-40y=-40$
$\underline{40x+36y=4}$
$\frac{-4y}{-4}=\frac{-36}{-4}$
$y=9$
$-8x-8(9)=-8$
$-8x-72=-8$
$\underline{+72 \quad +72}$
$\frac{-8x}{8}=\frac{64}{8}$
$x=8$
$(8,9)$

31) $9y=7-x$
$-18y+4x=-26$
$9y=7-x$
$\underline{+x \quad +x}$
$2(9y+x)=(7)2$
$-18y+4x=-26$
$\underline{18y+2x=14}$
$\frac{6x}{6}=\frac{-12}{6}$
$x=-2$
$9y=7-(-2)$
$\frac{9y}{9}=\frac{9}{9}$
$y=1$
$(-2,1)$

33) $0 = 9x + 5y$

$(7)y = \frac{2}{7}x(7)$

$7y = 2x$

$\underline{-7y \quad -7y}$

$(-9)0 = (2x - 7y)(-9)$

$2(0) = (9x + 5y)2$

$0 = -18x + 63y$

$0 = \underline{\quad 18x + 10y \quad}$

$\frac{0}{73} = \frac{73y}{73}$

$0 = y$

$0 = 9x + 5(0)$

$\frac{0}{9} = \frac{9x}{9}$

$0 = x$

$(0,0)$

4.4

1) $(I) \quad a - 2b + c = 5$

$(II) 2a + b - c = -1$

$(III) 3a + 3b - 2c = -4$

$(I) \quad a - 2b + c = 5$

$(II) \underline{2a + b - c = -1}$

$A: 3a - b = 4$

$(I)\ 2(a - 2b + c) = (5)2$

$(III) 3a + 3b - 2c = -4$

$\underline{2a - 4b + 2c = 10}$

$B: 5a - b = 6$

$A: -1(3a - b) = 4(-1)$

$B: 5a - b = 6$

$\underline{-3a + b = -4}$

$\frac{2a}{2} = \frac{2}{2}$

$a = 1$

$(1, -1, 2)$

$A: 3(1) - b = 4$

$3 - b = 4$

$\underline{-3 \qquad -3}$

$\frac{-b}{-1} = \frac{1}{-1}$

$b = -1$

$(I)\ (1) - 2(-1) + c = 5$

$1 + 2 + c = 5$

$3 + c = 5$

$\underline{-3 \qquad -3}$

$c = 2$

3) $(I)\ 3x + y - z = 11$

$(II)\ x + 3y = z + 13$

$(III)\ x + y - 3z = 11$

$(I) - 1(3x + y - z) = 11(-1)$

$(II) \quad x + 3y - z = 13$

$\underline{-3x - y + z = -11}$

$A: -2x + 2y = 2$

$(I) - 3(3x + y - z) = 11(-3)$

$(III) \quad x + y - 3z = 11$

$\underline{-9x - 3y + 3z = -33}$

$B: -8x - 2y = -22$

$(II) x + 3y = z + 13$

$\underline{\qquad -z - z}$

$(II)\ x + 3y - z = 13$

$A: -2x + 2y = 2$

$B: \underline{-8x - 2y = -22}$

$-\frac{10x}{-10} = -\frac{20}{-10}$

$x = 2$

$(2, 3, -2)$

$A: -2(2) + 2y = 2$

$-4 + 2y = 2$

$\underline{+4 \qquad +4}$

$\frac{2y}{2} = \frac{6}{2}$

$y = 3$

$(I)\ 3(2) + 3 - z = 11$

$6 + 3 - z = 11$

$9 - z = 11$

$\underline{-9 \qquad -9}$

$\frac{-z}{-1} = \frac{2}{-1}$

$z = -2$

5) $(I) x+6y+3z=4$

$(II)\ 2x+y+2z=3$

$(III) 3x-2y+z=0$

$(I)\ x+6y+3z=4$

$(III)\ 3(3x-2y+z)=(0)3$

$x+6y+3z=4$

$\underline{9x-6y+3z=0}$

$A: 10x+6z=4$

$(II) 2(2x+y+2z)=(3)2$

$(III)\ 3x-2y+z=0$

$\underline{4x+2y+4z=6}$

$B: 7x+5z=6$

$A: -5(10x+6z)=4(-5)$

$B: 6(7x+5z)=6(6)$

$-50x-30z=-20$

$\underline{42x+30z=36}$

$\frac{-8x}{-8}=\frac{16}{-8}$

$x=-2$

$A: 10(-2)+6z=4$

$-20+6z=4$

$\underline{+20 \qquad +20}$

$\frac{6z}{6}=\frac{24}{6}$

$z=4$

$(-2,-1,4)$

$(I)\ (-2)+6y+3(4)=4$

$-2+6y+12=4$

$10+6y=4$

$\underline{-10 \qquad -10}$

$\frac{6y}{6}=\frac{-6}{6}$

$y=-1$

7) $(I)\ x+y+z=6$

$(II) 2x-y-z=-3$

$(III)\ x-2y+3z=6$

$(I)\ x+y+z=6$

$(II) \underline{2x-y-z=-3}$

$\frac{3x}{3}=\frac{3}{3}$

$x=1$

$(II) 2(1)-y-z=-3$

$2-y-z=-3$

$\underline{-2 \qquad -2}$

$A: -y-z=-5$

$A: 3(-y-z)=(-5)3$

$B: -2y+3z=5$

$\underline{-3y-3z=-15}$

$-\frac{5y}{-5}=-\frac{10}{-5}$

$y=2$

$A: -(2)-z=-5$

$\underline{+2 \qquad +2}$

$-\frac{z}{-1}=-\frac{3}{-1}$

$z=3$

$(III) 1-2y+3z=6$

$\underline{-1 \qquad -1}$

$B: -2y+3z=5$

$(1,2,3)$

9) $(I)\ x+y-z=0$

$(II) x-y-z=0$

$(III) x+y+2x=0$

$(I)\ x+y-z=0$

$\underline{(II) x-y-z=0}$

$A: 2x-2z=0$

$(II) x-y-z=0$

$\underline{(III)\ x+y+2z=0}$

$B: 2x+z=0$

$A: (-1)(2x-2z)=0(-1)$

$B: 2x+z=0$

$\underline{-2x+2z=0}$

$\frac{3z}{0}=0$

$x=0$

$A: 2x-2(0)=0$

$\frac{2x}{2}=\frac{0}{2}$

$x=0$

$(I)\ 0+y-0=0$

$y=0$

$(0,0,0)$

11) $(I) -2x + y - 3z = 1$
$(II)\ \ x - 4y + z = 6$
$(III) 4x + 16y + 4z = 24$

$(I) -2x + y - 3z = 1$
$(II) 2(x - 4y + z) = 6(2)$
$-2x + y - 3z = 1$
$\underline{2x - 8y + 2z = 12}$
$A: \quad -7y - z = 13$

$(I) 2(-2x + y - 3z) = (1)2$
$(III) 4x + 16y + 4z = 24$
$\underline{-4x + 2y - 6z = 2}$
$B: 18y - 2z = 26$

$A: -2(-7y - z) = 13(-2)$
$B: 18y - 2z = 26$
$\underline{14y + 2z = -26}$
$\frac{32y}{32} = \frac{0}{32}$
$y = 0$

$A: -7(0) - z = 13$
$-\frac{z}{-1} = \frac{13}{-1}$
$z = -13$

$(I) -2x + 0 - 3(-13) = 1$
$-2x + 39 = 1$
$\underline{-39 \quad -39}$
$-\frac{2x}{-2} = -\frac{38}{-2}$
$x = 19$

$(19, 0, -13)$

13) $(I) 2x + y - 3z = 0$
$(II) x - 4y + z = 0$
$(III) 4x + 16y + 4z = 0$

$(I) 4(2x + y - 3z) = (0)4$
$(II)\ \ x - 4y + z = 0$
$\underline{-8x + 4y - 12z = 0}$
$A: 9x - 11z = 0$

$(II) 4(x - 4y + z) = (0)4$
$(III) 4x + 16y + 4z = 0$
$\underline{4x - 16y + 4z = 0}$
$B: \quad 8x + 8z = 2$

$A: 8(9x - 11z) = 0(8)$
$B: -9(8x + 8z) = 2(-9)$
$72x - 88z = 0$
$\underline{-72x - 72z = 0}$
$\frac{-160z}{-160} = 0$
$z = 0$

$A: 9x - 11(0) = 0$
$\frac{9x}{9} = \frac{0}{9}$
$x = 0$

$(I) 2(0) + y - 3(0) = 0$
$y = 0$

$(0, 0, 0)$

15) $(I)\ 3x + 2y + 2x = 3$
$(II)\ x + 2y - z = 5$
$(III) 2x - 4y + z = 0$

$(II)\ x + 2y - z = 5$
$\underline{(III)\ 2x - 4y + z = 0}$
$A: 3x - 2y = 5$

$(I)\ 3x + 2y + 2x = 3$
$(II)\ 2(x + 2y - z) = (5)2$
$3x + 2y + 2z = 3$
$\underline{2x + 4y - 2z = 10}$
$B: 5x + 6x = 13$

$A: 3(3x - 2y) = (5)5$
$B: 5x + 6y = 13$
$\underline{9x - 6y = 15}$
$\frac{14x}{14} = \frac{28}{14}$
$x = 2$

$A: 3(2) - 2y = 5$
$6 - 2y = 5$
$\underline{-6 \qquad -6}$
$-\frac{2y}{-2} = -\frac{1}{-2}$
$y = \frac{1}{2}$

$(I)\ 3(2) + 2\left(\frac{1}{2}\right) + 2z = 3$
$6 + 1 + 2z = 3$
$7 + 2z = 3$
$\underline{-7 \qquad -7}$
$\frac{2z}{2} = -\frac{4}{2}$
$z = -2$

$(2, \frac{1}{2}, -2)$

17) $(I)\, x - 2y + 3z = 4$
$(II)\ 2x - y + z = -1$
$(III)\ 4x + y + z = 1$

$(II)\ 2x - y + z = -1$
$\underline{(III)\, 4x + y + z = 1}$
$A: 6x + 2z = 0$

$(I)\ x - 2y + 3z = 4$
$(III)\ 2(4x + y + z) = (1)2$
$x - 2y + 3z = 4$
$\underline{8x + 2y + 2z = 2}$
$B: 9x + 5z = 6$

$A: 3(6x + 2y) = (0)3$
$B: -2(9x + 5z) = 6(-2)$
$18x + 6z = 0$
$\underline{-18x - 10z = -12}$
$-\frac{4z}{-4} = -\frac{12}{-4}$
$z = 3$

$A: 6x + 2(3) = 0$
$6x + 6 = 0$
$\underline{-6 \quad -6}$
$\frac{6x}{6} = \frac{-6}{6}$
$x = -1$

$(I)\ (-1) - 2y + 3(3) = 4$
$8 - 2y = 4$
$\underline{-8 \qquad -8}$
$\frac{-2y}{-2} = \frac{-4}{-2}$
$y = 2$

$(-1, 2, 3)$

19) $(I)\ x - y + 2z = 0$
$(II)\, x - 2y + 3z = -1$
$(III)\, 2x - 2y + z = -3$

$(I)(-1)(x - y + 2z) = 0(-1)$
$(II)\, x - 2y + 3z = -1$
$\underline{-x + y - 2z = 0}$
$A: -y + z = -1$
$-y + (1) = -1$
$\underline{-1 \quad -1}$
$-\frac{y}{-1} = -\frac{2}{-1}$
$y = 2$

$(I)\ (-2)(x - y + 2z) = 0(-2)$
$(III)\, 2x - 2y + z = -3$
$\underline{-2x + 2y - 4z = 0}$
$-\frac{3z}{-3} = -\frac{3}{-3}$
$z = 1$

$(I)\, x - (2) + 2(1) = 0$
$x - 2 + 2 = 0$
$x = 0$

$(0, 2, 1)$

21) $(I)\ 4x - 3y + 2z = 40$
$(II)\ 5x + 9y - 7z = 47$
$(III)\, 9x + 8y - 3z = 97$

$(I)\, 3(4x - 3y + 2z) = (40)3$
$(II)\, 5x + 9y - 7z = 47$
$\underline{12x - 9y + 6z = 120}$
$A: 17x - z = 167$

$(I)\ 8(4x - 3y + 2z) = (40)8$
$(III)\, 3(9x + 8y - 3z) = (97)3$
$32x - 24y + 16z = 320$
$\underline{27x + 24y - 9z = 291}$
$B: 59x + 7z = 611$

$A:\ 7(17x - z) = (167)7$
$B: 59x + 7z = 611$
$\underline{119x - 7z = 1169}$
$\frac{178x}{178} = \frac{1780}{178}$
$x = 10$

$A: 17(10) - z = 167$
$170 - z = 167$
$\underline{-170 \qquad -170}$
$-\frac{z}{-1} = -\frac{3}{-1}$
$z = 3$

$(I)\ 4(10) - 3y + 2(3) = 40$
$46 - 3y = 40$
$\underline{-46 \qquad -46}$
$-\frac{3y}{-3} = -\frac{6}{-3}$
$y = 2$

$(10, 2, 3)$

23) $(I) 3x + 3y - 2z = 13$
$(II)\ 6x + 2y - 5z = 13$
$(III) 5x - 2y - 5z = -1$

$(II) 6x + 2y - 5z = 13$
$(III) \underline{5x - 2y - 5z = -1}$
$A: 11x - 10z = 12$

$(I) 2(3x + 3y - 2z) = (13)2$
$(III) 3(5x - 2y - 5z) = (-1)3$
$6x + 6y - 4z = 26$
$\underline{15x - 6y - 15z = -3}$
$B: 21x - 19z = 23$

$A: 19(11x - 10z) = (12)19$
$B: (-10)(21x - 19z) = 23(-10)$
$209x - 190z = 228$
$\underline{-210 + 190z = -230}$
$-\frac{x}{-1} = -\frac{2}{-1}$
$x = 2$

$A: 11(2) - 10z = 12$
$22 - 10z = 12$
$\underline{-22 \qquad -22}$
$-\frac{10z}{-10} = -\frac{10}{-10}$
$z = 1$

$(I)\ 3(2) + 3y - 2(1) = 13$
$4 + 3y = 13$
$\underline{-4 \qquad -4}$
$\frac{3y}{3} = \frac{9}{3}$
$y = 3$

$(2, 3, 1)$

25) $(I)\ 3x - 4y + 2z = 1$
$(II) 2x + 3y - 3z = -1$
$(III)\ x + 10y - 8z = 7$

$(I)\ 3x - 4y + 2z = 1$
$(III)(-3)(\ x + 10y - 8z) = 7(-3)$
$3x - 4y + 2z = 1$
$\underline{-3x - 30y + 24z = -21}$
$A:\ -34y + 26z = -20$

$(II) 2x + 3y - 3z = -1$
$(III)(-2)(\ x + 10y - 8z) = 7(-2)$
$2x + 3y - 3z = -1$
$\underline{-2x - 20y + 16z = -14}$
$B:\ -17y + 13z = -15$

$A: -34y + 26z = -20$
$B: -2(-17y + 13z) = -15(-2)$
$-34y + 26z = -20$
$\underline{34y - 26z = 30}$
$0 = 10$

$false$
$No\ solution\ \theta$

27) $(I) m + 6n + 3p = 8$
$(II)\ 3m + 4n = -3$
$(III) 5m + 7n = 1$

$(II)(-5)(3m + 4n) = (-3)(-5)$
$(III) 3(5m + 7n) = (1)3$
$-15m - 20n = 15$
$\underline{15m + 21n = 3}$
$n = 18$

$(II) 3m + 4(18) = -3$
$3m + 72 = -3$
$\underline{-72 \qquad -72}$
$\frac{3m}{3} = -\frac{75}{3}$
$m = -25$

$(I)(-25) + 6(18) + 3p = 8$
$-25 + 108 + 3p = 8$
$83 + 3p = 8$
$\underline{-83 \qquad -83}$
$\frac{3p}{3} = -\frac{75}{3}$
$p = -25$

$(-25, 18, -25)$

29) $(I)\ -2w+2x+2y-2z=-10$
$(II)\ w+x+y+z=-5$
$(III)\ 3w+2x+2y+4z=-11$
$(IV)\ w+3x-2y+2z=-6$

$(I)\ -2w+2x+2y-2z=-10$
$(II)(-2)(w+x+y+z)=(-5)(-2)$
$-2w+2x+2y-2z=-10$
$\underline{-2w-2x-2y-2z=10}$
$A:\ -4w-4z=0$

$(I)\ (-1)(-2w+2x+2y-2z)=(-10)(-1)$
$(III)\ 3w+2x+2y+4z=-11$
$\underline{2w-2x-2y+2z=10}$
$B: 5w+6z=-1$

$A: 3(-4w-4z)=0(3)$
$B: 2(5w+6z)=(-1)2$
$-12w-12z=0$
$\underline{10w+12z=-2}$
$-\frac{2w}{-2}=-\frac{2}{-2}$
$w=1$

$A:\ -4(1)-4z=0$
$-4-4z=0$
$\underline{+4\qquad +4}$
$-\frac{4z}{-4}=\frac{4}{-4}$
$z=-1$

$(III)\ 3(1)+2x+2y+4(-1)=-11$
$3+2x+2y-4=-11$
$2x+2y-1=-11$
$\underline{\qquad +1\quad +1}$
$C: 2x+2y=-10$

$(IV)\ (1)+3x-2y+2(-1)=-6$
$1+3x-2y-2=-6$
$3x-2y-1=-6$
$\underline{\qquad +1\quad +1}$
$D: 3x-2y=-5$

$C: 2x+2y=-10$
$D: 3x-2y=-5$
$\frac{5x}{5}=-\frac{15}{5}$
$x=-3$

$C: 2(-3)+2y=-10$
$-6+2y=-10$
$\underline{+6\qquad +6}$
$\frac{2y}{2}=-\frac{4}{2}$
$y=-2$

$(1,-3,-2,-1)$

31) $(I)\ w + x + y + z = 2$

$(II)\ w + 2x + 2y + 4z = 1$

$(III)\ -w + x - y - z = -2$

$(IV) - w + 3x + y - z = -2$

$(I)\ w + x + y + z = 2$

$\underline{(III)\ -w + x - y - z = -6}$

$\frac{2x}{2} = -\frac{4}{2}$

$x = -2$

$(III)(-1)(-w + x - y - z) = (-6)(-1)$

$(IV) - w + 3x + y - z = -2$

$\underline{w - x + y + z = 6}$

$2x + 2y = 4$

$2(-2) + 2y = 4$

$-4 + 2y = 4$

$+4 \qquad + 4$

$\frac{2y}{2} = \frac{8}{2}$

$y = 4$

$(II)\ w + 2x + 2y + 4z = 1$

$\underline{(IV) - 2 + 3x + y - z = -2}$

$5x + 3y + 3z = -1$

$5(-2) + 3(4) + 3z = -1$

$-10 + 12 + 3z = -1$

$2 + 3z = -1$

$\underline{-2 \qquad - 2}$

$\frac{3z}{3} = \frac{-3}{3}$

$z = -1$

$(I)\ w + (-2) + (4) + (-1) = 2$

$w + 1 = 2$

$\underline{-1 \quad -1}$

$w = 1$

$(1, -2, 4, -1)$

4.5

1) A collection of dimes and quarters is worth S15.25. There are 103 coins in all. How many of each is there?

N	V	T
D	10	10D
Q	25	25Q
103		1525

$(-10)(D + Q) = (103)(-10)$

$10D + 25Q = 1525$

$\underline{-10D - 10Q = -1030}$

$\frac{15Q}{15} = \frac{495}{15}$

$Q = 33$

$D + 33 = 103$

$\underline{-33 \quad - 33}$

$D = 70$

$70\ dimes$

$33\ Quarters$

3) The attendance at a school concert was 578. Admission was $2.00 for adults and $1.50 for children. The total receipts were S985.00. How many adults and how many children attended?

N	V	T
A	2	2A
C	1.5	1.5C
578		985

$-2(A + C) = (578)(-2)$

$2A + 1.5C = 985$

$\underline{-2A - 2C = -1156}$

$\frac{-0.5C}{-0.5} = \frac{-1156}{-0.5}$

$C = 342$

$236\ Adults$

$342\ Children$

$A + 342 = 578$

$\underline{-342 \quad - 342}$

$A = 236$

5) A boy has $2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind has he?

N	V	T
N	5	5N
D=2N	10	20N
		225

$$5N + 20N = 225$$
$$\frac{25N}{25} = \frac{225}{25}$$
$$N = 9$$
$$D = 2(9) = 18$$

$9\ Nickels$

$18\ Dimes$

7) A collection of 27 coins consisting of nickels and dimes amounts to $2.25. How many coins of each kind are there?

N	V	T
N	S	SN
D	10	10D
27		225

$$(-10)(N + D) = (27)(-10)$$
$$5N + 10D = 225$$
$$\underline{-10N - 10D = -270}$$
$$\frac{-5N}{-5} = \frac{-45}{-5}$$
$$N = 9$$

$$9 + D = 27$$
$$\underline{-9 \qquad -9}$$
$$D = 18$$

$18\ Dimes$

$9\ Nickels$

9) There were 429 people at a play. Admission was $1 each for adults and 75 cents each for children. The receipts were $372.50. How many children and how many adults attended?

N	V	T
A	1	A
C	.75	.75C
429		372.50

$$(-1)(A + C) = (429)(-1)$$
$$A + .75C = 372.5$$
$$\underline{-A - C \quad = -429}$$
$$\frac{-.25C}{-.25} = \frac{-56.5}{-.25}$$
$$C = 226$$

$$A + 226 = 429$$
$$\underline{-226 \ -226}$$
$$A = 203$$

$203\ Adults$

$226\ Children$

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was $1.25 each and for non-card holders the price was $2 each. The total amount of money collected was $310. How many of each type of ticket was sold?

N	V	T
A	1.25	1.25A
N	2	2N
203		310

$$-2(A + N) = (203)(-2)$$
$$1.25A + 2N = 310$$
$$\underline{-2A - 2N = -406}$$
$$\frac{-.75A}{-.75} = \frac{-96}{-.75}$$
$$A = 128$$

$$128 + N = 203$$
$$\underline{-128 \qquad -128}$$
$$N = 75$$

$75\ Non\ Card$

$128\ Activity\ Card$

13) At a recent Vikings game $445 in admission tickets was taken in. The cost of a student ticket was $1.50 and the cost of a non-student ticket was $2.50. A total of 232 tickets were sold. How many students and how many nonstudents attended the game?

N	V	T
5	1.5	1.55
N	2.5	2.5N
232		445

$$-1.5(5+N) = (232)(-1.5)$$
$$1.55 + 2.5N = 445$$
$$\underline{-1.55 - 1.5N = 348}$$
$$N = 97$$

$$S + 97 = 232$$
$$\underline{-97 \quad -97}$$
$$S = 135$$

$97\ Non - Students$

$135\ Students$

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of $1.15. Find the number of nickels and dimes in the coin purse.

N	V	T
N	5	5N
D	10	100
18		115

$$-5(N+D) = (18)(-5$$
$$5N + 10D = 115$$
$$\underline{-5N - 5D = -90}$$
$$\frac{5D}{5} = \frac{25}{5}$$
$$D = 5$$

$$N + 5 = 18$$
$$\underline{-5 \quad -5}$$
$$N = 13$$

$13\ Nickels$

$5\ Dimes$

17)) A postal clerk sold some 15¢ stamps and some 25¢ stamps. Altogether, 15 stamps were sold for a total cost of $3.15. How many of each type of stamps were sold?

N	V	T
F	15	15F
T	25	25T
15		315

$$-15(F+T) = (15)(-15)$$
$$15F + 25T = 315$$
$$\underline{-15F - 15T = -225}$$
$$\frac{10T}{10} = \frac{90}{10}$$
$$T = 9$$

$$F + 9 = 15$$
$$\underline{-9 \quad -9}$$
$$F = 6$$

$6\ Fifteen\ cents, 9\ twenty - five\ cents$

19) The total value of dimes and quarters in a bank is $6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

N	V	T
D	10	10D
Q=D+6	25	25D+150
		605

$$10D + 25D + 150D = 605$$
$$35D + 150 = 605$$
$$\underline{-150 \quad -150}$$
$$\frac{35D}{35} = \frac{455}{35}$$
$$D = 13$$

$$Q = 13 + 6$$
$$Q = 19$$

$13\ Dimes$

$19\ Quarters$

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

N	V	T
N	5	5N
D=2N−10	10	20N−100
		275

$$5N + 20N - 100 = 275$$
$$25N - 100 = 275$$
$$\underline{+100 \quad +100}$$
$$\frac{25N}{25} = \frac{375}{25}$$
$$N = 15$$

$$D = 2(15) - 10$$
$$D = 30 - 20$$
$$D = 10$$

$20\ Dimes$

$15\ Nickels$

23) A bank teller cashed a check for \$200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

N	V	T
W	20	20W
T	10	10T
12		200

$$-10(W + T) = (12)(-10)$$
$$20W + 10T = 200$$
$$\underline{-10W - 10T = -120}$$
$$\frac{10W}{10} = \frac{80}{10}$$
$$W = 8$$

$$8 + T = 12$$
$$\underline{-8 \qquad -8}$$
$$T = 4$$

$4\ Tens$

$8\ Twenties$

25) A total of \$27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3385. How much was invested at each rate?

N	V	T
x	.12	.12x
y	.13	.13y
27000		3385

$$-.12(x + y) = (27000)(-.12)$$
$$.12x + .13y = 3385$$
$$\underline{-.12x - .12y = -3240}$$
$$\frac{-.01y}{-.01} = \frac{-145}{-.01}$$
$$y = 14500$$

$$x + 14500 = 27000$$
$$\underline{-14500 \quad -14500}$$
$$x = 12500$$

\$12,500 @12%

\$14,500 @ 13%

27) A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

N	V	T
x	.10	.1x
y	.12	.12y
9000		1030

$$-.1(x + y) = (9000)(-.1)$$
$$.1 + .12y = 1030$$
$$\underline{-.1x - .1y = -900}$$
$$\frac{.02y}{.02} = \frac{130}{.02}$$
$$y = 6500$$

$$x + 6500 = 9000$$
$$\underline{-6500 \quad -6500}$$
$$x = 2500$$

\$2500 @ 10%

\$6500 @ 12%

29) An inheritance of $10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was $1038.50. How much was invested at each rate?

N	V	T
x	.095	.095x
y	.11	.11y
10000		1038.50

$$-.095(x+y) = (10000)(-.095)$$
$$.095x + .11y = 1038.50$$
$$\underline{-.095x - .095y = -950}$$
$$\frac{0.015y}{.015} = \frac{88.5}{.015}$$
$$y = 5900$$

$$x + 5900 = 10000$$
$$\underline{-5900 \quad -5900}$$
$$x = 4100$$

$4100 @ 9.5%
$5900 @11%

31) Jason earned $256 interest last year on his investments. If $1600 was invested at a certain rate of return and $2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

N	V	T
1600	x	1600x
2400	2x	4800x
		256

$$1600x + 4800x = 256$$
$$\frac{6400x}{6400} = \frac{256}{6400}$$
$$x = 0.04$$
$$2x = 0.08$$

$1600 @ 4%
$2400 @ 8%

33) A total of $8500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is $385. How much was invested at each rate?

N	V	T
x	.06	.06x
y	.035	.035y
8500		385

$$-.035(x+y) = (8500)(-.035)$$
$$.06x + .035y = 385$$
$$\underline{-.035x - .035y = -297.5}$$
$$\frac{.025x}{.025} = \frac{87.5}{.025}$$
$$x = 3500$$

$$3500 + y = 8500$$
$$\underline{-3500 \quad -3500}$$
$$y = 5000$$

$3500 @ 6%
$5000 @ 3.5%

35) A total of $15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is $1455. How much was invested at each rate?

N	V	T
x	.08	.08x
y	.11	.11y
15000		1455

$$-.08(x+y) = (15000)(-.08)$$
$$.08x + .11y = 1455$$
$$\underline{-.08x - .08y = -1200}$$
$$\frac{.03y}{.03} = \frac{255}{.03}$$
$$y = 8500$$

$$x + 8500 = 15000$$
$$\underline{-8500 \quad -8500}$$
$$x = 6500$$

$6500 @ 8%
$8500 @ 11%

37) A total of \$6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$300. How much was invested at each rate?

N	V	T
x	.0425	.0425x
y	.0575	.0575y
6000		300

$$-.0425(x+y) = (6000)(-.0425) \qquad x + 3000 = 6000$$
$$.0425x + .0575y = 300 \qquad \underline{-3000 \quad -3000}$$
$$\underline{-.0425x - .0425y = -255} \qquad x = 3000$$
$$\frac{.015y}{.015} = \frac{45}{.014}$$
$$y = 3000$$

\$3000 @ 4.25%
\$3000 @ 5.75%

39) A total of \$11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

N	V	T
x	.068	.068x
y	.082	.082y
11000		797

$$-.068(x+y) = (11000)(-.068) \qquad x + 3500 = 11000$$
$$.068x + .082y = 797 \qquad \underline{-3500 \quad -3500}$$
$$\underline{-.068x - .068y = -748} \qquad x = 7500$$
$$\frac{.014y}{.014} = \frac{49}{.014}$$
$$y = 3500$$

\$7500 @ 6.8%
\$3500 @8.2%

42) Samantha earned \$1480 in interest last year on her investments. If \$5000 was invested at a certain rate of return and \$11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

N	V	T
5000	x	5000x
11000	$\frac{2}{3}$x	$\frac{22000}{3}$x
		1480

$$3\left(5000x + \frac{22000}{3}x\right) = (1480)3$$
$$\underline{15000x + 22000x = 4440}$$
$$\frac{37000x}{37000} = \frac{4440}{37000}$$
$$x = .12$$
$$\frac{2}{3}(.12) = .08$$

\$5000 @ 12%
\$11000 @ 8%

44) 30 coins having a value of \$3.30 consists of nickels, dimes and quarters. If there are twice as many quarters as dimes, how many coins of each kind were there?

N	V	T
N	5	5N
D	10	10D
Q=2D	25	25D
30		330

$$N + D + 2D = 30 \qquad (-5)(N + 3D) = (30)(-5)$$
$$5N + 10D + 50D = 330 \qquad 5N + 60D = 330$$
$$-5N - 15D = -150$$
$$\frac{45D}{45} = \frac{180}{45}$$
$$N + 3(4) = 30 \qquad D = 4$$
$$N + 12 = 30$$
$$-12 \quad -12 \qquad Q = 2(4) = 8$$
$$N = 18$$

18 *Nickels*
4 *Dimes*
8 *Quarters*

4.6

1) A tank contains 8000 liters of a solution that is 40% acid. How much water should be added to make a solution that is 30% acid?

A	P	T
8000	.4	3200
w	0	0
8000+w	.3	2400+.3w

$$3200 = 2400 + .3w$$
$$\underline{-2400 - 2400}$$
$$\frac{800}{.3} = \frac{.3w}{.3}$$
$$w = 2{,}666.67\ L.$$

3) Of 12 pounds of salt water 10% is salt; of another mixture 3% is salt. How many pounds of the second should be added to the first in order to get a mixture of 5% salt?

A	P	T
12	.1	1.2
x	.03	.03x
12+x	.05	.6+.05x

$$1.2 + .03x = .6 + .05x$$
$$\underline{-.03x \qquad -.03x}$$
$$1.2 = .6 + .02x$$
$$\underline{-.6 \quad -.6}$$
$$\frac{.6}{.02} = \frac{.02x}{.02}$$
$$x = 30\ lbs$$

5) How many pounds of a 4% solution of borax must be added to 24 pounds of a 12% solution of borax to obtain a 10% solution of borax?

A	P	T
x	.04	.04x
24	.12	2.88
x+24	.10	.1x+.24

$$.04x + 2.88 = .1x + 2.4$$
$$\underline{-.04x \qquad -.04x}$$
$$2.88 = .06x + 2.4$$
$$\underline{-2.4 \qquad -2.4}$$
$$\frac{.48}{.06} = \frac{.06x}{.06}$$
$$x = 8\ lbs$$

7) A 100 LB bag of animal feed is 40% oats. How many pounds of oats must be added to this feed to produce a mixture which is 50% oats?

A	P	T
100	.4	40
x	1	X
100+x	.5	50+.5x

$$40 + x = 50 + .5x$$
$$\underline{-.5x \qquad -.5x}$$
$$40 + .5x = 50$$
$$\underline{-40 \qquad -40}$$
$$\frac{.5x}{.5} = \frac{10}{.5}$$
$$x = 20\ lbs$$

9) How many pounds of tea that cost \$4.20 per pound must be mixed with 12 lb of tea that cost \$2.25 per pound to make a mixture that costs \$3.40 per pound?

A	P	T
x	4.2	4.2x
12	2.25	27
x+12	3.40	3.4x+40.8

$$4.2x + 27 = 3.4x + 40.8$$
$$\underline{-3.4x \qquad -3.4x}$$
$$0.8x + 27 = 40.8$$
$$\underline{-27 \quad -27}$$
$$\frac{0.8x}{0.8} = \frac{13.8}{.8}$$
$$x = 12.25\ lbs$$

11) How many kilograms of hard candy that cost \$7.50 per kilogram must be mixed with 24 kg of jelly beans that cost \$3.25 per kilogram to make a mixture that sells for \$4.50 per kilogram?

A	P	T
x	7.5	7.5x
24	3.25	78
x+24	4.5	4.5x+108

$$7.5x + 78 = 4.5x + 108$$
$$\underline{-4.5x \qquad -4.5x}$$
$$3x + 78 = 108$$
$$\underline{-78 \quad -78}$$
$$\frac{3x}{3} = \frac{30}{3}$$
$$x = 10kg$$

13) How many pounds of lima beans that cost 90¢ per pound must be mixed with 16 lb of corn that cost 50¢ per pound to make a mixture of vegetables that costs 65¢ per pound?

A	P	T
x	.9	.9x
16	.5	8
X+16	.65	.65x+10.4

$$.9x + 8 = .65x + 10.4$$
$$\underline{-.65x \quad -.65x}$$
$$.25x + 8 = 10.4$$
$$\underline{-8 \quad -8}$$
$$\frac{.25x}{.25} = \frac{2.4}{.25}$$
$$x = 9.6\ lbs$$

15) Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100cc. of a solution that is 68% acid?

A	P	T
A	.5	.5A
B	.8	.8B
100	.68	68

$$-.5(A+B) = (100)(-.5)$$
$$.5A + .8B = 68$$
$$\underline{-.5A - .5B = -50}$$
$$\frac{.3B}{.3} = \frac{18}{.3}$$
$$B = 60$$

$$A + 60 = 100$$
$$\underline{-60 \quad -60}$$
$$A = 40$$

$60\ cc\ of\ 80\%$

$40\ cc\ of\ 50\%$

17) A farmer has some cream which is 21% butterfat and some which is 15% butter fat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

A	P	T
A	.21	.21A
B	.15	.15B
60	.19	11.4

$-.15(A+B)=(60)(-.15)$

$.21A+.15B=11.4$

$\underline{-.15A-.15B=-9}$

$\frac{.06A}{.06}=\frac{2.4}{.06}$

$A=40$

$40+B=60$

$\underline{-40\quad -40}$

$B=20$

$40\ gal\ 21\%$

$20\ gal\ 15\%$

19) A chemist wants to make 50ml of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

A	P	T
x	.13	.13x
y	.18	.18y
50	.16	8

$-.13(x+y)=(50)(-.13)$

$.13x+.18y=8$

$\underline{-.13x-.13y=-6.5}$

$\frac{.05y}{.05}=\frac{1.5}{.05}$

$y=30$

$x+30=50$

$\underline{-30\quad -30}$

$x=20$

$20\ mL\ 13\%$

$30\ mL\ 18\%$

21) A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gal of paint that is 19% green dye?

A	P	T
x	.21	.21x
y	.15	.15y
60	.19	11.4

$-.15(x+y)=(60)(-.15)$

$.21x+.15y=11.4$

$\underline{-.15x-.15y=-9}$

$\frac{.06x}{.06}=\frac{2.5}{.06}$

$x=40$

$40+y=60$

$\underline{-40\quad -40}$

$y=20$

$40\ gal\ 21\%$

$20\ gal\ 15\%$

23) To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth \$4.00/lb. with a weed killer worth \$8.00/lb. The mixture will cost \$6.00/lb. How much of each should be used to prepare 500 lb. of the mixture?

A	P	T
x	4	4x
y	8	8y
500	6	3000

$-4(x+y)=(500)(-4)$

$4x+8y=3000$

$\underline{-4x-4y=-2000}$

$\frac{4y}{4}=\frac{1000}{4}$

$y=250$

$x+250=500$

$\underline{-250\quad -250}$

$x=250$

$250\ lbs\ @\ \$4$

$250\ lbs\ @\ \$8$

25) A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

A	P	T
x	9	9x
y	6	6y
60	7	420

$$-6(x+y) = (60)(-6)$$
$$9x + 6y = 420$$
$$\underline{-6x - 6y = -360}$$
$$\frac{3x}{3} = \frac{60}{3}$$
$$x = 20$$

$$20 + y = 60$$
$$\underline{-20 \qquad -20}$$
$$y = 40$$

$20\ lbs$ @ $9¢$

$40\ lbs$ @ $6¢$

27) A goldsmith combined an alloy that costs S4.30 per ounce with an alloy that costs $1.80 per ounce. How many ounces of each were used to make a mixture of 200 oz costing $2.50 per ounce?

A	P	T
x	4.30	4.3x
y	1.80	1.80y
200	2.50	500

$$-1.8(x+y) = (200)(-1.8)$$
$$4.3x + 1.8y = 500$$
$$\underline{-1.8x - 1.8y = -360}$$
$$\frac{2.5x}{2.5} = \frac{140}{2.5}$$
$$x = 56$$

$$56 + y = 200$$
$$\underline{-56 \qquad -56}$$
$$y = 144$$

$56\ oz.$ @ $4.30

$144\ oz.$ @ $1.80

29) The manager of a garden shop mixes grass seed that is 60% rye grass with 70 lb of grass seed that is 80% rye grass to make a mixture that is 74% rye grass. How much of the 60% mixture is used?

A	P	T
x	.6	.6x
70	.8	56
x+70	.74	.74x+51.8

$$.6x + 56 = .74x + 51.8$$
$$\underline{-.6x \qquad -.6x}$$
$$56 = .45x + 51.8$$
$$\underline{-51.8 \qquad -51.8}$$
$$\frac{4.2}{.14} = \frac{.14x}{.14}$$
$$30\ lbs = x$$

31) A caterer made an ice cream punch by combining fruit juice that cost $2.25 per gallon with ice cream that costs $3.25 per gallon. How many gallons of each were used to make 100 gal of punch costing $2.50 per pound?

A	P	T
x	2.25	2.25x
y	3.25	3.25y
100	2.5	250

$$-2.25(x+y) = (100)(-2.25)$$
$$2.25x + 3.25y = 250$$
$$\underline{-2.25x - 2.25y = -225}$$
$$y = 25$$

$$x + 25 = 100$$
$$\underline{-25 \quad -25}$$
$$x = 75$$

$75\ gal$ @ $2.25

$25\ gal$ @ $3.25

33) A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 lb of a fabric that is 28% wool?

A	P	T
x	.2	.2x
y	.5	.5y
600	.28	168

$$-.2(x+y) = (600)(-.2)$$
$$.2x + .5y = 168$$
$$-2.x - .2y = -120$$
$$\frac{.3y}{.3} = \frac{48}{.3}$$
$$y = 160$$

$$x + 160 = 600$$
$$-160 - 160$$
$$x = 440$$

$440\ lbs$ @ 20%
$160\ lbs$ @ 50%

35) The manager of a specialty food store combined almonds that cost \$4.50 per pound with walnuts that cost S2.50 per pound. How many pounds of each were used to make a 100 lb mixture that cost \$3.24 per pound?

A	P	T
x	4.50	4.5x
y	2.50	2.5y
100	3.24	324

$$-2.5(x+y) = (100)(-2.5)$$
$$4.5x + 2.5y = 324$$
$$-2.5x - 2.5y = -250$$
$$\frac{2x}{2} = \frac{74}{2}$$
$$x = 37$$

$$37 + y = 100$$
$$-37 \quad -37$$
$$y = 63$$

$37\ lbs$ @ $\$4.50$
$63\ lbs$ @ $\$2.50$

37) How many ounces of dried apricots must be added to 18 oz of a snack mix that contains 20% dried apricots to make a mixture that is 25% dried apricots?

A	P	T
x	1	x
18	.2	3.6
x+18	.25	.25x+4.5

$$x + 3.6 = .25x + 4.5$$
$$-.25x \quad -.25x$$
$$.75x + 3.6 = 4.5$$
$$-3.6 \quad -3.6$$
$$\frac{.75x}{.75} = \frac{0.9}{.75}$$
$$x = 1.2\ oz$$

39) How many ounces of pure bran flakes must be added to 50 oz. of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

A	P	T
x	1	x
50	.4	20
x+50	.5	.5x+25

$$x + 20 = .5x + 25$$
$$-.5x \quad -.5x$$
$$.5x + 20 = 25$$
$$-20 \quad -20$$
$$\frac{.5x}{.5} = \frac{5}{.5}$$
$$x = 10\ oz$$

41) How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

A	P	T
w	0	0
50	1	50
w+50	.4	.4w+20

$$50 = .4w + 20$$
$$\underline{-20 \qquad\qquad -20}$$
$$\frac{30}{.4} = \frac{.4w}{.4}$$
$$75g = w$$

43) How many ounces of pure water must be added to 50 oz of a 15% saline solution to make a saline solution that is 10% salt?

A	P	T
x	0	0
50	.15	7.5
x+50	.10	.1x+5

$$7.5 = .1x + 5$$
$$\underline{-5 \qquad\qquad -5}$$
$$\frac{2.5}{.1} = \frac{.1x}{.1}$$
$$25oz = x$$

Chapter 5: Polynomials

5.1

1) $4 \cdot 4^4 \cdot 4^4 = 4^9$

3) $4 \cdot 2^2 = 2^2 \cdot 2^2 = 2^4$

5) $3m \cdot 4mn = 12m^2n$

7) $2m^4n^2 \cdot 4nm^2 = 8m^6n^3$

9) $(3^3)^4 = 3^{12}$

11) $(4^4)^2 = 4^8$

13) $(2u^3v^2)^2 = 4u^6v^4$

15) $(2a^4)^4 = 2^4a^{16} = 16a^{16}$

17) $\frac{4^5}{4^3} = 4^2$

19) $\frac{3^2}{3} = 3$

21) $\frac{3nm^2}{3n} = m^2$

23) $\frac{4x^3y^4}{3xy^3} = \frac{4x^2y}{3}$

25) $(x^3y^4 \cdot 2x^2y^3)^2$
$(2x^5y^7)^2$
$2^2x^{10}y^{14}$
$4x^{10}y^{14}$

27) $2x(x^4y^4)^4$
$2x(x^{16}y^{16})$
$2x^{17}y^{16}$

29) $\frac{2x^7y^5}{3x^3y \cdot 4x^2y^3} = \frac{2x^7y^5}{12x^5y^4} = \frac{x^2y}{6}$

31) $\left(\frac{(2x)^3}{x^3}\right)^2 = \left(\frac{2^3x^3}{x^3}\right)^2 = \left(\frac{8x^3}{x^3}\right)^2 = 8^2 = 64$

33) $\left(\frac{2y^{17}}{(2x^2y^4)^4}\right)^3 = \left(\frac{2y^{17}}{2^4x^8y^{16}}\right)^3 = \left(\frac{2y^{17}}{16x^8y^{16}}\right)^3 =$
$\left(\frac{y}{8x^8}\right)^3 = \frac{y^3}{8^3x^{24}} = \frac{y^3}{512x^{24}}$

35) $\left(\frac{2mn^4 \cdot 2m^4n^4}{mn^4}\right)^3 = \left(\frac{4m^5n^8}{mn^4}\right)^3 = (4m^4n^4)^3 =$
$4^3m^{12}n^{12} = 64m^{12}n^{12}$

37) $\frac{2xy^5 \cdot 2x^2y^3}{2xy^4 \cdot y^3} = \frac{4x^3y^8}{2xy^7} = 2x^2y$

39) $rr\frac{q^3r^2 \cdot (2p^2q^2r^3)^2}{2p^3} = \frac{q^3r^2(2^2p^4q^4r^2)}{2p^3} =$
$\frac{q^3r^2 \cdot 4p^4p^4r^6}{2p^3} = \frac{4p^4q^7r^8}{2p^3} = 2pq^7r^8$

41) $\left(\frac{zy^3 \cdot z^3x^4y^4}{x^3y^3z^3}\right)^4 = \left(\frac{z^4y^7x^4}{x^3y^3z^3}\right)^4 = (xy^4z)^4 =$
$x^4y^{16}z^4$

43) $\frac{2x^2y^2z^6 \cdot 2zx^2y^2}{(x^2z^3)^2} = \frac{4x^4y^4z^7}{x^4z^6} = 4y^4z$

5.2

1) $2x^4y^{-2}(2xy^3)^4 = 2x^4y^{-2}(2^4x^4y^{12}) =$
$2^5x^8y^{10} = 32x^8y^{10}$

3) $(a^4b^{-3})^3\ 2a^3b^{-2} = a^{12}b^9 \cdot 2a^3b^{-2} =$
$2a^{15}b^{-11} = \frac{2a^{15}}{b^{11}}$

5) $(2x^2y^2)^4x^{-4} = 2^4x^8y^8x^{-4} = 16x^4y^8$

7) $(x^3y^4)\&3\ \ x^{-4}y^4 = x^9y^{12}x^{04}y^4 = x^5y^{16}$

9) $\frac{2x^{-3}y^2}{3x^{-3}y^3\cdot 3x^0} = \frac{2y^2x^3}{x^3\cdot 3y^3\cdot 3x^0} = \frac{2y^3x^3}{9x^3y^3} = \frac{2}{9y}$

11) $\frac{4xy^{-3}\cdot x^{-4}y^0}{4y^{-1}} = \frac{4xy^0y}{4y^3x^4} = \frac{4xy}{4y^3x^4} = \frac{1}{x^3y^2}$

13) $\frac{u^2v^{-1}}{2u^0v^4\cdot 2uv} = \frac{u^2}{v\cdot 2u^0v^4\cdot 2uv} = \frac{u^2}{4uv^6} = \frac{u}{4v^6}$

15) $\frac{u^2}{4u^0v^3\cdot 3v^2} = \frac{u^2}{12v^5}$

17) $\frac{2y}{(x^0y^2)^4} = \frac{2y}{x^0y^8} = \frac{2}{y^7}$

19) $\left(\frac{2a^2b^3}{a^{-1}}\right)^4 = (2a^2a\ b^3)^4 = (2a^3b^3)^4 =$
$2^4a^{12}b^{12} = 16a^{12}b^{12}$

21) $\frac{2nm^4}{(2m^2n^2)^4} = \frac{2nm^4}{2^4m^8n^8} = \frac{1}{2^3m^4n^7} = \frac{1}{8m^4n^7}$

23) $\frac{(2mn)^4}{m^0n^{-2}} = \frac{2^4m^4n^4}{m^0n^{-2}} = 2^4m^4n^4n^2 = 16m^4n^6$

25) $\frac{y^3\cdot x^{-3}y^2}{(x^4y^2)^3} = \frac{y^3x^{-3}y^2}{x^{12}y^6} = \frac{y^3y^2}{x^3x^{12}y^6} = \frac{y^5}{x^{15}y^6} =$
$\frac{1}{x^{15}y}$

27) $\frac{2u^{-2}v^3(2uv^4)^{-1}}{2u^{-4}v^0} = \frac{2u^{-2}v^3\cdot 2^{-1}u^{-1}v^{-4}}{2u^{-4}v^0} =$
$\frac{2v^3\cdot u^4}{u^2 2uv^4\cdot 2v^0} = \frac{2v^3u^4}{4u^3v^4} = \frac{u}{2v}$

29) $\left(\frac{2x^0y^4}{y^4}\right)^3 = (2)^3 = 8$

31) $\frac{y(2x^4y^2)^2}{2x^4y^0} = \frac{y(2^2x^8y^4)}{2x^4y^0} = \frac{4x^8y^5}{2x^4y^0} = 2x^4y^5$

33) $\frac{2yzx^2}{2x^4y^4z^{-2}(zy^2)^4} = \frac{2yzx^2}{2x^4y^4z^{-2}z^4y^8} = \frac{2yz^3x^2}{2x^4y^{12}z^4} =$
$\frac{1}{x^2y^{11}z}$

35) $\frac{2kh^0\cdot 2h^{-3}k^0}{(2kj^3)^2} = \frac{2kh^0\cdot 2h^{-3}k^0}{2^2k^2j^6} = \frac{2k\cdot 2}{h^3\cdot 4k^2j^6} =$
$\frac{4k}{4k^2h^3j^6} = \frac{1}{kh^3j^6}$

37) $\frac{(cb^3)^2\cdot 2a^{-3}b^2}{(a^3b^{-2}c^3)^3} = \frac{c^2b^6\cdot 2a^{-3}b^2}{a^9b^{-6}c^9} = \frac{c^2b^6 2b^2b^6}{a^3a^9c^9} =$
$\frac{2b^{14}c^2}{a^{12}c^9} = \frac{2b^{14}}{a^{12}c^7}$

39) $\frac{(yx^{-4}z^2)^{-1}}{z^3x^2y^3z^{-1}} = \frac{y^{-1}x^4z^{-2}}{x^3x^2y^3z^{-1}} = \frac{x^4z}{yz^2x^3x^2y^3} =$
$\frac{x^4z}{x^2y^4z^5} = \frac{x^2}{y^4z^4}$

5.3

1) 885
$8.85\ x\ 10^2$

3) 0.081
$8.1\ x\ 10^{-2}$

5) 0.039
$3.9\ x\ 10^{-2}$

7) $8.7\ x\ 10^5$
870,000

9) $9\ x\ 10^{-4}$
0.0009

11) $2\ x\ 10^0$
2

13) $(7\ x\ 10^{-1})(2\ x\ 10^{-3})$
$14\ x\ 10^{-4}$
$1.4\ x\ 10^1 x\ 10^{-4}$
$1.4\ x\ 10^{-3}$

15) $(5.26\ x\ 10^{-5})(3.16\ x\ 10^{-2})$
$16.6216\ x\ 10^{-7}$
$1.66216\ x\ 10^{1}\ x\ 10^{-7}$
$1.66216\ x\ 10^{-6}$

17) $(2.6\ x\ 10^{-2})(6\ x\ 10^{-2})$
$15.6\ x\ 10^{-4}$
$1.56\ x\ 10^{1} x\ 10^{-4}$
$1.56\ x\ 10^{-3}$

19) $\frac{4.9\ x\ 10^{1}}{2.7\ x\ 10^{-3}} = 1.81\ x\ 10^{4}$

21) $\frac{5.33\ x\ 10^{-6}}{9.62\ x\ 10^{-2}} = 0.554\ x\ 10^{-4} =$
$5.54\ x\ 10^{-1}\ x\ 10^{-4} = 5.54\ x\ 10^{-5}$

23) $(5.5\ x\ 10^{-5})^{2}$
$30.25\ x\ 10^{-10}$
$3.025\ x\ 10^{1} x\ 10^{-10}$
$3.025\ x\ 10\hat{} - 9$

25) $(7.8\ x\ 10^{-2})^{5}$
$28.872\ x\ 10^{-10}$
$2.8872\ x\ 10^{1}\ x\ 10^{-10}$
$2.8872\ x\ 10^{-9}$

27) $(8.03\ x\ 10^{4})^{-4}$
$0.000241\ x\ 10^{-16}$
$2.41\ x\ 10^{-4} x\ 10^{-16}$
$2.41\ x\ 10^{-20}$

29) $\frac{6.1\ x\ 10^{-6}}{5.1\ x\ 10^{-4}} = 1.196\ x\ 10^{-2}$

31) $(3.6\ x\ 10^{0})(6.1\ x\ 10^{-3})$
$21.96\ x\ 10^{-3}$
$2.196\ x\ 10^{1} x\ 10^{-3}$
$2.196\ x\ 10^{-2}$

33) $(1.8\ x\ 10^{-5})^{-3}$
$0.1715\ x\ 10^{15}$
$1.715\ x\ 10^{-1}\ x\ 10^{15}$
$1.715\ x\ 10^{14}$

35) $\frac{9\ x\ 10^{4}}{7.83\ x\ 10^{-2}} = 1.149\ x\ 10^{6}$

37) $\frac{3.22\ x\ 10^{-3}}{7\ x\ 10^{-6}} = 0.46\ x\ 10^{3} =$
$4.6\ x\ 10^{-1} x\ 10^{3} = 4.6\ x\ 10^{2}$

39) $\frac{2.4\ x\ 106-6}{6.5\ x\ 10^{0}} = 0.3692\ x\ 10^{-6} =$
$3.692\ x\ 10^{-1}\ x\ 10^{-6} = 3.692\ x\ 10^{-7}$

41) $\frac{6\ x\ 10^{3}}{5.8\ x\ 10^{-3}} = 1.034\ x\ 10^{6}$

5.4

1) $-a^3a^2 + 6a - 21\ at\ a = -4$
$-(-4)^3 - (-4)^2 + 6(-4) - 21$
$-(-64) - (16) + 6(-4) - 21$
$64 - 16 - 24 - 21$
3

3) $n^3 - 7n^2 + 15n - 20\ when\ n = 2$
$(2)^3 - 7(2)^2 + 15(2) - 20$
$8 - 7(4) + 15(2) - 20$
$8 - 28 + 30 - 20$
-10

5) $-5n^4 - 11n^3 - 9n^2 - n - 5\ when\ n = -1$
$-5(-1)^4 - 11(-1)^3 - 9(-1)^2 - (-1) - 5$
$-5(1) - 11(-1) - 9(1) - 1(-1) - 5$
$-5 + 11 - 9 + 1 - 5$
-7

7) $x^2 + 9x + 23\ \ when\ x = -3$
$(-3)^2 + 9(-3) + 23$
$9 + 9(-3) + 23$
$9 - 27 + 23$
5

9) $x^4 - 6x^3 + x^2 - 24 \text{ when } x = 6$
$(6)^4 - 6(6)^3 + (6)^2 - 24$
$1296 - 6(216 + 36 - 24$
$1296 - 1296 + 36 - 24$
12

11) $(5p - 5p^4) - (8p - 8p^4)$
$5p - 5p^5 - 8p + 8p^4$
$3p^4 - 3p$

13) $(3n^2 + n^3) - (2n^3 - 7n^2)$
$3n^2 + n^3 - 2n^3 - 7n^2$
$-n^3 + 10n^2$

15) $(8n + n^4) - (3n - 4n^4)$
$8n + n^4 - 3n + 4n^4$
$5n^4 + 5n$

17) $(1 + 5p^3) - (1 - 8p^3)$
$1 + 5p^3 - 1 + 8p^3$
$13p^3$

19) $(5n^4 + 6n^3) + (8 - 3n^3 - 5n^4)$
$3n^3 + 8$

21) $(3 + b^4) + (7 + 2b + b^4)$
$2b^4 + 2b + 10$

23) $(8x^3 + 1) - (5x^4 - 6x^3 + 2)$
$8x^3 + 1 - 5x^4 + 6x^3 - 2$
$-5x^4 + 14x^3 - 1$

25) $(2a + 2a^4) - (3a^2 - 5a^4 + 4a)$
$2a + 2a^4 - 3a^2 + 5a^4 - 4a$
$7a^4 - 3a^2 - 2a$

27) $(4p^2 - 3 - 2p) - (3p^2 - 6p + 3)$
$4p^2 - 3 - 2p - 3p^2 + 6p - 3$
$p^2 + 4p - 6$

29) $(4b^3 + 7b^2 - 3) + (8 + 5b^2 + b^3)$
$5b^3 + 12b^2 + 5$

31) $(3 + 2n^2 + 4n^4) + (n^3 - 7n^2 - 4n^4)$
$n^3 - 5n^2 + 3$

33) $(n - 5n^4 + 7) + (n^2 - 7n^4 - n)$
$-12n^4 + n^2 + 7$

35) $(8r^4 - 5r^3 + 5r^2) + (2r^2 + 2r^3 - 7r^4 + 1)$
$r^4 - 3r^3 + 7r^2 + 1$

37) $(2n^2 + 7n^4 - 2) + (2 + 2n^3 + 4n^2 + 2n^4)$
$9n^4 + 2n^3 + 6n^2$

39) $(8 - b + 7b^3) - (3b^4 + 7b - 8 - 7b^2) + (3 - 3b + 6b^3)$
$8 - b + 7b^3 - 3b^4 - 7b + 8 + 7b^2 + 3 - 3b + 6b^3$
$-3b^4 + 13b^3 - 7b^2 - 11b + 19$

41) $(8x^4 + 2x^3 + 2x) + (2x + 2 - 2x^3 - x^4) - (x^3 + 5x^4 + 8x)$
$8x^4 + 2x^3 + 2x + 2x + 2 - 2x^3 - x^4 - x^3 - 5x^4 - 8x$
$2x^4 - x^3 - 4x + 2$

5.5

1) $6(p-7)$
$6p-42$

3) $2(6x+3)$
$12x+6$

5) $5m^4(4m+4)$
$20m^5+20m^4$

7) $(4n+6)(8n+8)$
$32n^2+32n+48n+48$
$32n^2+80n+48$

9) $(8b+3)(7b-5)$
$56b^2-40b+21b-15$
$56b^2-19b-15$

11) $(4x+5)(2x+3)$
$8x^2+12x+10x+15$
$8x^2+22x+15$

13) $(3v-4)(5v-2)$
$15v^2-6v-20v+8$
$15v^2-26v+8$

15) $(6x-7)(4x+1)$
$24x^2+6x-28x-7$
$24x^2-22x-7$

17) $(5x+y)(6x-4y)$
$30x^2-20xy+6xy-4y^2$
$30x^2-14xy-4y^2$

19) $(x+3y)(3x+4y)$
$3x^2-4xy+9xy+12y^2$
$3x^2+13xy+12y^2$

21) $(7x+5y)(8x+3y)$
$56x^2+21xy+40xy+15y^2$
$56x^2+61xy+15y^2$

23) $(r-7)(6r^2-4+5)$
$6r^3-r^2+5r-42r^2+7r-35$
$6r^3-43r^2+12r-35$

25) $(6n-4)(2n^2-2n+5)$
$12n^3-12n^2+30n-8n^2+8n-20$
$12n^3-20n^2+38n-20$

27) $(6x+3y)(6x^2-7xy+4y^2)$
$36x^3-42x^2y+24xy^2+18x^2y-21xy^2+12y^3$
$36x^3-24x^2y+3xy^2+12y^3$

29) $(8n^2+4n+6)(6n^2-6n+6)$
$48n^4-40n^3+48n^2+24n^3-20n^2+24n+36n^2-30n+36$
$48n^4-16n^3+64n^2-6n+36$

31) $(5k^2+3k+3)(3k^2+3k+6)$
$15k^4+15k^3+30k^2+9k^3+9k^2+18k+9k^2+9k+18$
$15k^4+24k^3+48k^2+27k+18$

33) $3(3x-4)(2x+1)$
$3(6x^2+3x-8x-4)$
$3(6x^2-5x-5)$
$18x^2-15x-12$

35) $3(2x+1)(4x-5)$
$3(8x^2-10x+4x-5)$
$3(8x^2-6x-5)$
$24x^2-18x-15$

37) $7(x-5)(x-2)$
$7(x^2-2x-5x+10)$
$7(x^2-7x+10)$
$7x^2-49x+70$

39) $6(4x-1)(4x+1)$
$6(16x^2+4x-4x-1)$
$6(16x^2-1)$
$96x^2-6$

5.6

1) $(x+8)(x-8)$
x^2-64

3) $(1+3p)(1-3p)$
$1-9p^2$

5) $(1-7n)(1+7n)$
$1-49n^2$

7) $(5n-8)(5n+8)$
$25n^2-64$

9) $(4x+8)(4x-8)$
$16x^2-64$

11) $(4y-x)(4y+x)$
$16y^2-x^2$

13) $(4m-8n)(4m+8n)$
$16n^2-64n^2$

15) $(6x-2y)(6x+2y)$
$36x^2-4y^2$

17) $(a+5)^2$
$2(5a)=10a$
$a^2+10a+25$

19) $(x-8)^2$
$2(-8x)=-16x$
$x^2-16x+64$

21) $(p+7)^2$
$2(7p)=14p$
$p^2+14p+49$

23) $(7-5n)^2$
$2(-35n)=-70n$
$49-70n+25n^2$

25) $(5m-8)^2$
$2(-40m)=-80m$
$25m^2-80m+64$

27) $(5x+7y)^2$
$2(35xy)=70xy$
$25x^2+70xy+49y^2$

29) $(2x+2y)^2$
$2(4xy)=8xy$
$4x^2+8xy+4x=y^2$

31) $(5+2r)^2$
$2(10r)=20r$
$25+20r+4r^2$

33) $(2+5x)^2$
$2(10x)=20x$
$4+20x+25x^2$

35) $(4v-7)(4v+7)$
$16v^2-49$

37) $(n-5)(n+5)$
n^2-25

39) $(4k+2)^2$
$2(8k)=16k$
$16k^2+16k+4$

5.7

1) $\frac{20x^4+x^3+2x^2}{4x^3}=\frac{20x^4}{4x^3}+\frac{x^3}{4x^3}+\frac{2x^2}{4x^3}=5x+\frac{1}{4}+\frac{1}{2x}$

3) $\frac{20n^4+n^3+40n^2}{10n}=\frac{20n^4}{10n}+\frac{n^3}{10n}+\frac{40n^2}{10n}=2n^3+\frac{n^2}{10}+4n$

5) $\frac{12x^4+24x^3+3x^2}{6x}=\frac{12x^4}{6x}+\frac{24x^3}{6x}+\frac{3x^2}{6x}=2x^3+4x^2+\frac{x}{2}$

7) $\frac{10n^4+50n^3+2n^2}{10n^2} = \frac{10n^4}{10n^2} + \frac{50n^3}{10n^2} + \frac{2n^2}{10n^2} = n^2 + 5n + \frac{1}{5}$

9) $\frac{x^2-2x-71}{x+8}$

$$\begin{array}{r} x - 10 + \frac{9}{x+8} \\ x+8\,\overline{)\,x^2 - 2x - 71} \\ \underline{-x^2 + (-8x)} \\ -10x - 71 \\ \underline{+10x + 80} \\ 9 \end{array}$$

11) $\frac{n^2+13n+32}{n+5}$

$$\begin{array}{r} n + 8 - \frac{8}{n+5} \\ n+5\,\overline{)\,n^2 + 13n + 32} \\ \underline{-n^2 - 5n} \\ 8n + 32 \\ \underline{-8n - 40} \\ -8 \end{array}$$

13) $\frac{v^2-2v-89}{v-10}$

$$\begin{array}{r} v + 8 - \frac{9}{v-10} \\ v-10\,\overline{)\,v^2 - 2v - 89} \\ \underline{-v^2 + 10v} \\ 8v - 89 \\ \underline{-8v + 80} \\ 9 \end{array}$$

15) $\frac{a^2-4a-38}{a-8}$

$$\begin{array}{r} a + 4 - \frac{6}{a-8} \\ a-8\,\overline{)\,a^2 - 4a - 38} \\ \underline{-a^2 + 8a} \\ 4a - 38 \\ \underline{-4a + 32} \\ -6 \end{array}$$

17) $\frac{45p^2+56p+19}{9p+4}$

$$\begin{array}{r} 5p + 4 + \frac{3}{9p+4} \\ 9p+4\,\overline{)\,45p^2 + 56p + 19} \\ \underline{-45p^2 - 20p} \\ 36p + 19 \\ \underline{-36p - 16} \\ 3 \end{array}$$

19) $\frac{10x^2-32x+9}{10x-2}$

$$\begin{array}{r l}
 & x-3+\frac{3}{10x-2} \\
10x-2 & \overline{)10x^2-32x+9} \\
 & \underline{-10x^2+2x} \\
 & \quad -30x+9 \\
 & \quad \underline{+30x-6} \\
 & \qquad 3
\end{array}$$

21) $\frac{4r^2-r-1}{4r+3}$

$$\begin{array}{r l}
 & r-1+\frac{2}{4r+3} \\
4r+3 & \overline{)4r^2-r-1} \\
 & \underline{-4r^2-3r} \\
 & \quad -4r-1 \\
 & \quad \underline{+4r+3} \\
 & \qquad 2
\end{array}$$

23) $\frac{n^2-4}{n-2}$

$$\begin{array}{r l}
 & n+2 \\
n-2 & \overline{)n^2-0n-4} \\
 & \underline{-n^2-2n} \\
 & \quad -2n-4 \\
 & \quad +\underline{2n+4} \\
 & \qquad 0
\end{array}$$

25) $\frac{27b^2+87b+35}{3b+8}$

$$\begin{array}{r l}
 & 9b+5-\frac{5}{3b+8} \\
3b+8 & \overline{)27b^2+87b+35} \\
 & \underline{-27b^2-72b} \\
 & \quad 15b+35 \\
 & \quad \underline{-15b-40} \\
 & \qquad -5
\end{array}$$

27) $\frac{4x^2-33x+28}{4x-5}$

$$\begin{array}{r l}
 & x-7-\frac{7}{4x-5} \\
4x-5 & \overline{)4x^2-33x+28} \\
 & \underline{-4x^2+5x} \\
 & \quad 28x+28 \\
 & \quad \underline{-28x-35} \\
 & \qquad -7
\end{array}$$

29) $\frac{a^3+15a^2+49a-55}{a+7}$

$$
\begin{array}{r}
a^2+8a-7-\frac{6}{a+7} \\
a+7\,\overline{\big)\,a^3+15a^2+49a-55} \\
\underline{-a^3-7a^2} \qquad\qquad\quad \\
8a^2+49a \qquad\quad \\
-\underline{8a^2-56a} \qquad\quad \\
-7a-55 \\
\underline{+7a+55} \\
0
\end{array}
$$

31) $\frac{x^3-26x-41}{x+4}$

$$
\begin{array}{r}
x^2-4x-10-\frac{1}{x+4} \\
x+4\,\overline{\big)\,x^3-0x^2-26x-41} \\
\underline{-x^3-4x^2} \qquad\qquad\quad \\
-4x^2-26x \qquad \\
+\underline{4x^2+16x} \qquad \\
-10x-41 \\
\underline{+10x+40} \\
1
\end{array}
$$

33) $\frac{3n^3+9n^2-64n-68}{n+6}$

$$
\begin{array}{r}
3n^2-9n-10-\frac{8}{n+6} \\
n+6\,\overline{\big)\,3n^3+9n^2-64n-68} \\
\underline{-3n^3-18n^2} \qquad\qquad\quad \\
-9n^2-64n \qquad \\
+\underline{9n^2+54n} \qquad \\
-10n-68 \\
\underline{+10n+60} \\
-8
\end{array}
$$

35) $\frac{x^3-46x+22}{x+7}$

$$
\begin{array}{r}
x^2-7x+3+\frac{1}{x+7} \\
x+7\,\overline{\big)\,x^3+0x^2-46x+22} \\
\underline{-x^3-7x} \qquad\qquad\quad \\
-7x-46x \qquad \\
\underline{+7x+49x} \qquad \\
3x+22 \\
\underline{-3x-21} \\
1
\end{array}
$$

37) $\frac{9p^3+45p^2+27p-5}{9p+9}$

$$p^2+4p-1+\frac{4}{9p+9}$$
$$9p+9\,\overline{)\,9p^3+45p^2+27p-5}$$
$$\underline{-9p^3-9p}$$
$$36p^2+27p$$
$$-\underline{36p^2-36p}$$
$$-9p-5$$
$$\underline{+9p+9}$$
$$4$$

39) $\frac{r^3-r^2-16r+8}{r-4}$

$$r^2+3r-4-\frac{8}{r-4}$$
$$r-4\,\overline{)\,r^3-r^2-16r+8}$$
$$\underline{-r^3+4r^2}$$
$$3r^2-16r$$
$$-3\underline{r^2+12r}$$
$$-4r+8$$
$$\underline{+4r-16}$$
$$-8$$

41) $\frac{12n^3+12n^2-15n-4}{2n+3}$

$$6n^2-3n-3+\frac{5}{2n+3}$$
$$2n+3\,\overline{)\,12n^3+12n^2-15n-4}$$
$$\underline{-12n^3-18n^2}$$
$$-6n^2-15n$$
$$\underline{+6n^2+9n}$$
$$-6n-4$$
$$\underline{+6n+9}$$
$$5$$

43) $\frac{4v^3-21v^2+6v+19}{4v+3}$

$$v^2-6v+6+\frac{1}{4v+3}$$
$$4v+3\,\overline{)\,4v^3-21v^2+6v+19}$$
$$\underline{-4v^3-3v^2}$$
$$-24v^2+6v$$
$$+\underline{24v^2+18v}$$
$$24v+19$$
$$-\underline{24v-18}$$
$$1$$

Chapter 6: Factoring

6.1

1) $9 + 8x$
$1(9 + 8x)$

3) $45x^2 - 25$
$5(9x^2 - 5)$

5) $56 - 35p$
$7(8 - 5p)$

7) $7ab - 36a^2b$
$7ab(1 - 5a)$

9) $-3a^2b + 6a^3b^2$
$-3a^2b(1 - 2ab)$

11) $-5x^2 - 5x^3 - 15x^4$
$-5x^2(1 + x + 3x^2)$

13) $20x^4 - 30x + 30$
$10(2x^4 - 3x + 3)$

15) $28m^4 + 40m^3 + 8$
$4(7m^4 + 10m^3 + 2)$

17) $30b^9 + 5ab - 15a^2$
$5(6b^9 + ab - 3a^2)$

19) $-48a^2b^2 - 56a^3b - 56a^5b$
$-8a^2b(6b + 7a + 7a^3)$

21) $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
$5x^3y^2z(4x^5z + 3x^2 + 7y)$

23) $50x^2y + 10y^2 + 70xz^2$
$10(5x^2y + y^2 + 7xz^2)$

25) $30qpr - 5qp + 5q$
$5q(6pr - p + 1)$

27) $-18n^5 + 3n^3 - 21n + 3$
$-3(6n^5 - n^3 + 7n - 1)$

29) $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
$-10x^{11}(4 + 22x - 5x^2 + 5x^3)$

31) $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
$-4mn(8n^7 - m^5 - 3n^3 - 4)$

6.2

1) $40r^3 - 8r^2 - 25r + 5$
$8r^2(5r - 1) - 5(5r - 1)$
$(5r - 1)(8r^2 - 5)$

3) $3n^2 - 2n^2 - 9n + 6$
$n^2(3n - 2) - 3(3n - 2)$
$(3n - 2)(n^2 - 3)$

5) $15b^3 + 21b^2 - 35b - 49$
$3b^2(5b + 7) - 7(5b + 7)$
$(5b + 7)(3b^2 - 7)$

7) $3x^3 + 15x^2 + 2x + 10$
$3x^2(x + 5) + 2(x + 5)$
$(x + 5)(3x^2 + 2)$

9) $35x^3 - 28x^2 - 20x + 16$
$7x^2(5x - 4) - 4(5x - 4)$
$(5x - 4)(7x^2 - 4)$

11) $7xy - 49x + 5y - 35$
$7x(y - 7) + 5(y - 7)$
$(y - 7)(7x + 5)$

13) $32xy + 40x^2 + 12y + 15x$
$8x(4y + 5x) + 3(4y + 5x)$
$(4y + 5x)(8x + 3)$

15) $16xy - 56x + 2y - 7$
$8x(2y - 7) + 1(2y - 7)$
$(2y - 7)(8x + 1)$

17) $2xy - 8x^2 + 7y^3 - 28y^2x$
$2x(y - 4x) + 7y^2(y - 4x)$
$(y - 4x)(2x + 7y^2)$

19) $40xy + 35x - 8y^2 - 7y$
$5x(8y + 7) - y(8y + 7)$
$(8y + 7)(5x - y)$

21) $32uv - 20u + 24v - 15$
$4u(8v - 5) + 3(8v - 5)$
$(8v - 5)(4u + 3)$

23) $10xy + 30 + 25x + 12y$
$10(xy + 3) + 1(25x + 12y)$
$No!$
$10xy + 25x + 12y + 30$
$5x(2y + 5) + 6(2y + 5)$
$(2y + 5)(5x + 6)$

25) $3uv + 14u - 6u^2 - 7v$
$u(3v + 14) - 1(6u^2 + 7v)$
$No!$
$3uv - 6u^2 - 7v + 14u$
$3u(v - 2u) - 7(v - 2u)$
$(v - 2u)(3u - 7)$

27) $16xy - 3x - 6x^2 + 8y$
$x(16y - 3) - 1(6x^2 - 8y)$
$No!$
$16xy - 6x^2 + 8y - 3x$
$2x(8y - 3x) + 1(8y - 3x)$
$(8y - 3x)(2x + 1)$

6.3

1) $p^2 + 17p + 72$
$(p + 8)(p + 9)$
72
8 9
17

3) $n^2 - 9n + 8$
$(n - 8)(n - 1)$
8
-8 -1
-9

5) $x^2 - 9x - 10$
$(x - 10)(x + 1)$
-10
-10 1
-9

7) $b^2 + 12b + 32$
$(b + 4)(b + 8)$
32
4 8
12

9) $x^2 + 3x - 70$
$(x + 10)(x - 7)$
-70
10 -7
3

11) $n^2 - 8n + 15$
$(n - 3)(n - 5)$
15
-3 -5
-8

13) $p^2 + 15p + 54$
$(p + 9)(p + 6)$
54
9 6
15

15) $n^2 - 15n + 56$
$(n - 7)(n - 8)$
56
-7 -8
-15

17) $u^2 - 8uv + 15v^2$
$(u - 5v)(u - 3v)$
15
-5 -3
-8

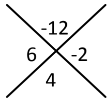
19) $m^2 - 2mn - 8n^2$
$(m + 4n)(m - 2n)$
-8
4 -2
2

21) $x^2 - 11xy - 12y^2$
$(x - 9y)(x - 2y)$
18
-9 -2
-11

24) $x^2 + xy - 12y^2$
$(x + 4y)(x - 3y)$
-12
4 -3
1

26) $x^2 + 4xy - 12y^2$

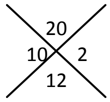
$(x + 6y)(x - 2y)$



27) $5a^2 + 60a + 100$

$5(a^2 + 12a + 20)$

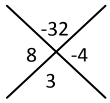
$5(a + 10)(a + 2)$



29) $6a^2 + 24a - 192$

$6(a^2 + 4a - 32)$

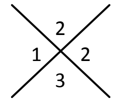
$6(a + 8)(a - 4)$



31) $6x^2 + 18xy + 12y^2$

$6(x^2 + 3xy + 2y^2)$

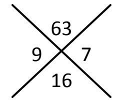
$6(x + y)(x + 2y)$



35) $6x^2 + 96xy + 378y^2$

$6(x^2 + 16xy + 63y^2)$

$6(x + 9y)(x + 7y)$



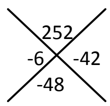
6.4

1) $7x^2 - 48x + 36$

$7x^2 - 6x - 42x + 36$

$x(7x - 6) - 6(7x - 6)$

$(7x - 6)(x - 6)$

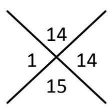


3) $7b^2 + 15b + 2$

$7b^2 + b + 14b + 2$

$b(7b + 1) + 2(7b + 1)$

$(7b + 1)(b + 2)$

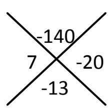


5) $5a^2 - 13a - 28$

$5a^2 + 7a - 20a - 28$

$a(5a + 7) - 4(5a + 7)$

$(5a + 7)(a - 4)$

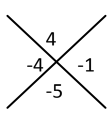


7) $2x^2 - 5x + 2$

$2x^2 - 4x - x + 2$

$2x(x - 2) - 1(x - 2)$

$(x - 2)(2x - 1)$

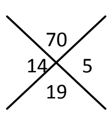


9) $2x^2 + 19x + 35$

$2x^2 + 14x + 5x + 35$

$2x(x + 7) + 5(x + 7)$

$(x + 7)(2x + 5)$

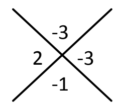


11) $2b^2 - b - 3$

$2b^2 + 2b - 3b - 3$

$2b(b + 1) - 3(b + 1)$

$(b + 1)(2b - 3)$

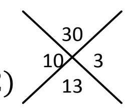


13) $5k^2 + 13k + 6$

$5k^2 + 10k + 3k + 6$

$5k(k + 2) + 3(k + 2)$

$(k + 2)(5k + 3)$

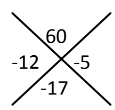


15) $3x^2 - 17x + 20$

$3x^2 - 12x - 5x + 20$

$3x(x - 4) - 5(x - 4)$

$(x - 4)(3x - 5)$

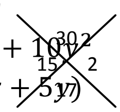


17) $3x^2 + 17xy + 10y^2$

$3x^2 + 15xy + 2xy + 10y^2$

$3x(x + 5y) + 2y(x + 5y)$

$(x + 5y)(3x + 2y)$

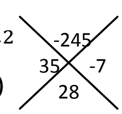


19) $5x^2 + 28xy - 49y^2$

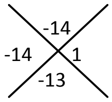
$5x^2 + 35xy - 7xy - 49y^2$

$5x(x + 7y) - 7y(x + 7y)$

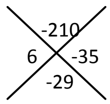
$(x + 7y)(5x - 7y)$



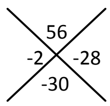
21) $6x^2 - 39x - 21$
$3(2x^2 - 13x - 7)$
$3(2x^2 - 14x + x - 7)$
$3(2x(x-7) + 1(x-7))$
$3(x-7)(2x+1)$



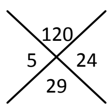
23) $21k^2 - 87k - 90$
$3(7k^2 - 29k - 30)$
$3(7k^2 + 6k - 35k - 30)$
$3(k(7k+6) - 5(7k+6))$
$3(7k+6)(k-5)$



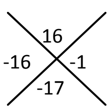
25) $14x^2 - 60x + 16$
$2(7x^2 - 30x + 8)$
$2(7x^2 - 2x - 28x + 8)$
$2\big(x(7x-2) - 4(7x-2)\big)$
$2(7x-2)(x-4)$



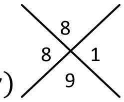
28) $6x^2 + 29x + 20$
$6x^2 + 5x + 24x + 20$
$x(6x+5) + 4(6x+5)$
$(6x+5)(x+4)$



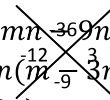
30) $4k^2 - 17k + 4$
$4k^2 - 16k - k + 4$
$4k(k-4) - 1(k-4)$
$(k-4)(4k-1)$



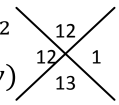
33) $4x^2 + 9xy + 2y^2$
$4x^2 + 8xy + xy + 2y^2$
$4x(x+2y) + y(x+2y)$
$(x+2y)(4+y)$



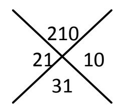
33) $4m^2 - 9mn - 9n^2$
$4m^2 - 12mn + 3mn - 9n^2$
$4m(m-3n) + 3n(m-3n)$
$(m-3n)(4m+3n)$



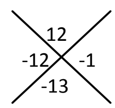
37) $4x^2 + 13xy + 3y^2$
$4x^2 + 12xy + xy + 3y^2$
$4x(x+3y) + y(x+3y)$
$(x+3y)(4x+y)$



39) $12x^2 + 62xy + 70y^2$
$2(6x^2 + 31xy + 35y^2)$
$2(6x^2 + 21xy + 10xy + 35y^2)$
$2\big(3x(2x+7y) + 5y(2x+7y)\big)$
$2(2x+7y)(3x+5y)$



40) $24x^2 - 52xy + 8y^2$
$4(6x^2 - 13xy + 2y^2)$
$4(6x^2 - 12xy - xy + 2y^2)$
$4\big(6x(x-2y) - y(x-2y)\big)$
$4(x-2y)(6x-y)$



6.5

1) $r^2 - 16$
$(r) \quad (4)$
$(r+4)(r-4)$

3) $v^2 - 25$
$(v) \quad (5)$
$(v+5)(v-5)$

5) $p^2 - 4$
$(p) \quad (2)$
$(p+2)(p-2)$

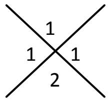
7) $9k^2 - 4$
$(3k) \quad (2)$
$(3k+2)(3k-2)$

9) $3x^2 - 27$
$3(x^2 - 9)$
$(x) \quad (3)$
$3(x+3)(x-3)$

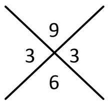
11) $16x^2 - 36$
$4(4x^2 - 9)$
$(2x) \quad (3)$
$4(2x+3)(2x-3)$

13) $18a^2 - 50b^2$
$2(9a^2 - 25b^2)$
$(3a) \quad (5b)$
$2(3a + 5b)(3a - 5b)$

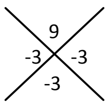
15) $a^2 - 2a + 1$
$(a - 1)^2$



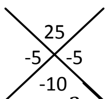
17) $x^2 + 6x + 9$
$(x + 3)^2$



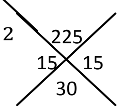
19) $x^2 - 6x + 9$
$(x + 3)^2$



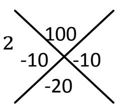
21) $25p^2 - 10p + 1$
$(5p - 1)^2$



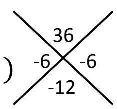
23) $25a^2 + 30ab + 9b^2$
$(5a + 3b)^2$



25) $4a^2 - 20ab + 25b^2$
$(2a - 5b)^2$



27) $8x^2 - 24xy + 18y^2$
$2(4x^2 - 12xy + 9y^2)$
$2(2x - 3y)^2$



29) $8 - m^3$
$(2)\ (m)$
$(2 - m)(4 + 2m + m^2)$

31) $x^3 - 64$
$(x) \quad (4)$
$(x - 4)(x^2 + 4x + 16)$

33) $216 - u^3$
$(6) \quad (u)$
$(6 - u)(36 + 6u + u^2$

35) $125a^3 - 64$
$(5a) \quad (4)$
$(5a - 4)(25a^2 + 20a + 16)$

37) $64x^3 + 27y^3$
$(4x) \quad (3y)$
$(4x + 3y)(16x^2 - 12xy + 9y^2)$

39) $54x^3 + 250y^3$
$2(27x^3 + 125y^3)$
$(3x) \quad (5y)$
$2(3x + 5y)(9x^2 - 15xy + 25y^2$

41) $a^4 - 81$
$(a^2)\ (9)$
$(a^2 + 9)(a^2 - 9)$
$(a) \quad (3)$
$(a^2 + 9)(a + 3)(a - 3)$

43) $16 - z^4$
$(4)\ (z^2)$
$(4 + z^2)(4 - z^2)$
$(2)\ (z)$
$(4 + z^2)(2 + z)(2 - z)$

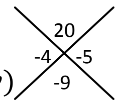
45) $x^4 - y^4$
$(x^2) \quad (y^2)$
$(x^2 + y^2)(x^2 - y^2)$
$(x) \quad (y)$
$(x^2 + y^2)(x + y)(x - y)$

47) $m^4 - 81b^4$
$(m^2) \quad (9b^2)$
$(m^2 + 9b^2)(m^2 - 9b^2)$
$(m) \quad (3b)$
$(m^2 + 9b^2)(m + 3b)(m - 3b)$

6.6

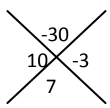
1) $24az - 18ah + 60yz - 45yh$
$3(8az - 6ah + 20yz - 15yh)$
$3(2a(4z - 3h) + 5y(4z - 3h)$
$3(4z - 3h)(2a + 5y)$

3) $5u^2 - 9uv + 4v^2$
$5u^2 - 4uv - 5uv + 4v^2$
$u(5u - 4v) - v(5u - 4v)$
$(5u - 4v)(u - v)$



5) $-2x^3 + 128y^3$
$-2(x^3 - 64y^3)$
$(x) \quad (4y)$
$-2(x - 4y)(x^2 + 4xy + 16y^2)$

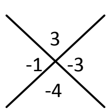
7) $5n^3 + 7n^2 - 6n$
$n(5n^2 + 7n - 6)$
$n(5n^2 + 10n - 3n - 6)$
$n\big(5n(n + 2) - 3(n + 2)\big)$
$n(n + 2)(5n - 3)$



9) $54u^3 - 16$
$2(27u^3 - 8)$
$(3u) \quad (2)$
$2(3u - 2)(9u^2 + 6u + 4)$

11) $n^2 - n$
$n(n - 1)$

13) $x^2 - 4xy + 3y^2$
$x^2 - xy - 3xy + 3y^2$
$x(x - y) - 3y(x - y)$
$(x - y)(x - 3y)$



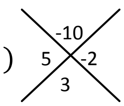
15) $9x^2 - 25y^2$
$(3x) \quad (5y)$
$(3x + 5y)(3x - 5y)$

17) $m^2 - 4n^2$
$(m)\ (2n)$
$(m + 2n)(m - 2n)$

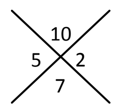
19) $36b^2c - 16xd - 24b^2d + 24xc$
$4(9b^2c - 4xd - 6b^2d + 6xc)$
$\cancel{4(1(9b^2c - 4xd) - 6(b^2d + xc)}$
$4(9b^2c - 6b^2d + 6xc - 4xd)$
$4(3b^2(3c - 2d) + 2x(3c - 2d)$
$4(3c - sd)(3b^2 + 2x)$

21) $128 + 54x^3$
$2(64 + 27x^3)$
$(4) \quad (3x)$
$2(4 + 3x)(16 - 12x + 9x^2)$

23) $2x^3 + 6x^2y - 20y^2x$
$2x(x^2 + 3xy - 10y^2)$
$2x(x + 5y)(x - 2y)$



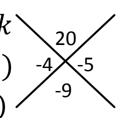
25) $n^3 + 7n^2 + 10n$
$n(n^2 + 7n + 10)$
$n(n + 5)(n + 2)$



27) $27x^3 - 64$
$(3x) \quad (4)$
$(3x - 4)(9x^2 + 12x + 16)$

29) $5x^2 + 2x$
$x(5x + 2)$

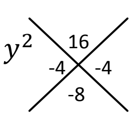
31) $3k^3 - 27k^2 + 60k$
$3k(k^2 - 9k + 20)$
$3k(k - 4)(k - 5)$



33) $mn - 12x + 3m - 4xn$

$\cancel{1(mn - 12x) + 1(3m - 4xn)}$
$mn + 3m - 4xn - 12x$
$m(n + 3) - 4x(n + 3)$
$(n + 3)(m - 4x)$

35) $16x^2 - 8xy + y^2$
$(4x - y)^2$



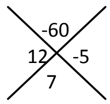
37) $27m^2 - 48n^2$

$3(9m^2 - 16n^2)$

$(3m) \quad (4n)$

$3(3m + 4n)(3m - 4n)$

39) $9x^3 + 21x^2y - 60y^2x$



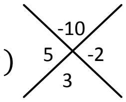
$3x(3x^2 + 7xy - 20y^2)$

$3x(3x^2 + 12xy - 5xy - 20y^2)$

$3x\big(3x(x + 4y) - 5y(x + 4y)\big)$

$3x(x + 4y)(3x - 5y)$

41) $2m^2 + 6mn - 20n^2$



$2(m^2 + 3mn - 10n^2)$

$2(m + 5n)(m - 2n)$

6.7

1) $(k - 7)(k + 2) = 0$

$k - 7 = 0 \quad k + 2 = 0$

$\underline{+7 \quad +7} \quad \underline{-2 \quad -2}$

$k = 7 \quad k = -2$

3) $(x - 1)(x + 4) = 0$

$x - 1 = 0 \quad x + 4 = 0$

$\underline{+1 \quad +1} \quad \underline{-4 \quad -4}$

$x = 1 \quad x = -4$

5) $6x^2 - 150 = 0$

$6(x^2 - 25) = 0$

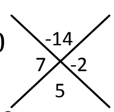
$6(x + 5)(x - 5) = 0$

$x + 5 = 0 \quad x - 5 = 0$

$\underline{-5 \quad -5} \quad \underline{+5 \quad +5}$

$x = -5 \quad x = 5$

7) $2n^2 + 10n - 28 = 0$



$2(n^2 + 5n - 14) = 0$

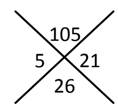
$2(n + 7)(n - 2) = 0$

$n + 7 = 0 \quad n - 2 = 0$

$\underline{-7 \quad -7} \quad \underline{+2 \quad +2}$

$n = -7 \quad n = 2$

9) $7x^2 + 26x + 15 = 0$



$7x^2 + 5x + 21x + 15 = 0$

$x(7x + 5) + 3(7x + 5) = 0$

$(7x + 5)(x + 3) = 0$

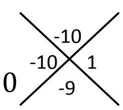
$7x + 5 = 0 \quad x + 3 = 0$

$\underline{-5 \quad -5} \quad \underline{-3 \quad -3}$

$\frac{7x}{7} = \frac{-5}{7} \quad x = -3$

$x = -\frac{5}{7}$

11) $5n^2 - 9n - 2 = 0$



$5n^2 - 10n + n - 2 = 0$

$5n(n - 2) + 1(n - 2) = 0$

$(n - 2)(5n + 1) = 0$

$n - 2 = 0 \quad 5n + 1 = 0$

$\underline{+2 \quad +2} \quad \underline{-1 \quad -1}$

$n = 2 \quad \frac{5n}{5} = \frac{-1}{5}$

$n = -\frac{1}{5}$

13) $x^2 - 4x - 8 = -8$

$\underline{+8 \quad +8}$

$x^2 - 4x = 0$

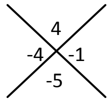
$x(x - 4) = 0$

$x = 0 \quad x - 4 = 0$

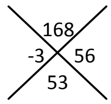
$\underline{+4 \quad +4}$

$x = 4$

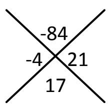
15) $x^2 - 4x - 1 = -5$
$\underline{\quad +5 \quad +5}$
$x^2 - 4x + 4 = 0$
$(x-4)(x-1) = 0$
$x - 4 = 0 \quad x - 1 = 0$
$\underline{+4 \;\; +4} \quad \underline{+1 \;\; +1}$
$x = 4 \qquad x = 1$



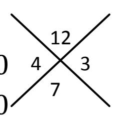
17) $49p^2 + 371p - 241 = 0$
$7(7p^2 + 53p - 24) = 0$
$7(7p^2 - 3p + 56p - 24) = 0$
$7(p(7p-3) + 8(7p-3) = 0$
$7(7p-3)(p+8) = 0$
$7p - 3 = 0 \quad p + 8 = 0$
$\underline{+3 \;\; +3} \quad \underline{-8 \;\; -8}$
$\frac{7p}{7} = \frac{3}{7} \qquad p = -8$
$p = \frac{3}{7}$



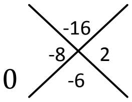
19) $7x^2 + 17x - 20 = -8$
$\underline{\quad +8 \quad +8}$
$7x^2 + 17x - 12 = 0$
$7x^2 - 4x + 21x - 12 = 0$
$x(7x-4) + 3(7x-4) = 0$
$(7x-4)(x+3) = 0$
$7x - 4 = 0 \quad x + 3 = 0$
$\underline{+4 \;\; +4} \quad \underline{-3 \;\; -3}$
$\frac{7x}{7} = \frac{4}{7} \qquad x = -3$
$x = \frac{4}{7}$



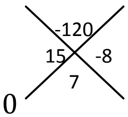
21) $7r^2 + 84 = -49r$
$\underline{+49r \quad +49r}$
$7r^2 + 49r + 84 = 0$
$7(r^2 + 7r + 12) = 0$
$7(r^2 + 7r + 12) = 0$
$7(r+4)(r+3) = 0$
$r + 4 = 0 \quad r + 3 = 0$
$\underline{-4 \;\; -4} \quad \underline{-3 \;\; -3}$
$r = -4 \quad r = -3$



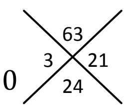
23) $x^2 - 6x = 16$
$\underline{-16 \;\; -16}$
$x^2 - 6x - 16 = 0$
$(x-8)(x+2) = 0$
$x - 8 = 0 \quad x + 2 = 0$
$\underline{+8 \;\; +8} \quad \underline{-2 \;\; -2}$
$x = 8 \qquad x = -2$



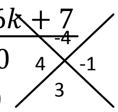
25) $3v^2 + 7v = 40$
$\underline{-40 \quad -40}$
$3v^2 + 7v - 40 = 0$
$3v^2 + 15v - 8v - 40 = 0$
$3v(v+5) - 8(v+5) = 0$
$(v+5)(3v-8) = 0$
$v + 5 = 0 \quad 3v - 8 = 0$
$\underline{-5 \;\; -5} \quad \underline{+8 \;\; +8}$
$v = -5 \qquad \frac{3v}{3} = \frac{8}{3}$
$v = \frac{8}{3}$



27) $35x^2 + 120x = -45$
$\underline{+45 \quad +45}$
$35x^2 + 120x + 45 = 0$
$5(7x^2 + 24x + 9) = 0$
$5(7x^2 + 3x + 21x + 9) = 0$
$5\big(x(7x+3) + 3(7x+3)\big) = 0$
$5(7x+3)(x+3) = 0$
$7x + 3 = 0 \quad x + 3 = 0$
$\underline{-3 \;\; -3} \quad \underline{-3 \;\; -3}$
$\frac{7x}{7} = \frac{-3}{7} \qquad x = -3$
$x = -\frac{3}{7}$



29) $4k^2 + 18k - 23 = 6k - 7$
$\underline{-6k + 7 \quad -6k + 7}$
$4k^2 + 12k - 16 = 0$
$4(k^2 + 3k - 4) = 0$
$4(k+4)(k-1) = 0$
$k + 4 = 0 \quad k - 1 = 0$
$\underline{-4 \;\; -4} \quad \underline{+1 \;\; +1}$
$k = -4 \quad k = 1$



31) $9x^2 - 46 + 7x = 7x + 8x^2 + 3$

$\underline{-8x^2 - 3 \quad -7x \quad -7x - 8x^2 - 3}$

$x^2 - 49 = 0$

$(x+7)(x-7) = 0$

$x + 7 = 0 \quad x - 7 = 0$

$\underline{-7 \quad -7} \quad \underline{+7 \quad +7}$

$x = -7 \qquad x = 7$

33) $2m^2 + 19m + 40 = -2m$

$\underline{+2m \qquad\qquad +2m}$

$2m^2 + 21m + 40 = 0$

$2m^2 + 4m + 16m + 40 = 0$

$m(2m + 5 + 8(2m+5) = 0$

$(2m+5)(m+8) = 0$

$2m + 5 = 0 \quad m + 8 = 0$

$\underline{-5 \quad -5} \quad \underline{-8 \quad -8}$

$\frac{2m}{2} = \frac{-5}{2} \qquad m = -8$

$m = -\frac{5}{2}$

80

5 16

21

35) $40p^2 + 183p - 168 = p + 5p^2$

$\underline{-5p^2 \qquad -p \qquad\qquad -p - 5p^2}$

$35p^2 + 182p - 168 = 0$

$7(5p^2 + 26p - 24) = 0$

$7(5p^2 - 4p + 30p - 24) = 0$

$7\big(p(5p-4) + 6(5p-4)\big) = 0$

$7(5p-4)(p+6) = 0$

$5p - 4 = 0 \quad p + 6 = 0$

$\underline{+4 \quad +4} \quad \underline{-6 \quad -6}$

$\frac{5p}{5} = \frac{4}{5} \qquad p = -6$

$p = \frac{4}{5}$

-120

-4 30

26

Chapter 7: Rational Expressions

7.1

1) $\frac{3k^2+30k}{k+10}$

$$k+10 \neq 0$$
$$\underline{\;-10 \quad -10\;}$$
$$k \neq -10$$

3) $\frac{15n^2}{10n+25}$

$$10n+25 \neq 0$$
$$\underline{\;-25 \quad -25\;}$$
$$\frac{10n}{10} \neq -\frac{25}{10}$$
$$n \neq -\frac{5}{2}$$

5) $\frac{10m^2+8m}{10m}$

$$\frac{10m}{10} \neq \frac{0}{10}$$
$$m \neq 0$$

7) $\frac{r^2+3r+12}{5r+10}$

$$5r+10 \neq 0$$
$$\underline{\;-10 \quad -10\;}$$
$$\frac{5r}{5} \neq -\frac{10}{5}$$
$$r \neq -2$$

9) $\frac{b^2+12b+32}{b^2+4b-32}$

$$b^2+4b-32 \neq 0$$
$$(b+8)(b-4) \neq 0$$
$$b+8 \neq 0 \qquad b-4 \neq 0$$
$$\underline{\;-8 \quad -8\;} \qquad \underline{\;+4 \quad +4\;}$$
$$b \neq -8 \qquad b \neq 4$$

11) $\frac{21x^2}{18x} = \frac{7x}{6}$

13) $\frac{24a}{40a^2} = \frac{3}{5a}$

15) $\frac{32x^3}{8x^4} = \frac{4}{x}$

17) $\frac{18m-24}{60} = \frac{6(3m-4)}{60} = \frac{3m-4}{10}$

19) $\frac{20}{4+2p} = \frac{20}{2(2+p)} = \frac{10}{2+p}$

21) $\frac{x+1}{x^2+8x+7} = \frac{x+1}{(x+7)(x+1)} = \frac{1}{x+7}$

23) $\frac{32x^2}{28x^2+28x} = \frac{32x^2}{28x(x+1)} = \frac{8x}{7(x+1)}$

25) $\frac{n^2+4n-12}{n^2-7n+10} = \frac{(n+6)(n-2)}{(n-5)(n-2)} = \frac{n+6}{n-5}$

27) $\frac{9v+54}{v^2-4v-60} = \frac{9(v+6)}{(v-10)(v+6)} = \frac{9}{v-10}$

29) $\frac{12x^2-42x}{30x^2-42x} = \frac{6x(2x-7)}{6x(5x-7)} = \frac{2x-7}{5x-7}$

31) $\frac{6a-10}{10a+4} = \frac{2(3a-5)}{2(5a+2)} = \frac{3a-5}{5a+2}$

33) $\frac{2n^2+19n-10}{9n+90} = \frac{(2n-1)(n+10)}{9(n+10)} = \frac{2n-1}{9}$

35) $\frac{8m+16}{20m-12} = \frac{8(m+2)}{4(5m-3)} = \frac{2(m+2)}{5m-3}$

37) $\frac{2x^2-10x+8}{3x^2-7x+4} = \frac{2(x-4)(x-1)}{(3x-4)(x-1)} = \frac{2(x-4)}{3x-4}$

39) $\frac{7n^2-32n+16}{4n-16} = \frac{(7n-4)(n-4)}{4(n-4)} = \frac{7n-4}{4}$

41) $\frac{n^2-2n+1}{6n+6} = \frac{(n-1)^2}{6(n+1)}$

43) $\frac{7a^2-26a-45}{6a^2-34a+20} = \frac{(7a+9)(a-5)}{2(3a-2)(a-5)} = \frac{7a+9}{2(3a-2)}$

7.2

1) $^{4}\frac{\cancel{8}x^2}{\cancel{9}}\cdot\frac{\cancel{9}}{\cancel{2}} = 4x^2$

3) $\frac{9\cancel{n}}{2\cancel{n}}\cdot\frac{7}{5n}=\frac{63}{10n}$

5) $\frac{\cancel{5}x^2}{{}_{2}\cancel{4}}\cdot\frac{\cancel{6}^{3}}{\cancel{5}}=\frac{3x^2}{2}$

7) $\frac{\cancel{7}\cancel{(m-6)}}{\cancel{m-6}}\cdot\frac{5m\cancel{(7m-5)}}{\cancel{7}\cancel{(7m-5)}}=5m$

9) $\frac{7r}{7r(r+10)}\div\frac{r-6}{(r-6)^2}=\frac{\cancel{7r}}{\cancel{7r}(r+10)}\cdot\frac{(r-6)^{\cancel{2}}}{\cancel{r-6}}=\frac{r-6}{r+10}$

11) $\frac{25n+25}{5}\cdot\frac{4}{30n+30}=\frac{\cancel{25}\cancel{(n+1)}}{\cancel{5}}\cdot\frac{\cancel{4}^{2}}{{}_{3}\cancel{30}\cancel{(n+1)}}=\frac{2}{3}$

13) $\frac{x-10}{35x+21}\div\frac{7}{3x+21}=\frac{x-10}{35x+21}\cdot\frac{35x+21}{7}=\frac{x-10}{\cancel{7}\cancel{(5x+3)}}\cdot\frac{\cancel{7}\cancel{(5x+3)}}{7}=\frac{x-10}{7}$

15) $\frac{x^2-6x-7}{x+5}\cdot\frac{x+5}{x-7}=\frac{\cancel{(x-7)}(x+1)}{\cancel{x+5}}\cdot\frac{\cancel{x+5}}{\cancel{x-7}}=x+1$

17) $\frac{8k}{24k^2-40k}\div\frac{1}{15k-25}=\frac{8k}{24k^2-40k}\cdot\frac{15k-25}{1}=\frac{\cancel{8k}}{\cancel{8k}\cancel{(3k-5)}}\cdot\frac{5\cancel{(3k-5)}}{1}=5$

19) $(n-8)\cdot\frac{6}{10n-80}=\frac{\cancel{n-8}}{1}\cdot\frac{\cancel{6}^{3}}{{}_{5}\cancel{10}\cancel{(n-8)}}=\frac{3}{5}$

21) $\frac{4m+36}{m+9}\cdot\frac{m-5}{5m^2}=\frac{4\cancel{(m+9)}}{\cancel{m+9}}\cdot\frac{m-5}{5m^2}=\frac{4(m-5)}{5m^2}$

23) $\frac{3x-6}{12x-24}\cdot(x+3)=\frac{\cancel{3}\cancel{(x-2)}}{{}_{4}\cancel{12}\cancel{(x-2)}}\cdot\frac{x+3}{1}=\frac{x+3}{4}$

25) $\frac{b+2}{40b^2-24b}\cdot(5b-3)=\frac{b+2}{8b\cancel{(5b-3)}}\cdot\frac{\cancel{5b-3}}{1}=\frac{b+2}{8b}$

27) $\frac{n-7}{6n-12}\cdot\frac{12-6n}{n^2+13n+42}=\frac{\cancel{n-7}}{\cancel{6}\cancel{(n-2)}}\cdot\frac{-\cancel{6}\cancel{(n-2)}}{\cancel{(n-7)}(n-6)}=\frac{-1}{n-6}$

29) $\frac{27a+36}{9a+63}\div\frac{6a+8}{2}=\frac{27a+36}{9a+63}\cdot\frac{2}{6a+8}=\frac{\cancel{9}\cancel{(3a+4)}}{\cancel{9}(a+7)}\cdot\frac{\cancel{2}}{\cancel{2}\cancel{(3a+4)}}=\frac{1}{a+7}$

31) $\frac{x^2-12x+32}{x^2-6x-16}\cdot\frac{7x^2+14x}{7x^2+21x}=\frac{\cancel{(x-8)}(x-4)}{\cancel{(x-8)}\cancel{(x+2)}}\cdot\frac{\cancel{7x}\cancel{(x+2)}}{\cancel{7x}(x+3)}=\frac{x-4}{x+3}$

33) $(10m^2+100m)\cdot\frac{18m^3-36m^2}{20m^2-40m}=\frac{\cancel{10m}(m+10)}{1}\cdot\frac{\overset{9}{\cancel{18}}m^2\cancel{(m-2)}}{\cancel{20m}\cancel{(m-2)}}=9m^2(m+10)$

35) $\frac{7p^2+25p+12}{6p+48}\cdot\frac{3p-8}{21p^2-44p-32}=\frac{\cancel{(7p+4)}(p+3)}{6(p+8)}\cdot\frac{\cancel{3p-8}}{\cancel{(7p+4)}\cancel{(3p-8)}}=\frac{p+3}{6(p+8)}$

37) $\frac{10b^2}{30b+20}\cdot\frac{30b+20}{2b^2+10b}=\frac{\cancel{10}b^{\cancel{2}}}{\cancel{10}\cancel{(3b+2)}}\cdot\frac{\overset{5}{\cancel{10}}\cancel{(3b+2)}}{\cancel{2b}(b+5)}=\frac{5b}{b+5}$

39) $\frac{7r^2-53r-24}{7r+2}\div\frac{49r+21}{49r+14}=\frac{7r^2-53r-24}{7r+2}\cdot\frac{49r+14}{49r+21}=\frac{(7r+3)(r-8)}{7r+2}\cdot\frac{7(7r+2)}{7(7r+3)}=r-8$

41) $\frac{x^2-1}{2x-4}\cdot\frac{x^2-4}{x^2-x-2}\div\frac{x^2+x-2}{3x-6}=\frac{x^2-1}{2x-4}\cdot\frac{x^2-4}{x^2-x-2}\cdot\frac{3x-6}{x^2+x-2}=\frac{\cancel{(x+1)}\cancel{(x-1)}}{2\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}\cancel{(x-2)}}{\cancel{(x-2)}\cancel{(x+1)}}\cdot\frac{3\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-1)}}=\frac{3}{2}$

43) $\frac{x^2+3x+9}{x^2+x-12}\cdot\frac{x^2+2x-8}{x^3-27}\div\frac{x^2-4}{x^2-6x+9}=\frac{x^2+3x+9}{x^2+x-12}\cdot\frac{x^2+2x-8}{x^3-27}\cdot\frac{x^2-6x+9}{x^2-4}=$

$\frac{\cancel{x^2+3x+9}}{\cancel{(x+4)}\cancel{(x-3)}}\cdot\frac{\cancel{(x+4)}\cancel{(x-2)}}{\cancel{(x-3)}\cancel{(x^2+3x+9)}}\cdot\frac{\cancel{(x-3)^2}}{\cancel{(x-2)}(x+2)}=\frac{1}{x+2}$

7.3

1) $\frac{(6)}{(6)}\frac{3}{8}=\frac{?}{48}$

$\frac{18}{48}$

3) $\frac{(y)}{(y)}\frac{a}{x}=\frac{?}{xy}$

$\frac{ay}{xy}$

5) $\frac{(3a^2c^3)}{(3a^2c^3)}\frac{2}{3a^3b^2c}=\frac{?}{9a^3b^2c^4}$

$\frac{6a^2c^3}{9a^3b^2c^4}$

7) $\frac{(x-4)}{(x-4)}\frac{2}{x+4}=\frac{?}{x^2-16}$

$(x+4)(x-4)$

$\frac{2x-8}{(x+4)(x-4)}$

9) $\frac{(x+3)}{(x+3)}\frac{(x-4)}{(x+2)}=\frac{?}{x^2+5x+6}$

$(x+2)(x+3)$

$\frac{x^2-4x+3x-12}{(x+2)(x+3)}=\frac{x^2-x-12}{(x+2)(x+3)}$

11) $2a^3, 6a^4b^2, 4a^3b^5$

$12a^4b^5$

13) $x^3-3x,\quad x-3, x$

$x(x^2-3)$

$x(x-3)$

15) $x+2,\ x-4$

$(x+2)(x-4)$

17) $x^2-25,\qquad x+5$

$(x+5)(x-5)$

$(x+5)(x-5)$

19) $x^2+3x+2,\quad x^2+5x+6$

$(x+1)(x+2)\ \ (x+2)(x+3)$

$(x+1)(x+2)(x+3)$

21) $\frac{(2a^3)}{(2a^3)}\frac{3a}{5b^2}\ ,\ \frac{2}{10a^3b}\frac{(b)}{(b)}$

$LCD=10a^3b^2$

$\frac{6a^4}{10a^3b^2}\ ,\ \frac{2b}{10a^3b^2}$

23) $\frac{(x+2)}{(x+2)}\frac{x+2}{x-3}, \frac{(x-3)}{(x+2)}\frac{(x-3)}{(x-3)}$

$LCD = (x-3)(x+2)$

$\frac{x^2+4x+4}{(x-3)(x+2)}, \frac{x^2-6x+9}{(x-3)(x+2)}$

25) $\frac{(x-4)}{(x-4)}\frac{x}{x^2-16}, \frac{3x}{x^2-8x+16}\frac{(x+4)}{(x+4)}$

$(x-4)(x+4) \quad (x-4)(x-4)$

$LCD = (x-4)^2(x+4)$

27) $\frac{4x}{x^2-x-6}, \quad \frac{x+2}{x-3}\frac{(x+2)}{(x+2)}$

$(x-3)(x+2)$

$LCD: (x-3)(x+2)$

$\frac{4x}{(x-3)(x+2)}, \frac{x^2+4x+4}{(x-3)(x+2)}$

7.4

1) $\frac{2}{a+3} + \frac{4}{a+3} = \frac{6}{a+3}$

3) $\frac{t^2+4t}{t-1} + \frac{2t-7}{t-1} = \frac{t^2+6t-7}{t-1} = \frac{(t+7)(t-1)}{t-1} = t+7$

5) $\frac{2x^2+3}{x^2-6x+5} + \frac{-x^2+5x+(-9)}{x^2-6x+5} = \frac{x^2+5x-6}{x^2-6x+5} =$
$\frac{(x+6)(x-1)}{(x+5)(x-1)} = \frac{x+6}{x+5}$

7) $\frac{(4)}{(4)}\frac{5}{6r} + \frac{-5}{8r}\frac{(3)}{(3)}$

$LCD: 24r$

$\frac{20}{24r} + \frac{-15}{24r} = \frac{5}{24r}$

9) $\frac{(2)}{(2)}\frac{8}{9t^3} + \frac{5}{6t^2}\frac{(3t)}{(3t)}$

$LCD: 18t^3$

$\frac{16}{18t^3} + \frac{15t}{18t^3} = \frac{16+15t}{18t^3}$

11) $\frac{(2)}{(2)}\frac{a+2}{2} + \frac{-a+4}{4}$

$LCD: 4$

$\frac{2a+4}{4} + \frac{-a+4}{4} = \frac{a+8}{4}$

13) $\frac{x-1}{4x} + \frac{-2x-3}{x}\frac{(4)}{(4)}$

$LCD: 4x$

$\frac{x-1}{4x} + \frac{-8x-12}{4x} = \frac{-7x-13}{4x}$

15) $\frac{(y)}{(y)}\frac{5x+3y}{2x^2y} + \frac{-3x-4y}{xy^2}\frac{(2x)}{(2x)}$

$LCD: 2x^2y^2$

$\frac{5xy+3y^2}{2x^2y^2} + \frac{-6x^2-8xy}{2x^2y^2} = \frac{-6x^2-3xy+3y^2}{2x^2y^2}$

17) $\frac{(z+1)}{(z+1)}\frac{2z}{z-1} + \frac{-3z}{z+1}\frac{(z-1)}{(z-1)}$

$LCD: (z-1)(z+1)$

$\frac{2z^2+2z}{(z-1)(z+1)} + \frac{-3z^2+3z}{(z-1)(z+1)} = \frac{-a^2+5z}{(z-1)(z+1)}$

19) $\frac{8}{x^2-4} + \frac{-3}{x+2}\frac{(x-2)}{(x-2)}$

$LCD: (x+2)(x-2)$

$\frac{8}{(x+2)(x-2)} + \frac{-3x+6}{(x+2)(x-2)} = \frac{-3x+14}{(x+2)(x-2)}$

21) $\frac{(4)}{(4)}\frac{t}{t-3} + \frac{-5}{4t-12}$

$LCD: 4(t-3)$

$\frac{4t}{4(t-3)} + \frac{-5}{4(t-3)} = \frac{4t-5}{4(t-3)}$

23) $\frac{(3)}{(3)}\frac{2}{5x^2+5x} + \frac{-4}{3x+3}\frac{(5x)}{(5x)}$

$5x(x+1) \quad 3(x+1)$

$LCD: 15x(x+1)$

$\frac{6}{15x(x+1)} + \frac{-20x}{15x(x+1)} = \frac{6-20x}{15x(x+1)}$

25) $\frac{(y+t)}{(y+t)}\frac{t}{y-t} + \frac{-y}{y+t}\frac{(y-t)}{(y-t)}$

$LCD: (y+t)(y-t)$

$\frac{yt+t^2}{(y+t)(y-t)} + \frac{-y^2+yt}{(y+t)(y-t)} = \frac{t^2+2yt-y^2}{(y+t)(y-t)}$

27) $\frac{(x+1)}{(x+1)}\frac{x}{x^2+5x+6}+\frac{-2}{x^2+3x+2}\frac{(x+3)}{(x+3)}$

$(x+2)(x+3)\quad(x+1)(x+2)$

$LCD: (x+1)(x+2)(x+3)$

$$\frac{x^2+x}{(x+1)(x+2)(x+3)}+\frac{-2x-6}{(x+1)(x+2)(x+3)}=\frac{x^2-x-6}{(x+1)(x+2)(x+3)}=\frac{(x-3)(x+2)}{(x+1)(x+2)(x+3)}=\frac{x-3}{(x+1)(x+3)}$$

29) $\frac{(x+6)}{(x+6)}\frac{x}{x^2+15x+56}+\frac{-7}{x^2+13x+42}\frac{(x+8)}{(x+8)}$

$(x+7)(x+8)\quad(x+7)(x+6)$

$LCD: (x+6)(x+7)(x+8)$

$$\frac{x^2+6x}{(x+6)(x+7)(x+8)}+\frac{-7x-56}{(x+6)(x+7)(x+8)}=\frac{x^2-x-56}{(x+6)(x+7)(x+8)}=\frac{(x-8)(x+7)}{(x+6)(x+7)(x+8)}=\frac{x-8}{(x+6)(x+8)}$$

31) $\frac{(x+3)}{(x+3)}\frac{5x}{x^2-x-6}+\frac{-18}{x^2-9}\frac{(x+2)}{(x+2)}$

$(x-3)(x+2)\quad(x+3)(x-3)$

$LCD: (x+2)(x-3)(x+3)$

$$\frac{5x^2+15x}{(x+2)(x-3)(x+3)}+\frac{-18x-36}{(x+2)(x-3)(x+3)}=\frac{5x^2-3x-36}{(x+2)(x-3)(x+3)}=\frac{(5x+12)(x-3)}{(x+2)(x-3)(x+3)}=\frac{5x+12}{(x+2)(x+3)}$$

33) $\frac{(x+3)}{(x+3)}\frac{2x}{x^2-1}+\frac{-4}{x^2+2x-3}\frac{(x+1)}{(x+1)}$

$(x+1)(x-1)\quad(x+3)(x-1)$

$LCD: (x+3)(x+1)(x-1)$

$$\frac{2x^2+6x}{(x+3)(x+1)(x-1)}+\frac{-4x-4}{(x+3)(x+1)(x-1)}=\frac{2x^2+2x-4}{(x+3)(x+1)(x-1)}=\frac{2(x+2)(x-1)}{(x+3)(x+1)(x-1)}=\frac{2(x+2)}{(x+3)(x+1)}$$

35) $\frac{(x+2)}{(x+2)}\frac{x+1}{x^2-2x-35}+\frac{x+6}{x^2+7x+10}\frac{(x-7)}{(x-7)}$

$(x-7)(x+5)\quad(x+5)(x+2)$

$LCD: (x+2)(x+5)(x-7)$

$$\frac{x^2+x+2x+2}{(x+2)(x+5)(x-7)}+\frac{x^2-7x+6x-42}{(x+2)(x+5)(x-7)}=\frac{2x^2+2x-40}{(x+2)(x+5)(x-7)}=\frac{2(x+5)(x-4)}{(x+2)(x+5)(x-7)}=\frac{2(x-4)}{(x-7)(x+2)}$$

37) $\frac{(-1)}{(-1)}\frac{4-a^2}{a^2-9}+\frac{-a+2}{3-a}\frac{(a+3)}{(a+3)}$

$(a+3)(a-3)\quad(-1)(a-3)$

$LCD: (-1)(a+3)(a-3)$

$$\frac{a^2-4}{(-1)(a+3)(a-3)}+\frac{-a^2-3a+2a+6}{(-1)(a+3)(a-3)}=\frac{-a+2}{(-1)(a+3)(a-3)}$$

39) $\frac{(2z+1)}{(2z+1)}\frac{2z}{1-2z}+\frac{(-1)(2z-1)}{(-1)(2z-1)}\frac{3z}{2z+1}+\frac{-3}{4z^2-1}\frac{(-1)}{(-1)}$

$(-1)(2z-1)\qquad(2z+1)(2z-1)$

$LCD: (-1)(2z-1)(2z+1)$

$$\frac{4z^2+2z}{(-1)(2z-1)(2z+1)}+\frac{-6z^2+3z}{(-1)(2z-1)(2z+1)}+\frac{3}{(-1)(2z-1)(2z+1)}=\frac{-2z^2+5z+3}{(-1)(2z-1)(2z+1)}=\frac{(-1)(2z+1)(z-3)}{(-1)(2z-1)(2z+1)}=\frac{z-3}{2z-1}$$

41) $\frac{(x+3)}{x+3}\frac{2x-3}{x^2+3x+2}+\frac{3x-1}{x^2+5x+6}\frac{(x+1)}{(x+1)}$

$(x+1)(x+2)\ (x+3)(x+2)$

$LCD: (x+1)(x+2)(x+3)$

$\frac{2x^2-3x+6x-9}{(x+1)(x+2)(x+3)}+\frac{3x^2+3x-x-1}{(x+1)(x+2)(x+3)}=\frac{(5x^2+5x-10)}{(x+1)(x+2)(x+3)}=\frac{5(x+2)(x-1)}{(x+1)(x+2)(x+3)}=\frac{5(x-1)}{(x+1)(x+3)}$

43) $\frac{(x+5)}{(x+5)}\frac{(2x+7)}{(x^2-2x-3)}+\frac{-3x+2}{x^2+6x+5}\frac{(x-3)}{(x-3)}$

$(x-3)(x+1)\quad (x+5)(x+1)$

$LCD: (x+1)(x-3)(x+5)$

$\frac{2x^2+7x+10x+35}{(x+1)(x-3)(x+5)}+\frac{(-3x^2+9x+2x-6)}{(x+1)(x-3)(x+5)}=\frac{-x^2+28x+29}{(x+1)(x-3)(x+5)}=\frac{-1(x-29)(x+1)}{(x+1)(x-3)(x+5)}=\frac{(-1)(x-29)}{(x-3)(x+5)}$

7.5

1) $\frac{(x^2)1+\frac{1}{x}(x^2)}{(x^2)1-\frac{1}{x^2}(x^2)}=\frac{x^2+x}{x^2-1}=\frac{x(x+1)}{(x+1)(x-1)}=\frac{x}{x-1}$

3) $\frac{(a)a-2(a)}{(a)\frac{4}{a}-a(a)}=\frac{a^2-2a}{4-a^2}=\frac{a(a-2)(-1)}{(2+a)(2-a)}=\frac{-a}{a+2}$

5) $\frac{(a^2)\frac{1}{a^2}-\frac{1}{a}(a^2)}{(a^2)\frac{1}{a^2}+\frac{1}{a}(a^2)}=\frac{1-a}{1+a}$

7) $\frac{(x+2)\ 2-\frac{4}{x+2}(x+2)}{(x+2)\ 5-\frac{10}{x+2}(x+2)}=\frac{2x+4-4}{5x+10-10}=\frac{2x}{5x}=\frac{2}{5}$

9) $\frac{(2a-3)\frac{3}{(2a-3)}+2\ (2a-3)}{(2a-3)\frac{-6}{(2a-3)}-4(2a-3)}=\frac{3+4a-6}{-6-8a+12}=\frac{4a-3}{-8a+6}=\frac{4a-3}{-2(4a-3)}=-\frac{1}{2}$

11) $\frac{x(x+1)\ \frac{x}{x+1}-\frac{1}{x}\ x(x+1)}{x(x+1)\ \frac{x}{x+1}+\frac{1}{x}\ x(x+1)}=\frac{x^2-x-1}{x^2+x+1}$

13) $\frac{(x^2)\frac{3}{x}}{(x^2)\frac{9}{x^2}}=\frac{3x}{9}=\frac{x}{3}$

15) $\dfrac{(16a^2b^2)\frac{a^2-b^2}{4a^2b}}{(16a^2b^2)\frac{a+b}{16ab^2}} = \dfrac{4b(a^2-b^2)}{a(a+b)} = \dfrac{4b(a+b)(a-b)}{a(a+b)} = \dfrac{4b(a-b)}{a}$

17) $\dfrac{(x^2)\ 1-\frac{3}{x}(x^2)-\frac{10}{x^2}(x^2)}{(x^2)1+\frac{11}{x}\ (x^2)+\frac{18}{x^2}(x^2)} = \dfrac{x^2-3x-10}{x^2+11x+18} = \dfrac{(x-5)(x+2)}{(x+9)(x+2)} = \dfrac{x-5}{x+9}$

19) $\dfrac{(3x-4)1-\frac{2x}{(3x-4)}\ (3x-4)}{(3x-4)x-\frac{32}{(3x-4)}(3x-4)} = \dfrac{3x-4-2x}{3x^2-4x-32} = \dfrac{x-4}{(3x+8)(x-4)} = \dfrac{1}{3x+8}$

21) $\dfrac{(x-4)x-(x-4)1+\frac{2}{(x-4)}\ (x-4)}{(x-4)\ x+(x-4)3+\frac{6}{(x-4)}(x-4)} = \dfrac{x^2-4x-x+4+2}{x^2-4x+3x-12+6} = \dfrac{x^2-5x+6}{x^2-x-6} = \dfrac{(x-2)(x+3)}{(x-3)(x+2)} = \dfrac{(x-2)}{x+2}$

23) $\dfrac{(2x+3)\ x-(2x+3)4+\frac{9}{(2x+3)}(2x+3)}{(2x+3)x+(2x+3)3-\frac{5}{(2x+3)}(2x+3)} = \dfrac{2x^2+3x-8x-12+9}{2x^2+3x+6x+9-5} = \dfrac{(2x+1)(x-3)}{(2x+1)(x+4)} = \dfrac{x-3}{x+4}$

25) $\dfrac{b(b+3)\frac{2}{b}-\frac{5}{b+3}b(b+3)}{b(b+3)\frac{3}{b}+\frac{3}{b+3}b(b+3)} = \dfrac{2b+6-5b}{3b+9+3b} = \dfrac{-3b+6}{6b+9} = \dfrac{(-3)(b-2)}{3(2b+3)} = \dfrac{(-1)(b-2)}{2b+3}$

27) $\dfrac{(a^2b^2)\frac{2}{b^2}-(a^2b^2)\frac{5}{ab}-\frac{3}{a^2}(a^2b^2)}{(a^2b^2)\frac{2}{b^2}-(a^2b^2)\frac{7}{ab}+\frac{3}{a^2}(a^2b^2)} = \dfrac{2a^2-5ab-3b^2}{2a^2+7ab+3b^2} = \dfrac{(2a+b)(a-3b)}{(2a+b)(a+3b)} = \dfrac{a-3b}{a+3b}$

29) $\dfrac{(y+2)(y-2)\frac{y}{y+2}-\frac{y}{y-2}(y+2)(y-2)}{(y+2)(y-2)\frac{y}{y+2}+\frac{y}{y-2}(y+2)(y-2)} = \dfrac{y^2-2y-y^2-2y}{y^2-2y+y^2+2y} = \dfrac{-4y}{2y^2} = -\dfrac{2}{y}$

31) $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}} = \dfrac{(x^2y^2)\frac{1}{x^2}+\frac{1}{y^2}(x^2y^2)}{(x^2y^2)\frac{1}{x}+\frac{1}{y}(x^2y^2)} = \dfrac{y^2-x^2}{xy^2+x^2y} = \dfrac{(y+x)(y-x)}{xy(y+x)} = \dfrac{y-x}{xy}$

33) $\dfrac{x^{-3}y-xy^{-3}}{x^{-2}-y^{-2}} = \dfrac{(x^3y^3)\frac{y}{x^3}-\frac{x}{y^3}(x^3y^3)}{(x^3y^3)\frac{1}{x^2}-\frac{1}{y^2}(x^3y^3)} = \dfrac{y^4-x^4}{xy^3-x^3y} = \dfrac{(y^2+x^2)(y^2-x^2)}{xy(y^2-x^2)} = \dfrac{y^2+x^2}{xy}$

35) $\dfrac{x^{-2}-6x^{-1}+9}{x^2-9} = \dfrac{(x^2)\frac{1}{x^2}-(x^2)\frac{6}{x}+9(x^2)}{(x^2)\ x^2-9} = \dfrac{1-6x+9x^2}{x^2(x^2-9)} = \dfrac{(1-3x)^2}{(x^2)(x+3)(x-3)}$

7.6

1) $\frac{10}{a}=\frac{6}{8}$

$\frac{80}{6}=\frac{6a}{6}$

$13.3=a$

3) $\frac{7}{6}=\frac{2}{k}$

$\frac{7k}{7}=\frac{12}{7}$

$k=1.71$

5) $\frac{6}{x}=\frac{8}{2}$

$\frac{12}{8}=\frac{8x}{8}$

$1.5=x$

7) $\frac{m-1}{5}=\frac{8}{2}$

$2(m-1)=40$

$2m-2=40$

$\underline{+2\quad +2}$

$\frac{2m}{2}=\frac{42}{2}$

$m=21$

9) $\frac{2}{9}=\frac{10}{p-4}$

$2(p-4)=90$

$2p-8=90$

$\underline{+8\quad +8}$

$\frac{2p}{2}=\frac{98}{2}$

$p=49$

11) $\frac{b-10}{7}=\frac{b}{4}$

$4(b-10)=7b$

$4b-40=7b$

$\underline{-4b\qquad -4b}$

$\frac{-40}{3}=\frac{3b}{3}$

$-13.3=b$

13) $\frac{x}{5}=\frac{x+2}{9}$

$9x=5(x+2)$

$9x=5x+10$

$\underline{-5x\ -5x}$

$\frac{4x}{4}=\frac{10}{4}$

$x=2.5$

15) $\frac{3}{10}=\frac{a}{a+2}$

$3(a+2)=10a$

$3a+6=10a$

$\underline{-3a\qquad -3a}$

$\frac{6}{7}=\frac{7a}{7}$

$0.86=a$

17) $\frac{v-5}{v+6}=\frac{4}{9}$

$9(v-5)=4(v+6)$

$9v-45=4v+24$

$\underline{-4v\qquad -4v}$

$5v-45=24$

$\underline{+45\ +45}$

$\frac{5v}{5}=\frac{69}{5}$

$v=13.8$

19) $\frac{7}{x-1}=\frac{4}{x-6}$

$7(x-6)=4(x-1)$

$7x-42=4x-4$

$\underline{-4x\qquad -4x}$

$3x-42=-4$

$\underline{+42\quad +42}$

$\frac{3x}{3}=\frac{38}{3}$

$x=12.67$

21) $\frac{x+5}{5} = \frac{6}{x-2}$

$$(x+5)(x-2) = 30$$
$$x^2 + 5x - 2x - 10 = 30$$
$$x^2 + 3x - 10 = 30$$
$$\underline{\quad -30 \; -30}$$
$$x^2 + 3x - 40 = 0$$
$$(x+8)(x-5) = 0$$
$$x + 8 = 0 \quad x - 5 = 0$$
$$\underline{-8 \; -8} \quad \underline{+5 \; +5}$$
$$x = -8 \quad x = 5$$

27) $\frac{n+4}{3} = \frac{-3}{n-2}$

$$(n+4)(n-2) = -9$$
$$n^2 - 2n + 4n - 8 = -9$$
$$n^2 + 2n - 8 = -9$$
$$\underline{\quad +9 \; +9}$$
$$n^2 + 2n + 1 = 0$$
$$(n+1)^2 = 0$$
$$n + 1 = 0$$
$$\underline{-1 \; -1}$$
$$n = -1$$

23) $\frac{m+3}{4} = \frac{11}{m-4}$

$$(m+3)(m-4) = 44$$
$$m^2 - 4m + 3m - 12 = 44$$
$$m^2 - m - 12 = 44$$
$$\underline{\quad -44 - 44}$$
$$m^2 - m - 56 = 0$$
$$(m-8)(m+7) = 0$$
$$m - 8 = 0 \quad m + 7 = 0$$
$$\underline{+8 \; +8} \quad \underline{-7 \; -7}$$
$$m = 8 \quad m = -7$$

29) $\frac{3}{x+4} = \frac{x+2}{5}$

$$15 = (x+4)(x+2)$$
$$15 = x^2 + 2x + 4x + 8$$
$$15 = x^2 + 6x + 8$$
$$\underline{-15 \qquad\qquad -15}$$
$$0 = x^2 + 6x - 7$$
$$0 = (x+7)(x-1)$$
$$x + 7 = 0 \quad x - 1 = 0$$
$$\underline{-7 \; -7} \quad \underline{+1 \; +1}$$
$$x = -7 \quad x = 1$$

25) $\frac{2}{p+4} = \frac{p+5}{3}$

$$6 = (p+4)(p+5)$$
$$6 = p^2 + 5p + 4p + 20$$
$$6 = p^2 + 9p + 20$$
$$\underline{-6 \qquad\qquad -6}$$
$$0 = p^2 + 9p + 14$$
$$0 = (p+7)(p+2)$$
$$p + 7 = 0 \quad p + 2 = 0$$
$$\underline{-7 \; -7} \quad \underline{-2 \; -2}$$
$$p = -7, -2$$

31) The currency in Western Samoa is the Tala. The exchange rate is approximately S0.70 to 1 Tala. At this rate, how many dollars would you get if you exchanged 13.3 Tala?

$\frac{T}{\$} = \frac{1}{0.70} = \frac{13.3}{x}$

$x = \$\,9.31$

39) Kali reduced the size of a painting to a height of 1.3 in. What is the new width if it was originally 5.2 in. tall and 10 in. wide?

$\frac{h}{w} = \frac{5.2}{10} = \frac{1.3}{x}$

$x = 2.5\ in$

41) A bird bath that is 5.3 ft tall casts a shadow that is 25.4 ft long. Find the length of the shadow that a 8.2 ft adult elephant casts.

$\frac{h}{s} = \frac{5.3}{25.4} = \frac{8.2}{x}$

$x = 39.3\ ft$

43) The Vikings led the Timberwolves by 19 points at the half. If the Vikings scored 3 points for every 2 points the Timberwolves scored, what was the half time score?

$\frac{V}{T} = \frac{(x+19)}{x} = \frac{3}{2}$

$$\begin{aligned} 2(x+19) &= 3x \\ 2x + 38 &= 3x \\ -2x \qquad &\quad -2x \\ \hline 38 &= x \end{aligned}$$

$Timberwolves: 38$

$Vikings: 57$

45) Computer Services Inc. charges S8 more for a repair than Low Cost Computer Repair. If the ratio of the costs is 3 : 6, what will it cost for the repair at Low Cost Computer Repair?

$\frac{CSI}{LCCR} = \frac{x+8}{x} = \frac{6}{3}$

$$\begin{aligned} 3(x+8) &= 6x \\ 3x + 24 &= 6x \\ -3x \qquad &\quad -3x \\ \frac{24}{3} &= \frac{3x}{3} \\ \$8 &= x \end{aligned}$$

7.7

1) $(2x)\,3x-(2x)\frac{1}{2}-(2x)\frac{1}{x}=0(2x)$

$LCD: 2x$

$\frac{2x}{2}\neq\frac{0}{2}$

$*\,x\neq 0\,*$

$6x^2-x-2=0$

$(2x+1)(3x-2)=0$

$2x+1=0 \qquad 3x-2=0$

$\underline{\;-1\;\;-1\;} \qquad \underline{\;+2\;\;+2\;}$

$\frac{2x}{2}=\frac{-1}{2} \qquad \frac{3x}{3}=\frac{2}{3}$

$x=-\frac{1}{2} \qquad x=\frac{2}{3}$

3) $x(x-4)+\frac{20}{(x-4)}\,(x-4)=\frac{5x}{(x-4)}\,(x-4)-2\,(x-4)$

$LCD: (x-4)$

$x-4\neq 0$

$\underline{\;+4\;\;+4\;}$

$*\,x\neq 4\,*$

$x^2-4x+20=5x-2x+8$

$x^2-4x+20=3x+8$

$\underline{\;-3x-8\;\;-3x-8\;}$

$x^2-7x+12=0$

$(x-4)(x-3)=0$

$x-4=0 \qquad x-3=0$

$\underline{\;+4\;\;+4\;} \qquad \underline{\;+3\;\;+3\;}$

$\cancel{x=4} \qquad x=3$

5) $x(x-3)+\frac{6}{(x-3)}(x-3)=\frac{2x}{(x-3)}\,(x-3)$

$LCD=x-3$

$x-3\neq 0$

$\underline{\;+3\;\;+3\;}$

$*\,x\neq 3\,*$

$x^2-3x+6=2x$

$\underline{\;-2x\qquad -2x\;}$

$x^2-5x+6=0$

$(x-2)(x-3)=0$

$x-2=0 \qquad x-3=0$

$\underline{\;+2\;\;+2\;} \qquad \underline{\;+3\;\;+3\;}$

$x=2 \qquad \cancel{x=3}$

7) $\frac{2x}{3x-4}(6x-1)(3x-4)=\frac{4x+5}{6x-1}(6x-1)(3x-4)-\frac{3}{3x-4}(6x-1)(3x-4)$

$LCD:\ (6x-1)(3x-4)$

$6x-1\neq 0 \quad 3x-4\neq 0$

$\underline{+1 \quad +1} \quad \underline{+4 \quad +4}$

$\frac{6x}{6}\neq\frac{1}{6} \qquad \frac{3x}{3}\neq\frac{4}{3}$

$*x\neq\frac{1}{6}**x\neq\frac{4}{3}*$

$12x^2-2x=12x^2-16x+15x-20-18x+3$

$12x^2-2x=12x^2-19x-17$

$\underline{-12x^2 \qquad -12x^2}$

$-2x=-19x-17$

$\underline{+19x \quad +19x}$

$\frac{17x}{17}=-\frac{17}{17}$

$x=-1$

9) $\frac{3m}{2m-5}(2)(2m-5)(3m+1)-\frac{7}{3m+1}(2)(2m-5)(3m+1)=\frac{3}{2}(2)(2m-5)(3m+1)$

$LCD: (2)(2m-5)(3m+1)$

$2m-5\neq 0 \quad 3m+1\neq 0$

$\underline{+5 \quad +5} \quad \underline{-1 \quad -1}$

$\frac{2m}{2}\neq\frac{5}{2} \qquad \frac{3m}{3}\neq-\frac{1}{3}$

$*x\neq\frac{5}{2}* \ *x\neq-\frac{1}{3}*$

$18m^2+6m-28m+70=18m^2-39m-15$

$18m^2-22m+70=18m^2-39m-15$

$\underline{-18m^2 \qquad -18m^2}$

$-22m+70=-39m-15$

$\underline{+39m \qquad +39m}$

$17m+70=-15$

$\underline{-70 \quad -70}$

$\frac{17m}{17}=\frac{-85}{17}$

$m=-5$

11) $\frac{4-x}{1-x}(1-x)(3-x)=\frac{12}{3-x}(1-x)(3-x)$

$LCD:\ (1-x)(3-x)$

$1-x\neq 0\quad 3-x\neq 0$

$\underline{\ +x\quad +x\ }\quad \underline{\ +x\quad +x\ }$

$*1\neq x*\ \ *3\neq x*$

$12-4x-3x+x^2=12-12x$

$x^2-7x+12=12-12x$

$\underline{\quad +12x-12\quad -12+12x\ }$

$x^2+5x=0$

$x(x+5)=0$

$x=0\quad x+5=0$

$\underline{\ -5\quad -5\ }$

$x=-5$

13) $\frac{7}{y-3}(2)(y-3)(y-4)-\frac{1}{2}(2)(y-3)(y-4)=\frac{y-2}{y-4}(2)(y-3)(y-4)$

$LCD:\ (2)(y-3)(y-4)$

$y-3\neq 0\quad y-4\neq 0$

$\underline{\ +3\quad +3\ }\quad \underline{\ +4\quad +4\ }$

$*y\neq 3**y\neq 4*$

$14y-56-y^2+3y+4y-12=2y^2-6y-4y+12$

$-y^2+21y-68=2y^2-10y+12$

$\underline{+y^2-21y+68\quad +y^2-21y+68}$

$0=3y^2-31y+80$

$0=(3y-16)(y-5)$

$3y-16=0\quad y-5=0$

$\underline{+16\ \ +16}\quad \underline{+5\quad +5}$

$\frac{3y}{3}=\frac{16}{3}\qquad y=5$

15) $\frac{1}{x+2}(x+2)(x-2)+\underset{x-2}{\frac{1}{2-x}}(x+2)(x-2)=\underset{(x-2)(x+2)}{\frac{3x+8}{x^2-4}}(x+2)(x-2)$

$LCD:\ (x+2)(x-2)$

$x+2\neq 0\quad x-2\neq 0$

$\underline{\ -2\quad -2\ }\quad \underline{+2\quad +2}$

$x\neq -2\qquad x\neq 2$

$x-2+x+2=3x+8$

$2x=3x+8$

$\underline{-3x\quad -3x}$

$-\frac{x}{-1}=\frac{8}{-1}$

$x=-8$

17) $\frac{(x+1)}{x-1}(6)(x-1)(x+1)+\frac{-x+1}{x+1}(6)(x-1)(x+1)=\frac{5}{6}(6)(x-1)(x+1)$

$LCD: (6)(x-1)(x+1)$

$x-1\neq 0 \quad x+1\neq 0$

$\underline{\quad +1 \quad +1 \quad -1 \quad -1}$

$*x\neq 1**x\neq -1*$

$6x^2+6x+6x+6-6x^2+6x+6x-6=5x^2-5$

$24x=5x^2-5$

$\underline{-24x \quad -24x}$

$0=5x^2-24x-5$

$0=(5x+1)(x-5)$

$5x+1=0 \quad x-5=0$

$\underline{\quad -1 \quad -1} \quad \underline{+5 \quad +5}$

$\frac{5x}{5}=\frac{-1}{5} \quad x=5$

$x=-\frac{1}{5}$

19) $\frac{3}{2x+1}\frac{(2x+1)(2x-1)}{1}+\frac{-2x-1}{2x-1}\frac{(2x+1)(2x-1)}{1}=1\frac{(2x+1)(2x-1)}{1}-\frac{8x^2}{4x^2-1}\frac{(2x+1)(2x-1)}{1}$

$LCD: (2x+1)(2x-1)$

$2x+1\neq 0 \quad 2x-1\neq 0$

$\underline{\quad -1 \quad -1} \quad \underline{+1 \quad +1}$

$\frac{2x}{2}\neq\frac{-1}{2} \quad \frac{2x}{2}\neq\frac{1}{2}$

$*x\neq -\frac{1}{2}* \quad *x\neq\frac{1}{2}*$

$6x-3-4x^2-2x-2x-1=4x^2-1-8x^2$

$-4x^2+2x-4=-4x^2-1$

$\underline{+4x^2 \qquad\qquad +4x^2}$

$2x-4=-1$

$\underline{+4 \quad +4}$

$\frac{2x}{2}=\frac{3}{2}$

$x=\frac{3}{2}$

21) $\frac{x-2}{x+3}(x+3)(x-2)-\frac{1}{x-2}(x+3)(x-2)=\frac{1}{x^2+x-6}(x+3)(x-2)$

$$(x-2)(x+3)$$

$LCD:\ (x+3)(x-2)$

$x+3\neq 0 \quad x-2\neq 0$

$\underline{\ -3\quad -3\ }\quad \underline{\ +2\quad +2\ }$

$*x\neq -3**x\neq 2*$

$x^2-4x+4-x-3=1$

$x^2-5x+1=1$

$\underline{-1\quad -1}$

$x^2-5x=0$

$x(x-5)=0$

$x=0 \quad x-5=0$

$\underline{+5\quad +5}$

$x=5$

23) $\frac{3}{x+2}+\frac{x-1}{x+5}=\frac{5x+20}{6x+24}$ $\quad \frac{5\cancel{(x+4)}}{6\cancel{(x+4)}}$

$\frac{3}{x+2}(6)(x+2)(x+5)+\frac{x-1}{x+5}(6)(x+2)(x+5)=\frac{5}{6}(6)(x+2)(x+5)$

$LCD:\ (6)(x+2)(x+5)$

$x+2\neq 0 \quad x+5\neq 0$

$\underline{\ -2\quad -2\ }\quad \underline{\ -5\quad -5\ }$

$*x\neq -2**x\neq -5*$

$18x+90+6x^2+12x-6x-12=5x^2+25x+10x+50$

$6x^2+24x+78=5x^2+35x+50$

$\underline{-5x^2-35x-50\quad -5x^2-35x-50}$

$x^2-11x+28=0$

$(x-7)(x-4)=0$

$x-7=0 \quad x-4=0$

$\underline{\ +7\quad +7\ }\quad \underline{\ +4\quad +4\ }$

$x=7 \qquad x=4$

25) $\frac{x}{x-1}(x+1)(x-1) - \frac{2}{x+1}(x+1)(x-1) = \frac{4x^2}{\underset{(x+1)(x-1)}{x^2-1}}(x+1)(x-1)$

$LCD: (x+1)(x-1)$

$x+1 \neq 0 \quad x-1 \neq 0$

$\underline{-1 \quad -1 \quad +1 \ +1}$

$*x \neq -1* \quad *x \neq 1*$

$x^2 + x - 2x + 2 = 4x^2$

$x^2 - x + 2 = 4x^2$

$\underline{-x^2 + x - 2 \quad -x^2 + x - 2}$

$0 = 3x^2 + x - x$

$0 = (3x-2)(x+1)$

$3x - 2 = 0 \quad x + 1 = 0$

$\underline{+2 \ +2} \quad \underline{-1 \ -1}$

$\frac{3x}{3} = \frac{2}{3} \qquad x = -1$

$x = \frac{2}{3}$

27) $\frac{2x}{x+1}(x+1)(x+5) - \frac{3}{x+5}(x+1)(x+5) = \frac{-8x^2}{\underset{(x+1)(x+5)}{x^2+6x+5}}(x+1)(x+5)$

$LCD: (x+1)(x+5)$

$x+1 \neq 0 \quad x+5 \neq 0$

$\underline{-1 \ -1} \quad \underline{-5 \ -5}$

$*x \neq -1**x \neq -5*$

$2x^2 + 10x - 3x - 3 = -8x^2$

$2x^2 + 7x - 3 = -8x^2$

$\underline{+8x^2 \qquad\qquad +8x^2}$

$10x^2 + 7x - 3 = 0$

$(10x-3)(x+1) = 0$

$10x - 3 = 0 \quad x + 1 = 0$

$\underline{+3 \ +3} \quad \underline{-1 \ -1}$

$\frac{10x}{10} = \frac{3}{10} \qquad x = -1$

$x = \frac{3}{10}$

29) $\frac{x-5}{x-9}\,(x-9)(x-3)+\frac{x+3}{x-3}(x-9)(x-3)=\frac{-4x^2}{\underset{(x-9)(x-3)}{x^2-12x+27}}\,(x-9)(x-3)$

$LCD:\ (x-9)(x-3)$

$x-9\ \neq 0 \quad x-3\neq 0$

$\underline{\ \ +9\ \ +9\ } \quad \underline{\ +3\ \ \ +3\ }$

$*\,x\neq 9**\,x\neq 3\,*$

$x^2-3x-5x+15+x^2-9x+3x-27=-4x^2$

$2x^2-14x-12=-4x^2$

$\underline{+4x^2 \qquad\qquad\qquad +4x^2}$

$6x^2-14x-12=0$

$2(3x^2-7x-6)=0$

$2(3x+2)(x-3)=0$

$3x+2=0 \quad x-3=0$

$\underline{\ \ -2\ \ -2\ } \quad \underline{\ +3\ \ +3\ }$

$\frac{3x}{3}=\frac{-2}{3} \qquad x=3$

$x=-\frac{2}{3}$

31) $\frac{x-3}{x-6}\,(x-6)(x+3)+\frac{x+5}{x+3}(x-6)(x+3)=\frac{-2x^2}{\underset{(x-6)(x+3)}{x^2-3x-18}}\,(x-6)(x+3)$

$LCD:\ (x-6)(x+3)$

$x-6\ \neq 0 \quad x+3\ \neq 0$

$\underline{\ \ +6\ \ +6\ } \quad \underline{\ -3\ \ \ -3\ }$

$*x\neq 6* \quad *\,x\neq -3\,*$

$x^2-9+x^2-6x+5x-30=-2x^2$

$2x^2-x-90=-2x^2$

$\underline{\ +2x^2\ } \quad \underline{\qquad +2x^2}$

$4x^2-x-90=0$

$(4x-13)(x+3)=0$

$4x-13=0 \quad x+3=0$

$\underline{+13\ \ +13} \quad \underline{\ -3\ \ -3\ }$

$\frac{4x}{4}=\frac{13}{4} \qquad x=-3$

$x=\frac{13}{4}$

33) $\frac{4x+1}{x+3}\,(x+3)(x-1)+\frac{5x-3}{x-1}(x+3)(x-1)=\frac{8x^2}{\underbrace{x^2+2x-3}_{(x+3)(x-1)}}(x+3)(x-1)$

$LCD:\ (x+3)(x-1)$

$x+3\neq 0 \quad x-1\neq 0$

$\underline{\ \ -3\ \ -3\ } \quad \underline{\ +1\ \ +1\ }$

$*\,x\neq -3\,*\,*\,x\neq 1\,*$

$4x^2-4x+x-1+5x^2+15x-3x-9=8x^2$

$9x^2+9x-10=8x^2$

$\underline{-8x^2\qquad} \quad \underline{\qquad -8x^2}$

$x^2+9x-10=0$

$(x+10)(x-1)=0$

$x+10=0 \quad x-1=0$

$\underline{-10\ \ -10} \quad \underline{+1\ \ +1}$

$x=-10 \quad \cancel{x=1}$

7.8

1) $7\ mi\ to\ yd$

$\left(\frac{7\cancel{mi}}{1}\right)\left(\frac{5280\cancel{ft}}{1\cancel{mi}}\right)\left(\frac{1\ yd}{(3\cancel{ft})}\right)=\frac{36960yd}{3}=12{,}320\ yd$

3) $11.2\ mg\ to\ g$

$\left(\frac{11.2\ \cancel{mg}}{1}\right)\left(\frac{1\ g}{1000\ \cancel{mg}}\right)=\frac{11.29}{1000}=0.0112\ g$

5) $9{,}800{,}000\ mm\ to\ mi$

$\left(\frac{9{,}800{,}000\ \cancel{mm}}{1}\right)\left(\frac{1\ \cancel{m}}{1000\ \cancel{mm}}\right)\left(\frac{3.29\ \cancel{ft}}{1\ \cancel{m}}\right)\left(\frac{1\ mi}{5280\ \cancel{ft}}\right)=\frac{32{,}144{,}000\ mi}{5280000}=6.088\ mi$

7) $435{,}000\ m^2\ to\ km^2$

$\left(\frac{435{,}000m^2}{1}\right)\left(\frac{1km}{1{,}000\ m}\right)^2$

$\left(\frac{435{,}000\cancel{m^2}}{1}\right)\left(\frac{1km^2}{1{,}000{,}000\cancel{m^2}}\right)=\frac{435{,}000\ km^2}{1{,}000{,}000}=0.435\ km^2$

9) $0.0065\ km^3 to\ m^3$

$\left(\frac{0.0065km^3}{1}\right)\left(\frac{1000m}{km}\right)^3$

$\left(\frac{0.0065\cancel{km^3}}{1}\right)\left(\frac{(1{,}000{,}000{,}000m^3)}{\cancel{km^3}}\right)=6{,}500{,}000\ m^3$

11) $5{,}500\ cm^3 to\ yd^3$

$\left(\frac{5{,}500cm^3}{1}\right)\left(\frac{1in}{2.54cm}\right)^3\left(\frac{1yd}{36in}\right)^3$

$\left(\frac{5{,}500\cancel{cm^3}}{1}\right)\left(\frac{1\cancel{in^3}}{16.387064\cancel{cm^3}}\right)\left(\frac{1yd^3}{46656\cancel{in^3}}\right)=\frac{5{,}500\ yd^3}{764554.858}=0.00719\ yd^3$

13) 185 $yd/\min$ $to\ min/hr$

$$\left(\frac{185\cancel{yd}}{\cancel{\min}}\right)\left(\frac{3\cancel{ft}}{1yd}\right)\left(\frac{1mi}{5280\cancel{ft}}\right)\left(\frac{60\cancel{min}}{1hr}\right)=\frac{33300mi}{5280hr}=6.307\ mi/hr$$

15) 248 $mi/hr\ to\ m/sec$

$$\left(\frac{248\cancel{mi}}{\cancel{hr}}\right)\left(\frac{1.61\cancel{km}}{1\cancel{mi}}\right)\left(\frac{1000m}{1\cancel{km}}\right)\left(\frac{1\cancel{hr}}{3600\sec}\right)=\frac{399.280m}{3600\sec}=110.9\ m/sec$$

17) $7.5\frac{T}{yd^2}\ to\ lbs/in^2$

$$\left(\frac{7.5T}{yd^2}\right)\left(\frac{2000lbs}{1T}\right)\left(\frac{1yd}{36in}\right)^2$$

$$\left(\frac{7.5\cancel{T}}{\cancel{yd^2}}\right)\left(\frac{2000lbs}{1\cancel{T}}\right)\left(\frac{1\cancel{yd^2}}{1296in^2}\right)=\frac{15000lbs}{1296in^2}=11.57\ lbs/in^2$$

19) On a recent trip, Jan traveled 260 miles using 8 gallons of gas. How many miles per 1-gallon did she travel? How many yards per 1-ounce?

$$\frac{260mi}{8gal}=32.5\ mi/gal$$

$$\left(\frac{32.5\cancel{mi}}{\cancel{gal}}\right)\left(\frac{5280\cancel{ft}}{1\cancel{mi}}\right)\left(\frac{1yd}{3\cancel{ft}}\right)\left(\frac{1\cancel{gal}}{4\cancel{qt}}\right)\left(\frac{1\cancel{qt}}{2\cancel{pt}}\right)\left(\frac{1\cancel{pt}}{2\cancel{c}}\right)\left(\frac{1\cancel{c}}{8oz}\right)=\frac{171{,}600yd}{384oz}=446.875\ yd/oz$$

21) A certain laser printer can print 12 pages per minute. Determine this printer's output in pages per day, and reams per month. (1 ream = 5000 pages)

$$\left(\frac{12pg}{1\cancel{\min}}\right)\left(\frac{60\cancel{min}}{1\cancel{hr}}\right)\left(\frac{24\cancel{hr}}{1day}\right)=17280\ pg/day$$

$$\left(\frac{17280\cancel{pg}}{\cancel{day}}\right)\left(\frac{30\cancel{day}}{mon}\right)\left(\frac{1ream}{5000\cancel{pg}}\right)=\frac{5184000reams}{5000months}=103.68\ reams/month$$

23) Blood sugar levels are measured in miligrams of gluclose per deciliter of blood volume. If a person's blood sugar level measured 128 mg/dL, how much is this in grams per liter?

$$\left(\frac{128\cancel{mg}}{\cancel{dL}}\right)\left(\frac{1g}{100\cancel{mg}}\right)\left(\frac{10\cancel{dL}}{1L}\right)=\frac{1280g}{1000L}=1.28\ g/L$$

25) A car travels 14 miles in 15 minutes. How fast is it going in miles per hour? in meters per second?

$$\left(\frac{14mi}{15\cancel{\min}}\right)\left(\frac{60\cancel{min}}{1hr}\right)=\frac{840mi}{15hr}=56\ mi/hr$$

$$\left(\frac{14\cancel{mi}}{15\cancel{\min}}\right)\left(\frac{1.61\cancel{km}}{1\cancel{mi}}\right)\left(\frac{1000m}{1\cancel{km}}\right)\left(\frac{1\cancel{min}}{60\sec}\right)=\frac{22540m}{900\ sec}=25.04\ m/sec$$

27) A local zoning ordinance says that a house's "footprint" (area of its ground floor) cannot occupy more than $\frac{1}{4}$ of the lot it is built on. Suppose you own a $\frac{1}{3}$ acre lot, what is the maximum allowed footprint for your house in square feet? in square inches? (1 acre = 43560 ft^2)

$$\left(\frac{1\cancel{acre}}{3}\right)\left(\frac{43560ft^2}{1\cancel{acre}}\right)\left(\frac{1}{4}\right) = \frac{43560ft^2}{12} = 3{,}630\ ft^2$$

$$\left(\frac{3630ft^2}{1}\right)\left(\frac{12in}{1ft}\right)^2$$

$$\left(\frac{3630\cancel{ft^2}}{1}\right)\left(\frac{144in^2}{1\cancel{ft^2}}\right) = 522{,}720\ in^2$$

29) In April 1996, the Department of the Interior released a "spike flood" from the Glen Canyon Dam on the Colorado River. Its purpose was to restore the river and the habitants along its bank. The release from the dam lasted a week at a rate of 25,800 cubic feet of water per second. About how much water was released during the 1-week flood

$$\left(\frac{25{,}800ft^3}{1\ \cancel{sec}}\right)\left(\frac{3600\cancel{sec}}{1\cancel{hr}}\right)\left(\frac{24\cancel{hr}}{1\cancel{day}}\right)\left(\frac{7\cancel{day}}{1wk}\right) = 15{,}603{,}840{,}000\ ft^3/week$$

Chapter 8: Radicals

8.1

1) $r\sqrt{245}$
$\sqrt{5\cdot 7^2}$
$7\sqrt{5}$

4) $\sqrt{36}$
$\sqrt{2^2\cdot 3^2}$
$2\cdot 3$
6

5) $\sqrt{12}$
$\sqrt{2^2\cdot 3}$
$2\sqrt{3}$

7) $3\sqrt{12}$
$3\sqrt{2^2\cdot 3}$
$3\cdot 2\sqrt{3}$
$6\sqrt{3}$

9) $6\sqrt{128}$
$6\sqrt{2^7}$
$6\cdot 2^3\sqrt{2}$
$6\cdot 8\sqrt{2}$
$48\sqrt{2}$

11) $-8\sqrt{392}$
$-8\sqrt{7^2\cdot 2^3}$
$-8\cdot 7\cdot 2\sqrt{2}$
$-112\sqrt{2}$

13) $\sqrt{192n}$
$\sqrt{2^6\cdot 3n}$
$2^3\sqrt{3n}$
$8\sqrt{3n}$

15) $\sqrt{196v^2}$
$\sqrt{2^2\cdot 7^2v^2}$
$2\cdot 7\cdot v$
$14v$

17) $\sqrt{252x^2}$
$\sqrt{2^2\cdot 3^2\cdot 7x^2}$
$2\cdot 3x\sqrt{7}$
$6x\sqrt{7}$

19) $-\sqrt{100k^4}$
$-\sqrt{2^2\cdot 5^2k^4}$
$-2\cdot 5\cdot k^2$
$-10k^2$

21) $-7\sqrt{64x^4}$
$-7\sqrt{2^6x^4}$
$-7\cdot 2^3x^2$
$-7\cdot 8x^2$
$-56x^2$

23) $-5\sqrt{36m}$
$-5\sqrt{2^2\cdot 3^2m}$
$-5\cdot 2\cdot 3\sqrt{m}$
$-30\sqrt{m}$

25) $\sqrt{45x^2y^2}$
$\sqrt{3^2\cdot 5\cdot x^2y^2}$
$3xy\sqrt{5}$

27) $\sqrt{16x^3y^3}$
$\sqrt{2^4x^3y^3}$
$2^2xy\sqrt{xy}$
$4xy\sqrt{xy}$

29) $\sqrt{320x^4y^4}$
$\sqrt{2^6\cdot 5x^4y^4}$
$2^3x^2y^2\sqrt{5}$
$8x^2y^2\sqrt{5}$

31) $6\sqrt{80xy^2}$
$6\sqrt{2^4\cdot 5xy^2}$
$6\cdot 2^2y\sqrt{5x}$
$6\cdot 4y\sqrt{5x}$
$24y\sqrt{5x}$

33) $5\sqrt{245x^2y^3}$
$5\sqrt{5\cdot 7^2x^2y^3}$
$5\cdot 7xy\sqrt{5y}$
$35xy\sqrt{5y}$

35) $-2\sqrt{180u^3v}$
$-2\sqrt{2^2\cdot 3^2\cdot 5u^3v}$
$-2\cdot 2\cdot 3u\sqrt{5uv}$
$-12u\sqrt{5uv}$

37) $-8\sqrt{180x^4y^2z^4}$

$-8\sqrt{2^2\cdot 3^2\cdot 5x^4y^2z^4}$
$-8\cdot 2\cdot 3x^2yz^2\sqrt{5}$
$-48x^2yz^2\sqrt{5}$

39) $2\sqrt{80hj^4k}$
$2\sqrt{2^4\cdot 5hj^4k}$
$2\cdot 2^2j^2\sqrt{5hk}$
$2\cdot 4j^2\sqrt{5hk}$
$8j^2\sqrt{5hk}$

41) $-4\sqrt{54mnp^2}$
$-4\sqrt{2\cdot 3^2mnp^2}$
$-4\cdot 3p\sqrt{2\cdot 3mn}$
$-12p\sqrt{60n}$

8.2

1) $\sqrt[3]{625}$
$\sqrt[3]{5^4}$
$5\sqrt[3]{5}$

3) $r\sqrt[3]{750}$
$\sqrt[3]{2\cdot 3\cdot 5^3}$
$5\sqrt[3]{2\cdot 3}$
$5\sqrt[3]{6}$

5) $\sqrt[3]{875}$
$\sqrt[3]{5^3\cdot 7}$
$5\sqrt[3]{7}$

7) $-4\sqrt[4]{96}$
$-4\sqrt[4]{2^5\cdot 3}$
$-4\cdot 2\sqrt[4]{2\cdot 3}$
$-8\sqrt[4]{6}$

9) $6\sqrt[4]{112}$
$6\sqrt[4]{2^4\cdot 7}$
$6\cdot 2\sqrt[4]{7}$
$12\sqrt[4]{7}$

11) $-\sqrt[4]{112}$
$-\sqrt[4]{2^4\cdot 7}$
$-2\sqrt[4]{7}$

13) $r\sqrt[4]{648a^2}$
$\sqrt[4]{2^3\cdot 3^4a^2}$
$3\sqrt[4]{2^3a^2}$
$3\sqrt[4]{8a^2}$

15) $\sqrt[5]{224n^3}$
$\sqrt[5]{2^5\cdot 7n^3}$
$2\sqrt[5]{7n^3}$

17) $r\sqrt[5]{224p^5}$
$\sqrt[5]{2^5\cdot 7p^5}$
$2p\sqrt[5]{7}$

19) $-3\sqrt[7]{896r}$
$-3\sqrt[7]{2^7\cdot 7r}$
$-3\cdot 2\sqrt[7]{7r}$
$-6\sqrt[7]{7r}$

21) $r-2\sqrt[3]{-48v^7}$
$-2\sqrt[3]{-1^3\cdot 2^4\cdot 3v^7}$
$-2\cdot -1\cdot 2v^2\sqrt[3]{2\cdot 3v}$
$4v^2\sqrt[3]{6v}$

23) $-7\sqrt[3]{320n^6}$
$-7\sqrt[3]{2^6\cdot 5n^6}$
$-7\cdot 2^2n^2\sqrt[3]{5}$
$-7\cdot 4n^2\sqrt[3]{5}$
$-28n^2\sqrt[3]{5}$

25) $\sqrt[3]{-135x^5y^3}$
$\sqrt[3]{-1^3\cdot 3^3\cdot 5x^5y^3}$
$-1\cdot 3xy\sqrt[3]{5x^2}$
$-3xy\sqrt[3]{5x^2}$

27) $\sqrt[3]{-32x^4y^4}$
$\sqrt[3]{-1^3\cdot 2^5x^4y^4}$
$-1\cdot 2xy\ \sqrt[3]{2^2xy}$
$-2xy\ \sqrt[3]{4xy}$

29) $r\sqrt[3]{256x^4y^6}$
$\sqrt[3]{2^8x^4y^6}$
$2^2xy^2\ \sqrt[3]{2^2x}$
$4xy^2\ \sqrt[3]{4x}$

31) $7\sqrt[3]{-81x^3y^7}$
$7\sqrt[3]{-1^3\cdot 3^4x^3y^7}$
$7\cdot -1\cdot 3xy^2\sqrt[3]{3y}$
$-21xy^2\ \sqrt[3]{3y}$

33) $2\sqrt[3]{375u^2v^8}$
$2\sqrt[3]{3\cdot 5^3u^2v^8}$
$2\cdot 5v^2\sqrt[3]{3u^2v^2}$
$10v^2\sqrt[3]{3u^2v^2}$

35) $-3\sqrt[3]{192ab^2}$
$-3\sqrt[3]{2^6\cdot 3ab^2}$
$-3\cdot 2^2\ \sqrt[3]{3ab^2}$
$-3\cdot 4\ \sqrt[3]{3ab^2}$
$-12\ \sqrt[3]{3ab^2}$

37) $6\sqrt[3]{-54m^8n^3p^7}$

$6\sqrt[3]{-1^3\cdot 2\cdot 3^3m^8n^3p^7}$

$6\cdot -1\cdot$
$3m^2np^2\sqrt[3]{2m^2p}$
$-18m^2np^2\ \sqrt[3]{2m^2p}$

39) $6\sqrt[4]{648x^5y^7z^2}$
$6\sqrt[4]{2^3\cdot 3^4x^5y^7z^2}$
$6\cdot 3xy\sqrt[4]{2^3xy^3z^2}$
$18xy\ \sqrt[4]{8xy^3z^2}$

41) $7\sqrt[4]{128h^6j^8k^8}$
$7\sqrt[4]{2^7h^6j^8k^8}$
$7\cdot 2hj^2k^2\ \sqrt[4]{2^3h^2}$
$14hj^2k^2\ \sqrt[4]{8h^2}$

8.3

1) $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$
$6\sqrt{5}$

3) $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$
$-3\sqrt{2}+6\sqrt{5}$

6) $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$
$-5\sqrt{6}$

8) $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$
$3\sqrt{6}+5\sqrt{5}$

10) $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$
$2\sqrt{2}-3\sqrt{3^2\cdot 2}-\sqrt{2}$
$2\sqrt{2}-3\cdot 3\sqrt{2}-\sqrt{2}$
$2\sqrt{2}-9\sqrt{2}-\sqrt{2}$
$-8\sqrt{2}$

12) $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$
$-3\sqrt{6}-\sqrt{2^2\cdot 3}+3\sqrt{3}$
$-3\sqrt{6}-2\sqrt{3}+3\sqrt{3}$
$-3\sqrt{6}+\sqrt{3}$

14) $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$
$3\sqrt{2}+2\sqrt{2^3}-3\sqrt{2\cdot 3^2}$
$3\sqrt{2}+2\cdot 2\sqrt{2}-3\cdot 3\sqrt{2}$
$3\sqrt{2}+4\sqrt{2}-9\sqrt{2}$
$-2\sqrt{2}$

16) $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$
$3\sqrt{2\cdot 3^2}-\sqrt{2}-3\sqrt{2}$
$3\cdot 3\sqrt{2}-\sqrt{2}-3\sqrt{2}$
$9\sqrt{2}-2\sqrt{2}-3\sqrt{2}$
$5\sqrt{2}$

19) $-3\sqrt{6}-3\sqrt{6}-\sqrt{3}+3\sqrt{6}$
$-3\sqrt{6}-\sqrt{3}$

20) $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$
$-2\sqrt{2\cdot 3^2}-3\sqrt{2^3}-\sqrt{2^2\cdot 5}+2\sqrt{2^2\cdot 5}$
$-2\cdot 3\sqrt{2}-3\cdot 2\sqrt{2}-2\sqrt{5}+2\cdot 2\sqrt{5}$
$-6\sqrt{2}-6\sqrt{2}-2\sqrt{5}+4\sqrt{5}$
$-12\sqrt{2}-2\sqrt{5}$

21) $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$
$-2\sqrt{2^3\cdot 3}-2\sqrt{6}+2\sqrt{6}+2\sqrt{2^2\cdot 5}$
$-2\cdot 2\sqrt{2\cdot 3}-2\sqrt{6}+2\sqrt{6}+2\cdot 2\sqrt{5}$
$-4\sqrt{6}-2\sqrt{6}+2\sqrt{6}+4\sqrt{5}$
$-4\sqrt{6}+4\sqrt{5}$

23) $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$
$3\sqrt{2^3\cdot 3}-3\sqrt{3^3}+2\sqrt{6}+2\sqrt{2^3}$
$3\cdot 2\sqrt{2\cdot 3}-3\cdot 3\sqrt{3}+2\sqrt{6}+2\cdot 2\sqrt{2}$
$6\sqrt{6}-9\sqrt{3}+2\sqrt{6}+4\sqrt{2}$
$8\sqrt{6}-9\sqrt{3}+4\sqrt{2}$

25) $-2\sqrt[3]{16}+2\sqrt[3]{16}+2\sqrt[3]{2}$
$-2\sqrt[3]{2^4}+2\sqrt[3]{2^4}+2\sqrt[3]{2}$
$-2\cdot 2\sqrt[3]{2}+2\cdot 2\sqrt[3]{2}+2\sqrt[3]{2}$
$-4\sqrt[3]{2}+4\sqrt[3]{2}+2\sqrt[3]{2}$
$2\sqrt[3]{2}$

27) $2\sqrt[4]{243}-2\sqrt[4]{253}-\sqrt[4]{3}$
$2\sqrt[4]{3^5}-2\sqrt[4]{3^5}-\sqrt[4]{3}$
$2\cdot 3\sqrt[4]{3}-2\cdot 3\sqrt[4]{3}-\sqrt[4]{3}$
$6\sqrt[4]{3}-6\sqrt[4]{3}-\sqrt[4]{3}$
$-\sqrt[4]{3}$

29) $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$
$3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{3^5}$
$3\sqrt[4]{2}-2\sqrt[4]{2}-3\sqrt[4]{3}$
$\sqrt[4]{2}-3\sqrt[4]{3}$

31) $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

$-\sqrt[4]{3^5}+3\sqrt[4]{3^5}-3\sqrt[4]{4}$

$-3\sqrt[4]{3}+3\cdot 3\sqrt[4]{3}-3\sqrt[4]{4}$

$-3\sqrt[4]{3}+9\sqrt[4]{3}-3\sqrt[4]{4}$

$6\sqrt[4]{3}-3\sqrt[4]{4}$

33) $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

$2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{2^6}-\sqrt[4]{3}$

$2\sqrt[4]{2}+2\sqrt[4]{3}+3\cdot 2\sqrt[4]{2^2}-\sqrt[4]{3}$

$2\sqrt[4]{2}+2\sqrt[4]{3}+6\sqrt[4]{4}-\sqrt[4]{3}$

$2\sqrt[4]{2}+\sqrt[4]{3}+6\sqrt[4]{4}$

35) $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{64}$

$-3\sqrt[5]{6}-\sqrt[5]{2^6}+2\sqrt[5]{2^6\cdot 3}-2\sqrt[5]{2^6}$

$-3\sqrt[5]{6}-2\sqrt[5]{2}+2\cdot 2\sqrt[5]{2\cdot 3}-2\cdot 2\sqrt[5]{2}$

$-3\sqrt[5]{6}-2\sqrt[5]{2}+4\sqrt[5]{6}-4\sqrt[5]{2}$

$\sqrt[5]{6}-6\sqrt[5]{2}$

37) $2\sqrt[5]{160}-2\sqrt[5]{192}-\sqrt[5]{160}-\sqrt[5]{-160}$

$2\sqrt[5]{2^5\cdot 5}-2\sqrt[5]{2^6\cdot 3}-\sqrt[5]{2^5\cdot 5}-\sqrt[5]{-1^5\cdot 2^5\cdot 5}$

$2\cdot 2\sqrt[5]{5}-2\cdot 2\sqrt[5]{2\cdot 3}-2\sqrt[5]{5}-(-1)\cdot 2\sqrt[5]{5}$

$4\sqrt[5]{5}-4\sqrt[5]{6}-2\sqrt[5]{5}+2\sqrt[5]{5}$

$4\sqrt[5]{5}-4\sqrt[5]{6}$

39) $-\sqrt[6]{256}-2\sqrt[6]{4}-3\sqrt[6]{320}-2\sqrt[6]{128}$

$-\sqrt[6]{2^8}-2\sqrt[6]{4}-3\sqrt[6]{2^6\cdot 5}-2\sqrt[6]{2^7}$

$-2\sqrt[6]{2^2}-2\sqrt[6]{4}-3\cdot 2\sqrt[6]{5}-2\cdot 2\sqrt[6]{2}$

$-2\sqrt[6]{4}-2\sqrt[6]{4}-6\sqrt[6]{5}-4\sqrt[6]{2}$

$-4\sqrt[6]{4}-6\sqrt[6]{5}-4\sqrt[6]{2}$

8.4

2) $3\sqrt{5}\cdot -4\sqrt{16}$
$-12\sqrt{80}$
$-12\sqrt{2^4\cdot 5}$
$-12\cdot 2^2\sqrt{5}$
$-12\cdot 4\sqrt{5}$
$-48\sqrt{5}$

5) $\sqrt{12m}\sqrt{15m}$
$\sqrt{180m^2}$
$\sqrt{2^2\cdot 3^2\cdot 5m^2}$
$2\cdot 3m\sqrt{5}$
$6m\sqrt{5}$

7) $\sqrt[3]{4x^3}\,\sqrt[3]{2x^4}$
$\sqrt[3]{8x^7}$
$\sqrt[3]{2^3x^7}$
$2x^2\sqrt[3]{x}$

9) $\sqrt{6}(\sqrt{2}+2)$
$\sqrt{12}+2\sqrt{6}$
$\sqrt{2^2\cdot 3}+2\sqrt{6}$
$2\sqrt{3}+2\sqrt{6}$

11) $-5\sqrt{15}(3\sqrt{3}+2)$
$-15\sqrt{45}-10\sqrt{15}$
$-15\sqrt{3^2\cdot 5}-10\sqrt{15}$
$-15\cdot 3\sqrt{5}-10\sqrt{15}$
$-45\sqrt{5}-10\sqrt{15}$

13) $5\sqrt{10}(5n+\sqrt{2})$
$25n\sqrt{10}+5\sqrt{20}$
$25n\sqrt{10}+5\sqrt{2^2+5}$
$25n\sqrt{10}+5\cdot 2\sqrt{5}$
$25n\sqrt{10}+10\sqrt{5}$

15) $(2+2\sqrt{2})(-3+\sqrt{2})$
$-6+2\sqrt{2}-6\sqrt{2}+2\sqrt{4}$
$-6+2\sqrt{2}-6\sqrt{2}+2\sqrt{2^2}$
$-6+2\sqrt{2}-6\sqrt{2}+2\cdot 2$
$-6+2\sqrt{2}-6\sqrt{2}+4$
$-2-4\sqrt{2}$

17) $(\sqrt{5}-5)(2\sqrt{5}-1)$
$2\sqrt{25}-\sqrt{5}-10\sqrt{5}+5$
$2\sqrt{5^2}-\sqrt{5}-10\sqrt{5}+5$
$2\cdot 5-\sqrt{5}-10\sqrt{5}+5$
$10-\sqrt{5}-10\sqrt{5}+5$
$15-11\sqrt{5}$

20) $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$
$3\sqrt{4a^2}+\sqrt{10a^2}+6\sqrt{6a^2}+2\sqrt{15a^2}$
$3\sqrt{2^2a^2}+\sqrt{10a^2}+6\sqrt{6a^2}+2\sqrt{15a^2}$
$3\cdot 2a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$
$6a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$

21) $(-5-4\sqrt{3})(-3-4\sqrt{3})$
$15+20\sqrt{3}+12\sqrt{3}+16\sqrt{9}$
$15+20\sqrt{3}+12\sqrt{3}+16\sqrt{3^2}$
$15+20\sqrt{3}+12\sqrt{3}+16\cdot 3$
$15+20\sqrt{3}+12\sqrt{3}+48$
$63+32\sqrt{3}$

21) $\frac{\sqrt{12}}{5\sqrt{100}}=\frac{\sqrt{3}}{5\sqrt{25}}=\frac{\sqrt{3}}{5\sqrt{5^2}}=\frac{\sqrt{3}}{5\cdot 5}=\frac{\sqrt{3}}{25}$

23) $\frac{\sqrt{5}}{4\sqrt{125}}=\frac{1}{4\sqrt{25}}=\frac{1}{4\sqrt{5^2}}=\frac{1}{4\cdot 5}=\frac{1}{20}$

25) $\frac{\sqrt{10}}{\sqrt{6}}=\frac{\sqrt{5}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{\sqrt{15}}{3}$

27) $\frac{2\sqrt{4}}{3\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{2\sqrt{12}}{3\cdot 3}=\frac{2\sqrt{2^2\cdot 3}}{9}=\frac{2\cdot 2\sqrt{3}}{9}=\frac{4\sqrt{3}}{9}$

29) $\frac{5x^2}{4\sqrt{3x^3y^3}} = \frac{5x^2}{4xy\sqrt{3xy}} = \frac{5x}{4y\sqrt{3xy}}\left(\frac{\sqrt{3xy}}{\sqrt{3xy}}\right) = \frac{5x\sqrt{3xy}}{4y\cdot 3xy} = \frac{5x\sqrt{3xy}}{12xy^2}$

31) $\frac{\sqrt{2p^2}}{\sqrt{3p}} = \frac{\sqrt{2p}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{6p}}{3}$

33) $\frac{3\sqrt[3]{10}}{5\sqrt[3]{27}} = \frac{3\sqrt[3]{10}}{5\sqrt[3]{3^2}} = \frac{3\sqrt[3]{10}}{5\cdot 3} = \frac{3\sqrt[3]{10}}{15} = \frac{\sqrt[3]{10}}{5}$

35) $\frac{\sqrt[3]{5}}{4\sqrt[3]{4}} = \frac{\sqrt[3]{5}}{4\sqrt[3]{2^2}}\left(\frac{\sqrt[3]{2}}{\sqrt[3]{2}}\right) = \frac{3\sqrt[3]{10}}{4\cdot 2} = \frac{3\sqrt[3]{10}}{8}$

37) $\frac{5\sqrt[4]{5r^4}}{\sqrt[4]{8r^2}} = \frac{5\sqrt[4]{5r^2}}{\sqrt[4]{8}} = \frac{5\sqrt[4]{5r^2}}{\sqrt[4]{2^3}}\left(\frac{\sqrt[4]{2}}{\sqrt[4]{2}}\right) = \frac{5\sqrt[4]{10r}}{2}$

8.5

1) $\frac{4+2\sqrt{3}}{\sqrt{9}} = \frac{4+2\sqrt{3}}{3}$

3) $\frac{4+2\sqrt{3}}{5\sqrt{4}} = \frac{4+2\sqrt{3}}{5\cdot 2} = \frac{4+2\sqrt{3}}{10} = \frac{2(2+\sqrt{3})}{10} = \frac{2+\sqrt{3}}{5}$

5) $\frac{2-5\sqrt{5}}{4\sqrt{13}}\left(\frac{\sqrt{13}}{\sqrt{13}}\right) = \frac{2\sqrt{13}-5\sqrt{65}}{4\cdot 13} = \frac{2\sqrt{13}-5\sqrt{65}}{52}$

7) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{6}-3\sqrt{9}}{3} = \frac{\sqrt{6}-3\cdot 3}{3} = \frac{\sqrt{6}-9}{3}$

9) $\frac{2p+3\sqrt{5p^4}}{5\sqrt{20p^2}} = \frac{2p+3\sqrt{5p^4}}{5\sqrt{2^2\cdot 5p^2}} = \frac{2p+3p^2\sqrt{5}}{5\cdot 2p\sqrt{5}} = \frac{2p+3p^2\sqrt{5}}{10p\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{2p\sqrt{5}+3p^2\sqrt{25}}{10p(5)} = \frac{2p\sqrt{5}+3p^2\cdot 5}{50p} =$
$\frac{2p\sqrt{5}+15p^2}{50p} = \frac{p(2\sqrt{5}+15p)}{50p} = \frac{2\sqrt{5}+15p}{50}$

11)
$\frac{\sqrt{3m^2}-4\sqrt{2m^4}}{5\sqrt{12m^4}} = \frac{\sqrt{3m^2}-4\sqrt{2m^4}}{5\sqrt{2^2\cdot 3m^4}} = \frac{m\sqrt{3}-4m^2\sqrt{2}}{2\cdot 5m^2\sqrt{3}} = \frac{m\sqrt{3}-4m^2\sqrt{2}}{10m^2\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{m\sqrt{9}-4m^2\sqrt{6}}{10m^2\cdot 3} =$
$\frac{3m-4m^2\sqrt{6}}{30m^2} = \frac{m(3-4m\sqrt{6})}{30m^2} = \frac{3-4m\sqrt{6}}{30m}$

13) $\frac{5}{3\sqrt{5}+\sqrt{2}}\left(\frac{3\sqrt{5}+2}{3\sqrt{5}+2}\right) = \frac{15\sqrt{5}-5\sqrt{2}}{9\cdot 5-2} = \frac{15\sqrt{5}-5\sqrt{2}}{45-2} = \frac{15\sqrt{5}-5\sqrt{2}}{43}$

15) $\frac{2}{5+\sqrt{2}}\left(\frac{5-\sqrt{2}}{5-\sqrt{2}}\right)=\frac{10-2\sqrt{2}}{25-2}=\frac{10-2\sqrt{2}}{23}$

17) $\frac{3}{4-3\sqrt{3}}\left(\frac{4+3\sqrt{3}}{4+3\sqrt{3}}\right)=\frac{12+9\sqrt{3}}{16-9\cdot 3}=\frac{12+9\sqrt{3}}{16-27}=\frac{12+9\sqrt{3}}{-11}$

19) $\frac{4}{3+\sqrt{5}}\left(\frac{3-\sqrt{5}}{3-\sqrt{5}}\right)=\frac{12-4\sqrt{5}}{9-5}=\frac{12-4\sqrt{5}}{4}=\frac{4(3-\sqrt{5})}{4}=3-\sqrt{5}$

21) $\frac{-4}{4-4\sqrt{2}}\left(\frac{4+4\sqrt{2}}{4+4\sqrt{2}}\right)=\frac{-16-16\sqrt{2}}{16-16\cdot 2}=\frac{-16-16\sqrt{2}}{16-32}=\frac{-16-16\sqrt{2}}{-16}=\frac{-16(1+\sqrt{2})}{-16}=1+\sqrt{2}$

23) $\frac{5}{\sqrt{n^4}-5}=\frac{5}{n^2-5}$

25) $\frac{4p}{3-5\sqrt{p^4}}=\frac{4p}{3-5p^2}$

27) $\frac{4}{5+\sqrt{5x^2}}=\frac{4}{5+x\sqrt{5}}\left(\frac{5-x\sqrt{5}}{5-x\sqrt{5}}\right)=\frac{20-4x\sqrt{5}}{25+5x^2}$

29) $\frac{5}{2+\sqrt{5r^3}}=\frac{5}{2+r\sqrt{5r}}\left(\frac{2-r\sqrt{5r}}{2-r\sqrt{5r}}\right)=\frac{10-5r\sqrt{5r}}{4-r^2(5r)}=\frac{10-5r\sqrt{5r}}{4-5r^3}$

31) $\frac{5}{-5v-3\sqrt{v}}\left(\frac{-5v+3\sqrt{v}}{-5v+3\sqrt{v}}\right)=\frac{-25v+15\sqrt{v}}{25v^2-9v}$

33) $\frac{4\sqrt{2}+3}{3\sqrt{2}+\sqrt{3}}\left(\frac{3\sqrt{2}-\sqrt{3}}{3\sqrt{2}-\sqrt{3}}\right)=\frac{12\sqrt{4}-4\sqrt{6}+9\sqrt{2}-3\sqrt{3}}{9\cdot 2-3}=\frac{12\cdot 2-4\sqrt{6}+9\sqrt{2}-3\sqrt{3}}{18-3}=\frac{24-4\sqrt{6}+9\sqrt{2}-3\sqrt{3}}{15}$

35) $\frac{2-\sqrt{5}}{-3+\sqrt{5}}\left(\frac{-3-\sqrt{5}}{-3-\sqrt{5}}\right)=\frac{-6-2\sqrt{5}+3\sqrt{5}+\sqrt{25}}{9-5}=\frac{-6-2\sqrt{5}+3\sqrt{5}+5}{4}=\frac{-1+\sqrt{5}}{4}$

37) $\frac{5\sqrt{2}+\sqrt{3}}{5+5\sqrt{2}}\left(\frac{5-5\sqrt{2}}{5-5\sqrt{2}}\right)=\frac{25\sqrt{2}-25\sqrt{4}+5\sqrt{3}-5\sqrt{6}}{25-25\cdot 2}=\frac{25\sqrt{2}-25\cdot 2+5\sqrt{3}-5\sqrt{6}}{25-50}=$

$\frac{25\sqrt{2}-50+5\sqrt{3}-5\sqrt{6}}{-25}=\frac{5(5\sqrt{2}-10+\sqrt{3}-\sqrt{6})}{-25}=\frac{5\sqrt{2}-10+\sqrt{3}-\sqrt{6}}{-5}$

39) $\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}\left(\frac{2\sqrt{3}+\sqrt{2}}{2\sqrt{3}+\sqrt{2}}\right)=\frac{2\sqrt{9}+\sqrt{6}+2\sqrt{6}+\sqrt{4}}{4\cdot 3-2}=\frac{2\cdot 3+\sqrt{6}+2\sqrt{6}+2}{12-2}=\frac{6+\sqrt{6}+2\sqrt{6}+2}{10}=\frac{8+3\sqrt{6}}{10}$

41) $\frac{\sqrt{3}-\sqrt{2}}{4+\sqrt{5}}\left(\frac{4-\sqrt{5}}{4-\sqrt{5}}\right)=\frac{4\sqrt{3}-\sqrt{15}-4\sqrt{2}+\sqrt{10}}{16-5}=\frac{4\sqrt{3}-\sqrt{15}-4\sqrt{2}+\sqrt{10}}{11}$

43) $\frac{4+2\sqrt{2x^2}}{5+2\sqrt{5x^3}}=\frac{4+2x\sqrt{2}}{5+2x\sqrt{5x}}\left(\frac{5-2x\sqrt{5x}}{5-2x\sqrt{5x}}\right)=\frac{20+8x\sqrt{5x}+10x\sqrt{2}-4x^2\sqrt{10x}}{25-4x^2(5x)}=$
$\frac{20+8x\sqrt{5x}+10x\sqrt{2}-4x^2\sqrt{10x}}{25-20x^3}$

45) $\frac{\left(2\sqrt{3m^2}-\sqrt{2m^4}\right)}{5-\sqrt{3m^2}}=\frac{\left(2m\sqrt{3}-m^2\sqrt{2}\right)}{5-m\sqrt{3}}\left(\frac{5+m\sqrt{3}}{5+m\sqrt{3}}\right)=\frac{10m\sqrt{3}+2m^2\sqrt{9}-5m^2\sqrt{2}-m^3\sqrt{6}}{25-3m^2}=$
$\frac{10m\sqrt{3}+2m^2(3)-5m^2\sqrt{2}-m^3\sqrt{6}}{25-3m^2}=\frac{10m\sqrt{3}+6m^2-5m^2\sqrt{2}-m^3\sqrt{6}}{25-3m^2}$

47) $\frac{2b-5\sqrt{2b}}{-1+\sqrt{2b^4}}=\frac{2b-5\sqrt{2b}}{-1+b^2\sqrt{2}}\left(\frac{-1-\sqrt{2}}{-1-\sqrt{2}}\right)=\frac{-2b-2b^3\sqrt{2}+5\sqrt{2b}-5b^2\sqrt{4b}}{1-2b^4}=$
$\frac{-2b-2b^3\sqrt{2}+5\sqrt{2b}-2\cdot 5b^2\sqrt{b}}{1-2b^4}=\frac{-2b-2b^3\sqrt{2}+5\sqrt{2b}-10b^2\sqrt{b}}{1-2b^4}$

49) $\frac{2-\sqrt{2x}}{4x-5\sqrt{3x^3}}=\frac{2-\sqrt{2x}}{4x-5x\sqrt{3x}}\left(\frac{4x+5x\sqrt{3x}}{4x+5x\sqrt{3x}}\right)=\frac{8x+10x\sqrt{3x}-4x\sqrt{2x}-5x\sqrt{6x^2}}{16x^2+25x^2(3x)}=$
$\frac{8x+10x\sqrt{3x}-4x\sqrt{2x}-5x^2\sqrt{6}}{16x^2+75x^2}=\frac{x(8+10\sqrt{3x}-4\sqrt{2x}-5x\sqrt{6})}{x(16x+75x)}=\frac{8+10\sqrt{3x}-4\sqrt{2x}-5x\sqrt{6}}{16x+75x}$

51) $\frac{-4p-\sqrt{p}}{-p-\sqrt{p^3}}=\frac{-4p-\sqrt{p}}{-p-p\sqrt{p}}\left(\frac{-p+p\sqrt{p}}{-p+p\sqrt{p}}\right)=\frac{4p^2-4p^2\sqrt{p}+p\sqrt{p}-p\sqrt{p^2}}{p^2-p^2\cdot p}=\frac{4p^2-4p^2\sqrt{p}+p\sqrt{p}-p^2}{p^2-p^3}=$
$\frac{3p^2-4p^2\sqrt{p}+p\sqrt{p}}{p^2-p^3}=\frac{p(3p-4p\sqrt{p}+\sqrt{p})}{p(p-p^2)}=\frac{3p-4p\sqrt{p}+\sqrt{p}}{p-p^2}$

8.6

1) $m^{\frac{3}{5}}=\left(\sqrt[5]{m}\right)^3$

3) $r(7x)^{\frac{3}{2}}=\left(\sqrt{7x}\right)^3$

5) $\frac{1}{\left(\sqrt{6x}\right)^3}=(6x)^{-\frac{3}{2}}$

7) $\frac{1}{\left(\sqrt[4]{n}\right)^7}=n^{-\frac{7}{4}}$

9) $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

11) $4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

13) $yx^{\frac{1}{3}} \cdot xy^{\frac{3}{2}}$

$y^{\frac{2}{2}}x^{\frac{1}{3}}\, x^{\frac{3}{3}}y^{\frac{3}{2}}$

$x^{\frac{4}{3}}y^{\frac{5}{2}}$

15) $\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^{-1}$

$a^{-\frac{1}{2}}b^{-\frac{1}{2}}$

$\frac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

17) $\frac{a^2b^0}{3a^4} = \frac{1}{3a^2}$

19) $uv \cdot u\left(v^{\frac{3}{2}}\right)^3$

$uv \ \cdot uv^{\frac{9}{2}}$

$uv^{\frac{2}{2}}uv^{\frac{9}{2}}$

$u^2v^{\frac{11}{2}}$

21) $\left(x^0y^{\frac{1}{3}}\right)^{\frac{3}{2}}\ x^0$

$y^{\frac{1}{2}}$

23) $\frac{a^{\frac{3}{4}}b^{-1}b^{\frac{7}{4}}}{3b^{-1}} = \frac{a^{\frac{3}{4}}b^{\frac{7}{4}}}{3}$

25) $\frac{3y^{-\frac{5}{4}}}{y^{-1}2y^{-\frac{1}{3}}} = \frac{3yy^{\frac{1}{3}}}{2y^{\frac{5}{4}}} = \frac{3y^{\frac{3}{3}}y^{\frac{1}{3}}}{2y^{\frac{5}{4}}} = \frac{3y^{\frac{4}{3}}}{2y^{\frac{5}{4}}} = \frac{3y^{\frac{16}{12}}}{2y^{\frac{15}{12}}} = \frac{3y^{\frac{1}{12}}}{2}$

27) $\left(\frac{m^{\frac{3}{2}}n^{-2}}{\left(mn^{\frac{4}{3}}\right)^{-1}}\right)^{\frac{7}{4}} = \left(\frac{m^{\frac{3}{2}}n^{-2}}{m^{-1}n^{-\frac{4}{3}}}\right)^{\frac{7}{4}} = \left(\frac{m^{\frac{3}{2}}mn^{\frac{4}{3}}}{n^2}\right)^{\frac{7}{4}} = \left(\frac{m^{\frac{3}{2}}m^{\frac{2}{2}}n^{\frac{4}{3}}}{n^{\frac{6}{3}}}\right)^{\frac{7}{4}} = \left(\frac{m^{\frac{5}{2}}}{n^{\frac{2}{3}}}\right)^{\frac{7}{4}} = \frac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

29) $r\dfrac{\left(m^2n^{\frac{1}{2}}\right)^0}{n^{\frac{3}{4}}} = \dfrac{1}{n^{\frac{3}{4}}}$

31) $r\dfrac{\left(x^{-\frac{4}{3}}y^{-\frac{1}{3}}y\right)^{-1}}{x^{\frac{1}{3}}y^{-2}} = \dfrac{\left(x^{-\frac{4}{3}}y^{-\frac{1}{3}}y^{\frac{3}{3}}\right)^{-1}}{x^{\frac{1}{3}}y^{-2}} = \dfrac{\left(x^{-\frac{4}{3}}y^{\frac{2}{3}}\right)^{-1}}{x^{\frac{1}{3}}y^{-2}} = \dfrac{x^{\frac{4}{3}}y^{-\frac{2}{3}}}{x^{\frac{1}{3}}y^{-2}} = \dfrac{x^{\frac{4}{3}}y^{2}}{x^{\frac{1}{3}}y^{\frac{2}{3}}} = \dfrac{x^{\frac{4}{3}}y^{\frac{6}{3}}}{x^{\frac{1}{3}}y^{\frac{2}{3}}} = xy^{\frac{4}{3}}$

33) $\dfrac{(uv^2)^{\frac{1}{2}}}{v^{-\frac{1}{4}}v^2} = \dfrac{u^{\frac{1}{2}}v}{v^{-\frac{1}{4}}v^2} = \dfrac{u^{\frac{1}{2}}v^{\frac{1}{4}}v}{v^2} = \dfrac{u^{\frac{1}{2}}v^{\frac{1}{4}}v^{\frac{4}{4}}}{v^2} = \dfrac{u^{\frac{1}{2}}v^{\frac{5}{4}}}{v^2} = \dfrac{u^{\frac{1}{2}}v^{\frac{5}{4}}}{v^{\frac{8}{4}}} = \dfrac{u^{\frac{1}{2}}}{v^{\frac{3}{4}}}$

8.7

1) $\sqrt[8]{16x^4y^6}$
$\sqrt[8]{2^4x^4y^6}$
$\sqrt[4]{2^2x^2y^3}$
$\sqrt[4]{4x^2y^3}$

3) $\sqrt[12]{64\,x^4y^6z^8}$
$\sqrt[12]{2^6x^4y^6z^8}$
$\sqrt[6]{2^3x^2y^3z^4}$
$\sqrt[6]{8x^2y^3z^4}$

5) $\sqrt[6]{\dfrac{16x^2}{9y^4}} = \sqrt[6]{\dfrac{2^4x^2}{3^2y^4}} = \sqrt[3]{\dfrac{2^2x}{3y^2}}\left(\dfrac{\sqrt[3]{3^2y}}{\sqrt[3]{3^2y}}\right) =$
$\dfrac{\sqrt[3]{36xy}}{3y}$

7) $\sqrt[12]{x^6y^9}$
$\sqrt[4]{x^2y^3}$

9) $\sqrt[8]{x^6y^4z^2}$
$\sqrt[4]{x^3y^2z}$

11) $\sqrt[9]{8x^3y^6}$
$\sqrt[9]{2^3x^3y^6}$
$\sqrt[3]{2xy^2}$

13) $\sqrt[3]{5}\sqrt{6}$
$\sqrt[6]{5^2\cdot 6^3}$
$\sqrt[6]{25\ \cdot 216}$
$\sqrt[6]{5400}$

15) $\sqrt{x}\sqrt[3]{7y}$
$\sqrt[6]{x^3\cdot 7^2y^2}$
$\sqrt[6]{49x^3y^2}$

17) $\sqrt{x}\ \sqrt[3]{x-2}$
$\sqrt[6]{x^3(x-2)^2}$

19) $\sqrt[5]{x^2y}\ \sqrt{xy}$
$\sqrt[10]{x^4y^2\cdot x^5y^5}$
$\sqrt[10]{x^9y^7}$

21) $\sqrt[4]{xy^2}\ \sqrt[3]{x^2y}$
$\sqrt[12]{x^3y^6\cdot x^8y^4}$
$\sqrt[12]{x^{11}y^{10}}$

23) $\sqrt[4]{a^2bc^2}\ \sqrt[5]{a^2b^3c}$

$\sqrt[20]{a^{10}b^5c^{10}\cdot a^8b^{12}c^4}$
$\sqrt[20]{a^{18}b^{17}c^{14}}$

25) $\sqrt{a}\ \sqrt[4]{a^3}$
$\sqrt[4]{a^2\cdot a^3}$
$\sqrt[4]{a^5}$
$a\sqrt[4]{a}$

27) $\sqrt[5]{b^2}\ \sqrt{b^3}$
$\sqrt[10]{b^4\cdot b^{15}}$
$\sqrt[10]{b^{19}}$
$b\sqrt[10]{b^9}$

29) $\sqrt{xy^3}\ \sqrt[3]{x^2y}$
$\sqrt[6]{x^3y^9x^4y^2}$
$\sqrt[6]{x^7y^{11}}$
$xy\sqrt[6]{xy^5}$

31) $\sqrt[4]{9ab^3}\ \sqrt{3a^4b}$
$\sqrt[4]{3^2ab^3}\ \sqrt{3a^4b}$
$\sqrt[4]{3^2ab^3\cdot 3^2a^8b^2}$
$\sqrt[4]{3^4a^9b^5}$
$3a^2b\sqrt[4]{ab}$

33) $\sqrt[3]{3xy^2z}\ \sqrt[4]{9x^3yz^2}$
$\sqrt[3]{3xy^2z}\ \sqrt[4]{3^2x^3yz^2}$
$\sqrt[12]{3^4x^4y^8z^4\cdot 3^6x^9y^3z^6}$
$\sqrt[12]{3^{10}x^{13}y^{11}z^{10}}$
$x\sqrt[12]{59049xy^{11}z^{10}}$

35) $\sqrt{27a^5(b+1)}\ \sqrt[3]{81a(b+1)^4}$
$\sqrt{3^3a^5(b+1)}\ \sqrt[3]{3^4a(b+1)^4}$
$\sqrt[6]{3^9a^{15}(b+1)^3\cdot 3^8a^2(b+1)^8}$
$\sqrt[6]{3^{17}a^{17}(b+1)^{11}}$
$3^2a^2(b+1)\sqrt[6]{3^5a^5(b+1)^5}$
$9a^2(b+1)\sqrt[6]{243a^5(b+1)^5}$

37) $\frac{\sqrt[3]{a^2}}{\sqrt[4]{a}} = \sqrt[12]{\frac{a^8}{a^3}} = \sqrt[2]{a^5}$

39) $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}} = \sqrt[12]{\frac{x^6y^9}{x^4y^4}} = \sqrt[12]{x^2y^5}$

41) $\frac{\sqrt{ab^3c}}{\sqrt[5]{a^2b^3c^{-1}}} = \sqrt[10]{\frac{a^5b^{15}c^5}{a^4b^6c^{-2}}} = \sqrt[10]{ab^9c^7}$

43) $\frac{\sqrt[4]{(3x-1)^3}}{\sqrt[5]{(3x-1)^3}} = \sqrt[20]{\frac{(3x-1)^{15}}{(3x-1)^{12}}} = \sqrt[20]{(3x-1)^3}$

45) $\frac{\sqrt[3]{(2x+1)^2}}{\sqrt[5]{(2x+1)^2}} = \sqrt[15]{\frac{(2x+1)^{10}}{(2x+1)^6}} = \sqrt[15]{(2x+1)^4}$

8.8

1) $3-(-8+4i)$
$3+8-4i$
$11-4i$

3) $r7i-(3-2i)$
$7i-3+2i$
$-3+9i$

5) $-6i-(3+7i)$
$-6i-3-7i$
$-3-13i$

7) $(3-3i)+(-7-8i)$
$3-3i-7-8i$
$-4-11i$

9) $i-(2+3i)-6$
$i-2-3i-6$
$-8-2i$

11) $(6i)(-8i)$
$-48i^2$
$-48(-1)$
48

13) $(-5i)(8i)$
$-40i^2$
$-40(-1)$
40

15) $(-7i)^2$
$49i^2$
$49(-1)$
-49

17) $(6+5i)^2$
$36+60i+25i^2$
$36+60i+25(-1)$
$36+60i-25$
$11+60i$

19) $(-7-4i)(-8+6i)$
$56-42i+32i-24i^2$
$56-42i+32i-24(-1)$
$56-42i+32i+24$
$80-10i$

21) $(-4+5i)(2-7i)$
$-8+28i+10i-35i^2$
$-8+28i+10i-35(-1)$
$-8+28i+10i+35$
$27+38i$

23) $(-8-6i)(-4+2i)$
$32-16i+24i-12i^2$
$32-16i+24i-12(-1)$
$32-16i+24i+12$
$44+8i$

25) $(1+5i)(2+i)$
$2+i+10i+5i^2$
$2+i+10i+5(-1)$
$2+i+10i-5$
$-3+11i$

27) $\frac{(-9+5i)}{i}\frac{(i)}{(i)}=\frac{-9i+5i^2}{i^2}=\frac{-9i+5(-1)}{-1}=\frac{-9i-5}{-1}=$
$9i+5$

29) $\frac{(-10-9i)}{6i}\frac{(i)}{(i)}=\frac{-10i-9i^2}{6i^2}=\frac{-10i-9(-1)}{6(-1)}=\frac{-10i+9}{-6}$

31) $\frac{(-3-6i)}{4i}\frac{(i)}{(i)}=\frac{-3i-6i^2}{4i^2}=\frac{-3i-6(-1)}{4(-1)}=\frac{-3i+6}{-4}$

33) $\frac{(10-i)}{-i}\frac{(i)}{(i)}=\frac{10i-i^2}{-i^2}=\frac{10i-(-1)}{-(-1)}=\frac{10i+1}{1}=10i+1$

35) $\frac{4i}{-10+i}\frac{(-10-i)}{(-10-i)}=\frac{-40i-4i^2}{100-i^2}=\frac{-40i-4(-1)}{100-(-1)}=\frac{-40i+4}{100+1}=\frac{-40i+4}{101}$

37) $\frac{8}{7-6i}\frac{(7+6i)}{(7+6i)}=\frac{56+48i}{49-36i^2}=\frac{56+48i}{49-36(-1)}=\frac{56+48i}{49+36}=\frac{56+48i}{85}$

39) $\frac{7}{10-7i}\frac{(10+7i)}{(10+7i)}=\frac{70+49i}{100-49i^2}=\frac{70+49i}{100-49(-1)}=\frac{70+49i}{100+49}=\frac{70+49i}{149}$

41) $\frac{5i}{-6-i}\frac{(-6+i)}{(-6+i)}=\frac{-30i+5i^2}{36-i^2}=\frac{-30i+5(-1)}{36-1(-1)}=\frac{-30i+5(-1)}{36-1(-1)}=\frac{-30i-5}{36+1}=\frac{-30i-5}{37}$

43) $\sqrt{-81}$
$\sqrt{-1\cdot 3^2}$
$3^2 i$
$9i$

45) $\sqrt{-10}\,\sqrt{-2}$
$\sqrt{-1\cdot 10}\,\sqrt{-1\cdot 2}$
$i\sqrt{10}\cdot i\sqrt{2}$
$i^2\sqrt{20}$
$-1\sqrt{2^2\cdot 5}$
$-1\cdot 2\sqrt{5}$
$-2\sqrt{5}$

47) $\frac{3+\sqrt{-27}}{6}=\frac{3+\sqrt{-1\cdot 3^3}}{6}=\frac{3+3i\sqrt{3}}{6}=\frac{3\left(1+i\sqrt{3}\right)}{6}=\frac{1+\sqrt{3}}{2}$

49) $\frac{8-\sqrt{-16}}{4}=\frac{8-\sqrt{-1\cdot 2^4}}{4}=\frac{8-2^2 i}{4}=\frac{8-4i}{4}=\frac{4(2-i)}{4}=2-i$

51) $i^{73}=i^1=i$

53) $i^{48}=i^0=1$

55) $i^{62}=i^2=-1$

57) $i^{154}=i^2=-1$

Chapter 9: Quadratics

9.1

1) $\sqrt{2x+3}-3=0$

$\underline{+3 \quad +3}$

$\left(\sqrt{2x+3}\right)^2=3^2$

$2x+3=9$

$\underline{-3 \quad -3}$

$\frac{2x}{2}=\frac{6}{2}$

$x=3$

Check: $\sqrt{2(3)+3}-3=0$

$\sqrt{6+3}-3=0$

$\sqrt{9}-3=0$

$3-3=0$

$0=0$ √

$x=3$

3) $\sqrt{6x-5}-x=0$

$\underline{+x \quad +x}$

$\left(\sqrt{6x-5}\right)^2=x^2$

$6x-5=x^2$

$-6x+5 \quad -6x+5$

$0=x^2-6x+5$

$0=(x-1)(x-5)$

$x-1=0 \quad x-5=0$

$\underline{+1 \quad +1} \quad \underline{+5 \quad +5}$

$x=1 \quad x=5$

Check: $\sqrt{6(5)-5}-5=0$

$\sqrt{30-5}-5=0$

$\sqrt{25}-5=0$

$5-5=0$

$0=0$ √

Check: $\sqrt{6(1)-5}-1=0$

$\sqrt{6-5}-1=0$

$\sqrt{1}-1=0$

$1-1=0$

$0=0$√

$x=5,1$

5) $(3+x)^2=\left(\sqrt{6x+13}\right)^2$

$9+6x+x^2=6x+13$

$-\underline{13-6x} \quad \underline{-6x-13}$

$x^2-4=0$

$(x+2)(x-2)=0$

$x+2=0 \quad x-2=0$

$\underline{-2 \quad -2} \quad \underline{+2 \quad +2}$

$x=-2 \quad x=2$

Check: $3+(-2)=\sqrt{6(-2)+13}$

$1=\sqrt{-12+13}$

$1=\sqrt{1}$

$1=1$ √

Check: $3+(2)=\sqrt{6(2)+13}$

$5=\sqrt{12+13}$

$5=\sqrt{25}$

$5=5$ √

$x=-2,2$

7) $\sqrt{3-3x}-1=2x$

$\underline{+1 \quad +1}$

$\left(\sqrt{3-3x}\right)^2=(2x+1)^2$

$3-3x=4x^2+4x+1$

$\underline{-3+3x} \quad \underline{+3x-3}$

$0=4x^2+7x-2$

$0=(4x-1)(x+2)$

$4x-1=0 \quad x+2=0$

$\underline{+1 \quad +1} \quad \underline{-2 \quad -2}$

$\frac{4x}{4}=\frac{1}{4} \quad x=-2$

$x=\frac{1}{4}$

Check: $\sqrt{3-3(\frac{1}{4})}-1=2\left(\frac{1}{4}\right)$

$\sqrt{3-\frac{3}{4}}-1=\frac{1}{2}$

$\sqrt{\frac{9}{4}}-1=\frac{1}{2}$

$\frac{3}{2}-1=\frac{1}{2}$

$\frac{1}{2}=\frac{1}{2}$ √

Check: $\sqrt{3-3(-2)}-1=2(-2)$

$\sqrt{3+6}-1=-4$

$\sqrt{9}-1=-4$

$3-1=-4$

$2=-4 \quad No!$

$x=\frac{1}{4}$

9) $\sqrt{4x+5}-\sqrt{x+4}=2$

$\underline{\quad +\sqrt{x+4} \quad +\sqrt{x+4}}$

$\left(\sqrt{4x+5}\right)^2=\left(2+\sqrt{x+4}\right)^2$

$4x+5=4+4\sqrt{x+4}+x+4$

$4x+5=8+x+4\sqrt{x+4}$

$\underline{-x-8 \quad -8-x}$

$(3x-3)^2=\left(4\sqrt{x+4}\right)^2$

$9x^2-18x+9=16(x+4)$

$9x^2-18x+9=16x+64$

$\underline{\quad -16x-64 \quad -16x-64}$

$9x^2-34x-55=0$

$(9x+11)(x-5)=0$

$9x+11=0 \quad x-5=0$

$\underline{-11 \quad -11} \quad \underline{+5 \quad +5}$

$\frac{9x}{9}=\frac{(-11)}{9} \qquad x=5$

$x=-\frac{11}{9}$

Check: $\sqrt{4\left(-\frac{11}{9}\right)+5}-\sqrt{-\frac{11}{9}+4}=2$

$\sqrt{-\frac{44}{9}+5}-\sqrt{\frac{25}{9}}=2$

$\sqrt{\frac{1}{9}}-\frac{5}{3}=2$

$\frac{1}{3}-\frac{5}{3}=2$

$-2=2 \quad No!$

Check: $\sqrt{4(5)+5}-\sqrt{(5)+4}=2$

$\sqrt{20+5}-\sqrt{9}=2$

$\sqrt{25}-3=2$

$5-3=2$

$2=2 \ \checkmark$

$x=5$

11) $\sqrt{2x+4}-\sqrt{x+3}=1$

$\underline{+\sqrt{x+3} \quad +\sqrt{x+3}}$

$\left(\sqrt{2x+4}\right)^2=\left(1+\sqrt{x+3}\right)^2$

$2x+4=1+2\sqrt{x+3}+x+3$

$2x+4=4+x+2\sqrt{x+3}$

$\underline{-x-4 \quad -4-x}$

$(x)^2=\left(2\sqrt{x+3}\right)^2$

$x^2=4(x+3)$

$x^2=4x+12$

$-4x-12 \quad -4x-12$

$x^2-4x-12=0$

$(x-6)(x+2)=0$

$x-6=0 \quad x+2=0$

$\underline{+6 \quad +6} \quad \underline{-2 \quad -2}$

$x=6 \qquad x=-2$

Check: $\sqrt{2(6)+4}-\sqrt{(6)+3}=1$

$\sqrt{12+4}-\sqrt{9}=1$

$\sqrt{16}-3=1$

$4-3=1$

$1=1 \ \checkmark$

Check: $\sqrt{2(-2)+4}-\sqrt{(-2)+3}=1$

$\sqrt{-4+4}-\sqrt{1}=1$

$\sqrt{0}-1=1$

$0-1=1$

$-1=1 \quad No!$

$x=6$

13) $\sqrt{2x+6}-\sqrt{x+4}=1$

$\underline{\quad +\sqrt{x+4} \quad +\sqrt{x+4}}$

$\left(\sqrt{2x+6}\right)^2=\left(1+\sqrt{x+4}\right)^2$

$2x+6=1+2\sqrt{x+4}+x+4$

$2x+6=5+x+2\sqrt{x+4}$

$\underline{-x-5 \quad -5-x}$

$(x+1)^2=\left(2\sqrt{x+4}\right)^2$

$x^2+2x+1=4(x+4)$

$x^2+2x+1=4x+16$

$\underline{-4x-16 \quad -4x-16}$

$x^2-2x-15=0$

$(x-5)(x+3)=0$

$x-5=0 \quad x+3=0$

$\underline{+5 \quad +5} \quad \underline{-3 \quad -3}$

$x=5 \qquad x=-3$

Check: $\sqrt{2(5)+6}-\sqrt{(5)+4}=1$

$\sqrt{10+6}-\sqrt{9}=1$

$\sqrt{16}-3=1$

$4-3=1$

$1=1$ √

Check: $\sqrt{2(-3)+6}-\sqrt{(-3)+4}=1$

$\sqrt{-6+6}-\sqrt{1}=1$

$\sqrt{0}-1=1$

$0-1=1$

$-1=1 \quad No!$

$x=5$

15) $\sqrt{6-2x}-\sqrt{2x+3}=3$

$\underline{+\sqrt{2x+3} \quad +\sqrt{2x+3}}$

$\left(\sqrt{6-2x}\right)^2=\left(3+\sqrt{2x+3}\right)^2$

$6-2x=9+6\sqrt{2x+3}+2x+3$

$6-2x=2x+12+6\sqrt{2x+3}$

$\underline{-12-2x \quad -2x-12}$

$(-6-4x)^2=\left(6\sqrt{2x+3}\right)^2$

$36+48x+16x^2=36(2x+3)$

$16x^2+48x+36=72x+108$

$\underline{-72x-108 \quad -72x-108}$

$16x^2-24x-72=0$

$8(2x^2-3x-9)=0$

$8(2x+3)(x-3)=0$

$2x+3=0 \quad x-3=0$

$\underline{-3 \quad -3} \quad \underline{+3 \quad +3}$

$\frac{2x}{2}=\frac{-3}{2} \qquad x=3$

$x=-\frac{3}{2}$

Check: $\sqrt{6-2(-\frac{3}{2})}-\sqrt{2\left(-\frac{3}{2}\right)+3}=3$

$\sqrt{6+3}-\sqrt{-3+3}=3$

$\sqrt{9}-\sqrt{0}=3$

$3-0=3$

$3=3$ √

Check: $\sqrt{6-2(3)}-\sqrt{2(3)+3}=3$

$\sqrt{6-6}-\sqrt{6+3}=3$

$\sqrt{0}-\sqrt{9}=3$

$0-3=3$

$-3=3 \quad No!$

$x=-\frac{3}{2}$

9.2

1) $\sqrt{x^2}=\sqrt{75}$

$x=\pm\sqrt{75}$

$x=\pm\sqrt{5^2\cdot 3}$

$x=\pm 5\sqrt{3}$

3) $x^2+5=13$

$\underline{-5 \quad -5}$

$\sqrt{x^2}=\sqrt{8}$

$x=\pm\sqrt{2^3}$

$x=\pm 2\sqrt{2}$

5) $3x^2 + 1 = 73$

$\underline{-1 \quad -1}$

$\frac{3x^2}{3} = \frac{72}{3}$

$\sqrt{x^2} = \sqrt{24}$

$x = \pm\sqrt{24}$

$x = \pm\sqrt{2^3 \cdot 3}$

$x = \pm 2\sqrt{2 \cdot 3}$

$x = \pm 2\sqrt{6}$

7) $\sqrt[5]{(x+2)^5} = \sqrt[5]{-243}$

$x + 2 = -3$

$\underline{-2 \quad -2}$

$x = -5$

9) $(2x+5)^3 - 6 = 21$

$+6 \quad +6$

$\sqrt[3]{(2x+5)^3} = \sqrt[3]{27}$

$2x + 5 = 3$

$\underline{-5 \quad -5}$

$\frac{2x}{2} = \frac{(-2)}{2}$

$x = -1$

11) $(x-1)^{\frac{2}{3}} = 16$ \`

$\sqrt{\left(\sqrt[3]{x-1}\right)^2} = \sqrt{16}$

$\left(\sqrt[3]{x-1}\right)^3 = (\pm 4)^3$

$x - 1 = \pm 64$

$\underline{+1 \quad +1}$

$x = 1 \pm 64$

$x = 65. -63$

13) $(2-x)^{\frac{3}{2}} = 27$

$\sqrt[3]{\left(\sqrt{2-x}\right)^3} = \sqrt[3]{27}$

$\left(\sqrt{2-x}\right)^2 = 3^2$

$2 - x = 9$

$\underline{-2 \qquad -2}$

$\frac{-x}{-1} = \frac{7}{-1}$

$x = -7$

Check: $\left(2-(-7)\right)^{\frac{3}{2}} = 27$

$9^{\frac{3}{2}} = 27$

$\left(\sqrt{9}\right)^3 = 27$

$3^3 = 27$

$27 = 27$ √

15) $(2x-3)^{\frac{2}{3}} = 4$

$\sqrt{\left(\sqrt[3]{2x-3}\right)^2} = \sqrt{4}$

$\left(\sqrt[3]{2x-3}\right)^3 = (\pm 2)^3$

$2x - 3 = \pm 8$

$\underline{+3 \qquad +3}$

$\frac{2x}{2} = \frac{3 \pm 8}{2}$

$x = \frac{11}{2}, -\frac{5}{2}$

17) $\left(x + \frac{1}{2}\right)^{-\frac{2}{3}} = 4$

$\sqrt{\left(\sqrt[3]{\frac{1}{x+\frac{1}{2}}}\right)^2} = \sqrt{4}$

$\left(\sqrt[3]{\frac{1}{x+\frac{1}{2}}}\right)^3 = (\pm 2)^3$

$\frac{1}{x+\frac{1}{2}} = \pm 8$

$\left(x + \frac{1}{2}\right)\frac{1}{x+\frac{1}{2}} =$

$\pm 8 \left(x + \frac{1}{2}\right)$

$\frac{1}{\pm 8} = \frac{\pm 8\left(x+\frac{1}{2}\right)}{\pm 8}$

$\pm\frac{1}{8} = x + \frac{1}{2}$

$\underline{-\frac{1}{2} \qquad -\frac{1}{2}}$

$-\frac{1}{2} \pm \frac{1}{8} = x$

$x = -\frac{3}{8}, -\frac{5}{8}$

19) $r(x-1)^{-\frac{5}{2}} = 32$

$\left(\frac{1}{x-1}\right)^{\frac{5}{2}} = 32$

$\sqrt[5]{\left(\sqrt{\frac{1}{x-1}}\right)^5} = \sqrt[5]{32}$

$\left(\sqrt{\frac{1}{x-1}}\right)^2 = (2)^2$

$(x-1)\frac{1}{x-1} = 4(x-1)$

$1 = 4x - 4$

$\underline{+4 \qquad +4}$

$\frac{5}{4} = \frac{4x}{4}$

$\frac{5}{4} = x$

Check: $\left(\frac{5}{4} - 1\right)^{-\frac{5}{2}} = 32$

$\left(\frac{1}{4}\right)^{-\frac{5}{2}} = 32$

$4^{\frac{5}{2}} = 32$

$\left(\sqrt{4}\right)^5 = 32$

$2^5 = 32$

$32 = 32$ √

$x = \frac{5}{4}$

21) $(3x-2)^{\frac{4}{5}} = 16$

$\sqrt[4]{\left(\sqrt[5]{3x-2}\right)^4} = \sqrt[4]{16}$

$\left(\sqrt[5]{3x-2}\right)^5 = +2^5$

$3x - 2 = \pm 32$

$\underline{+2 \quad +2}$

$\frac{3x}{3} = \frac{2 \pm 32}{3}$

$x = \frac{34}{3}, -10$

23) $(4x+2)^{\frac{3}{5}} = -8$

$$\sqrt[3]{\left(\sqrt[5]{4x+2}\right)^3} = \sqrt[3]{-8}$$

$$\left(\sqrt[5]{4x+2}\right)^5 = (-2)^5$$

$$4x + 2 = -32$$

$$\underline{\quad -2 \quad -2}$$

$$\frac{4x}{4} = \frac{-34}{4}$$

$$x = -\frac{17}{2}$$

9.3

1) $x^2 - 30x + \_\_\_$

$$\left(-30 \cdot \frac{1}{2}\right)^2$$

$$(-15)^2 = 225$$

$$x^2 - 30x + 225$$

$$(x-15)^2$$

3) $m^2 - 36m + \_\_\_$

$$\left(-36 \cdot \frac{1}{2}\right)^2$$

$$(-18)^2 = 324$$

$$m^2 - 36m + 324$$

$$(m-18)^2$$

5) $x^2 - 15x + \_\_$

$$\left(-15 \cdot \frac{1}{2}\right)^2$$

$$\left(-\frac{15}{2}\right)^2 = \frac{225}{4}$$

$$x^2 - 15x + \frac{225}{4}$$

$$\left(x - \frac{15}{2}\right)^2$$

7) $y^2 - y + \_\_$

$$\left(-1 \cdot \frac{1}{2}\right)^2$$

$$\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y^2 - y + \frac{1}{4}$$

$$\left(y - \frac{1}{2}\right)^2$$

9) $x^2 - 16x + 55 = 0$

$$\underline{\quad -55 \quad -55}$$

$$x^2 - 16x \qquad = -55$$

$$\left(-16 \cdot \frac{1}{2}\right)^2$$

$$(-8)^2 = 64$$

$$x^2 - 16x + 64 = -55 + 64f$$

$$\sqrt{(x-8)^2} = \sqrt{9}$$

$$x - 8 = \pm 3$$

$$\underline{\quad +8 \quad +8}$$

$$x = 8 \pm 3$$

$$x = 11, 5$$

11) $v^2 - 8v + 45 = 0$

$$\underline{\quad -45 \quad -45}$$

$$v^2 - 8v \qquad = -45$$

$$\left(-8 \cdot \frac{1}{2}\right)^2$$

$$(-4)^2 = 16$$

$$v^2 - 8v + 16 = -45 + 16$$

$$\sqrt{(v-4)^2} = \sqrt{-29}$$

$$v - 4 = \pm i\sqrt{29}$$

$$\underline{\quad +4 \quad +4}$$

$$v = 4 \pm i\sqrt{29}$$

13) $6x^2 + 12x + 63 = 0$

$$\underline{-63 \quad -63}$$

$$\frac{6x^2}{6} + \frac{12x}{6} \quad = -\frac{63}{6}$$

$$x^2 + 2x \quad = -\frac{21}{2}$$

$$\left(2 \cdot \frac{1}{2}\right)^2$$

$$(1)^2 = 1$$

$$x^2 + 2x + 1 = -\frac{21}{2} + 1$$

$$\sqrt{(x+1)^2} = \sqrt{-\frac{19}{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$x + 1 = \pm\frac{i\sqrt{38}}{2}$$

$$\underline{-1 \quad -1}$$

$$x = \frac{-2 \pm i\sqrt{38}}{2}$$

15) $5k^2 - 10k + 48 = 0$

$$\underline{-48 \quad -48}$$

$$\frac{5k^2}{5} - \frac{10k}{5} \quad = \frac{-48}{5}$$

$$k^2 - 2k \quad = \frac{-48}{5}$$

$$\left(-2 \cdot \frac{1}{2}\right)^2$$

$$(-1)^2 = 1$$

$$k^2 - 2k + 1 = -\frac{48}{5} + 1$$

$$\sqrt{(k-1)^2} = \sqrt{-\frac{48}{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$$

$$k - 1 = \pm\frac{i\sqrt{215}}{5}$$

$$\underline{+1 \quad +1}$$

$$k = \frac{5 \pm i\sqrt{215}}{5}$$

17) $x^2 + 10x - 57 = 4$

$$\underline{+57 \quad +57}$$

$$x^2 + 10x \quad = 61$$

$$\left(10 \cdot \frac{1}{2}\right)^2$$

$$(5)^2 = 25$$

$$x^2 + 10x + 25 = 61 + 25$$

$$\sqrt{(x+5)^2} = \sqrt{86}$$

$$x + 5 = \pm\sqrt{86}$$

$$\underline{-5 \quad -5}$$

$$x = -5 \pm \sqrt{86}$$

19) $n^2 - 16n + 67 = 4$

$$\underline{-67 \quad -67}$$

$$n^2 - 16n \quad = -63$$

$$\left(-16 \cdot \frac{1}{2}\right)^2 = (-8)^2 = 64$$

$$n^2 - 16n + 64 = 63 + 64$$

$$\sqrt{(n-8)^2} = \sqrt{1}$$

$$n - 8 = \pm 1$$

$$\underline{+8 \quad +8}$$

$$n = 9, 7$$

21) $2x^2 + 4x + 38 = -6$

$$\underline{-38 \quad -38}$$

$$\frac{2x^2}{2} + \frac{4x}{2} \quad = \frac{-44}{2}$$

$$x^2 + 2x \quad = -22$$

$$\left(2 \cdot \frac{1}{2}\right)^2 = 1^1 = 1$$

$$x^2 + 2x + 1 = -22 + 1$$

$$\sqrt{(x+1)^2} = \sqrt{-21}$$

$$x + 1 = \pm i\sqrt{21}$$

$$\underline{-1 \quad -1}$$

$$x = -1 \pm i\sqrt{21}$$

23) $8b^2 + 16b - 37 = 5$

$$\underline{+37 \quad +37}$$

$$\frac{8b^2}{8} + \frac{16b}{8} \quad = \frac{42}{8}$$

$$b^2 + 2b \quad = \frac{21}{4}$$

$$\left(2 \cdot \frac{1}{2}\right)^2 = 1^1 = 1$$

$$b^2 + 2b + 1 = \frac{21}{4} + 1$$

$$\sqrt{(b+1)^2} = \sqrt{\frac{25}{4}}$$

$$b + 1 = \pm\frac{5}{2}$$

$$\underline{-1 \quad -1}$$

$$b = -1 \pm \frac{5}{2}$$

$$b = \frac{3}{2}, -\frac{7}{2}$$

25) $r \quad x^2 = -10x - 29$

$\underline{+10x \ + 10x}$

$x^2 + 10x \qquad = -29$

$\left(10 \cdot \frac{1}{2}\right)^2 = (5)^2 = 25$

$x^2 + 10x + 25 = -29 + 25$

$\sqrt{(x+5)^2} = \sqrt{-4}$

$x + 5 = \pm 2i$

$\underline{-5 \qquad -5}$

$x = -5 \pm 2i$

27) $n^2 = -21 + 10n$

$\underline{-10n \qquad -10n}$

$n^2 - 10n \qquad = -21$

$\left(-10 \cdot \frac{1}{2}\right)^2 = (-5)^2 = 25$

$n^2 - 10n + 25 = -21 + 25$

$\sqrt{(n-5)^2} = \sqrt{4}$

$n - 5 = \pm 2$

$\underline{+5 \quad +5}$

$n = 5 \pm 2$

$n = 7, 3$

29) $3k^2 + 9 = 6k$

$\underline{-6k - 9 \ \ -6k - 9}$

$\frac{3k^2}{3} - \frac{6k}{3} \quad = \frac{-9}{3}$

$k^2 - 2k \quad = -3$

$\left(-2 \cdot \frac{1}{2}\right)^2 = (-1)^2 = 1$

$k^2 - 2k + 1 = -3 + 1$

$\sqrt{(k-1)^2} = \sqrt{-2}$

$k - 1 = \pm i\sqrt{2}$

$\underline{+1 \quad +1}$

$k = 1 \pm i\sqrt{2}$

31) $2x^2 + 63 = 8x$

$\underline{-8x - 63 \ \ -8x - 63}$

$\frac{2x^2}{2} - \frac{8x}{2} \qquad = -\frac{63}{2}$

$x^2 - 4x \qquad = -\frac{63}{2}$

$\left(-4 \cdot \frac{1}{2}\right)^2 = (-2)^2 = 4$

$x^2 - 4x + 4 = -\frac{63}{2} + 4$

$\sqrt{(x-2)^2} = \sqrt{-\frac{55}{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$x - 2 = \pm \frac{i\sqrt{110}}{2}$

$\underline{+2 \quad +2}$

$x = \frac{4 \pm i\sqrt{110}}{2}$

33) $p^2 - 8p \qquad = -55$

$\left(-8 \cdot \frac{1}{2}\right)^2 = (-4)^2 = 16$

$p^2 - 8p + 16 = -55 + 16$

$\sqrt{(p-4)^2} = \sqrt{-39}$

$p - 4 = \pm i\sqrt{39}$

$\underline{+4 \qquad +4}$

$p = 4 \pm i\sqrt{39}$

35) $7n^2 - n + 7 \quad = \quad 7n + 6n^2$

$\underline{-6n^2 - 7n - 7 \ \ -6n^2 - 7n - 7}$

$n^2 - 8n \qquad = -7$

$\left(-8 \cdot \frac{1}{2}\right)^2 = (-4)^2 = 16$

$n^2 - 8n + 16 = -7 + 16$

$\sqrt{(n-4)^2} = \sqrt{9}$

$n - 4 = \pm 3$

$\underline{+4 \quad +4}$

$n = 4 \pm 3$

$n = 7, 1$

37) $13b^2+15b+44=-5+7b^2+3b$

$\underline{-7b^2-3b-44\quad -44-7b^2-3b}$

$\frac{6b^2}{6}+\frac{12b}{6}=-\frac{49}{6}$

$b^2+2b=-\frac{49}{6}$

$\left(2\cdot\frac{1}{2}\right)^2=1^2=1$

$b^2+2b+1=-\frac{49}{6}+1$

$\sqrt{(b+1)^2}=\sqrt{-\frac{43}{6}}\left(\frac{\sqrt{6}}{\sqrt{6}}\right)$

$b+1=\pm\frac{i\sqrt{256}}{6}$

$\underline{-1\quad -1}$

$b=\frac{-6\pm i\sqrt{256}}{6}$

39) $5x^2+5x=-31-5x$

$\underline{+5x\qquad +5x}$

$\frac{5x^2}{5}+\frac{10x}{5}=-\frac{31}{5}$

$x^2+2x=-\frac{31}{5}$

$\left(2\cdot\frac{1}{2}\right)^2=1^2=1$

$x^2+2x+1=-\frac{31}{5}+1$

$\sqrt{(x+1)^2}=\sqrt{-\frac{26}{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$

$x+1=\pm\frac{i\sqrt{130}}{5}$

$\underline{-1\quad -1}$

$x=\frac{-5\pm i\sqrt{130}}{5}$

41) $v^2+5v+28=0$

$\underline{-28\quad -28}$

$v^2+5v=-28$

$\left(5\cdot\frac{1}{2}\right)^2=\left(\frac{5}{2}\right)^2=\frac{25}{4}$

$v^2+5v+\frac{25}{4}=-28+\frac{25}{4}$

$\sqrt{\left(v+\frac{5}{2}\right)^2}=\sqrt{-\frac{87}{4}}$

$v+\frac{5}{2}=\pm\frac{i\sqrt{87}}{2}$

$\underline{-\frac{5}{2}\quad -\frac{5}{2}}$

$v=\frac{-5\pm i\sqrt{87}}{2}$

43) $7x^2-6x+40=0$

$\underline{-40\quad -40}$

$\frac{7x^2}{7}-\frac{6x}{7}=-\frac{40}{7}$

$x^2-\frac{6}{7}x=-\frac{40}{7}$

$\left(-\frac{6}{7}\cdot\frac{1}{2}\right)^2=\left(-\frac{3}{7}\right)^2=\frac{9}{49}$

$x^2-\frac{6}{7}x+\frac{9}{49}=-\frac{40}{7}+\frac{9}{49}$

$\sqrt{\left(x-\frac{3}{7}\right)^2}=\sqrt{-\frac{271}{49}}$

$x-\frac{3}{7}=\pm\frac{i\sqrt{271}}{7}$

$\underline{+\frac{3}{7}\quad +\frac{3}{7}}$

$x=\frac{3\pm i\sqrt{271}}{7}$

45) $k^2-7k+50=3$

$\underline{-50\quad -50}$

$k^2-7k=-47$

$\left(-7\cdot\frac{1}{2}\right)^2=\left(-\frac{7}{2}\right)^2=\frac{49}{4}$

$k^2-7k+\frac{49}{4}=-47+\frac{49}{4}$

$\sqrt{\left(k-\frac{7}{2}\right)^2}=\sqrt{-\frac{139}{4}}$

$k-\frac{7}{2}=\pm\frac{i\sqrt{139}}{2}$

$\underline{+\frac{7}{2}\quad +\frac{7}{2}}$

$k=\frac{7\pm i\sqrt{139}}{2}$

47) $5x^2+8x-40=8$

$\underline{+40\quad +40}$

$\frac{5x^2}{5}+\frac{8x}{5}=\frac{48}{5}$

$x^2+\frac{8}{5}x=\frac{48}{5}$

$\left(\frac{8}{5}\cdot\frac{1}{2}\right)^2=\left(\frac{4}{5}\right)^2=\frac{16}{25}$

$x^2+\frac{8}{5}x+\frac{16}{25}=\frac{48}{5}+\frac{16}{25}$

$\sqrt{\left(x+\frac{4}{5}\right)^2}=\sqrt{\frac{256}{25}}$

$x+\frac{4}{5}=\pm\frac{16}{5}$

$\underline{-\frac{4}{5}\quad -\frac{4}{5}}$

$x=\frac{-4\pm16}{5}$

49) $m^2 = -15 + 9m$

$\underline{-9m \qquad -9m}$

$m^2 - 9m \quad = -15$

$\left(-9 \cdot \frac{1}{2}\right)^2 = \left(-\frac{9}{2}\right)^2 = \frac{81}{4}$

$m^2 - 9m + \frac{81}{4} = -15 + \frac{81}{4}$

$\sqrt{\left(m - \frac{9}{2}\right)^2} = \sqrt{\frac{21}{4}}$

$m - \frac{9}{2} = \pm\frac{\sqrt{21}}{2}$

$\underline{+\frac{9}{2} \quad +\frac{9}{2}}$

$m = \frac{9 \pm \sqrt{21}}{2}$

51) $\frac{8r^2}{8} + \frac{10r}{8} \quad = -\frac{55}{8}$

$r^2 + \frac{5}{4}r \quad = -\frac{55}{8}$

$\left(\frac{5}{4} \cdot \frac{1}{2}\right)^2 = \left(\frac{5}{8}\right)^2 = \frac{25}{64}$

$r^2 + \frac{5}{4}r + \frac{25}{64} = -\frac{55}{8} + \frac{25}{64}$

$\sqrt{\left(r + \frac{5}{8}\right)^2} = \sqrt{\frac{-415}{64}}$

$r + \frac{5}{8} = \pm\frac{i\sqrt{416}}{8}$

$\underline{-\frac{5}{8} \quad -\frac{5}{8}}$

$r = \frac{-5 \pm i\sqrt{415}}{8}$

53) $5n^2 - 8n + 60 = -3n - 6 + 4n^2$

$\underline{-4n^2 + 3n - 60 \ + 3n - 60 - 4n^2}$

$n^2 + 5n \quad = -54$

$\left(5 \cdot \frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

$n^2 + 5n + \frac{25}{4} = -54 + \frac{25}{4}$

$\sqrt{\left(n + \frac{5}{2}\right)^2} = \sqrt{-\frac{191}{4}}$

$n + \frac{5}{2} = \pm\frac{i\sqrt{191}}{2}$

$\underline{-\frac{5}{2} \quad -\frac{5}{2}}$

$n = \frac{-5 \pm i\sqrt{191}}{2}$

55) $2x^2 + 3x - 5 = -4x^2$

$\underline{+4x^2 \quad + 5 \quad + 4x^2 + 5}$

$\frac{6x^2}{6} + \frac{3x}{6} \quad = \frac{5}{6}$

$x^2 + \frac{1}{2}x \quad = \frac{5}{6}$

$\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$x^2 + \frac{1}{2}x + \frac{1}{16} = \frac{5}{6} + \frac{1}{16}$

$\sqrt{\left(x + \frac{1}{4}\right)^2} = \sqrt{\frac{43}{48}} = \frac{\sqrt{43}}{4\sqrt{3}} \left(\frac{\sqrt{3}}{\sqrt{3}}\right)$

$x + \frac{1}{4} = \pm\frac{\sqrt{129}}{12}$

$\underline{-\frac{1}{4} \quad -\frac{1}{4}}$

$x = \frac{(-3 \pm \sqrt{129})}{12}$

57) $-2x^2 + 3x - 5 = -4x^2$

$\underline{+4x^2 \qquad + 5 \quad + 4x^2 + 5}$

$\frac{2x^2}{2} + \frac{3x}{2} \quad = \frac{5}{2}$

$x^2 + \frac{3}{2}x \quad = \frac{5}{2}$

$\left(\frac{3}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{5}{2} + \frac{9}{16}$

$\sqrt{\left(x + \frac{3}{4}\right)^2} = \sqrt{\frac{49}{16}}$

$x + \frac{3}{4} = \pm\frac{7}{4}$

$\underline{-\frac{3}{4} \quad -\frac{3}{4}}$

$x = -\frac{3}{4} \pm \frac{7}{4}$

$x = 1, -\frac{5}{2}$

9.4

1) $4a^2 + 6 = 0$
$a = 4, b = 0, c = 6$
$\frac{-0 \pm \sqrt{0^2 - 4(4)(6)}}{2(4)} = \frac{\pm\sqrt{-96}}{8} = \frac{\pm\sqrt{-16 \cdot 6}}{8} = \frac{\pm 4i\sqrt{6}}{8} = \frac{\pm i\sqrt{6}}{2}$

3) $2x^2 - 8x - 2 = 0$
$a = 2, b = -8, c = -2$
$\frac{8 \pm \sqrt{(-8)^2 - 4(2)(-2)}}{2(2)} = \frac{8 \pm \sqrt{64+16}}{4} = \frac{8 \pm \sqrt{80}}{4} = \frac{8 \pm \sqrt{16 \cdot 5}}{4} = \frac{8 \pm 4\sqrt{5}}{4} = 2 \pm \sqrt{5}$

5) $2m^2 - 3 = 0$
$a = 2, b = 0, c = -3$
$\frac{-0 \pm \sqrt{(0)^2 - 4(2)(-3)}}{2(2)} = \frac{\pm\sqrt{24}}{4} = \frac{\pm\sqrt{4 \cdot 6}}{4} = \frac{(\pm 2\sqrt{6})}{4} = \frac{\pm\sqrt{6}}{2}$

7) $3r^2 - 2r - 1 = 0$
$a = 3, b = -2, c = -1$
$\frac{2 \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)} = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm \sqrt{16}}{6} = \frac{2 \pm 4}{6} = 1, -\frac{1}{3}$

9) $4n^2 - 36 = 0$
$a = 4, b = 0, c = -36$
$\frac{-0 \pm \sqrt{0^2 - 4(4)(-36)}}{2(4)} = \frac{\pm\sqrt{576}}{8} = \frac{\pm 24}{8} = \pm 3$

11) $v^2 - 4v - 5 = -8$
$\underline{\quad +8 \quad +8}$
$v^2 - 4v + 3 = 0$
$a = 1, b = -4, c = 3$
$\frac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = 3, 1$

13) $2a^2 + 3a + 14 = 6$
$\underline{\quad -14 \quad -14}$
$2a^2 + 3a + 8 = 0$
$a = 2, b = 3, c = 8$
$\frac{-3 \pm \sqrt{(3^2 - 4(2)(8)}}{2(2)} = \frac{-3 \pm \sqrt{9-64}}{4} = \frac{-3 \pm \sqrt{-55}}{4} = \frac{-3 \pm i\sqrt{55}}{4}$

15) $3k^2 + 3k - 4 = 7$

$\underline{\quad -7 \quad -7}$

$3k^2 + 3k - 11 = 0$

$a = 3, b = 3, c = -11$

$\frac{-3 \pm \sqrt{3^2 - 4(3)(-11)}}{2(3)} = \frac{-3 \pm \sqrt{9 + 132}}{6} = \frac{-3 \pm \sqrt{141}}{6}$

17) $7x^2 + 3x - 16 = -2$

$\underline{\quad +2 \quad +2}$

$7x^2 + 3x - 14 = 0$

$a = 7, b = 3, c = -14$

$\frac{-3 \pm \sqrt{3^2 - 4(7)(-14)}}{2(7)} = \frac{-3 \pm \sqrt{9 + 392}}{14} = \frac{-3 \pm \sqrt{401}}{14}$

19) $2p^2 + 6p - 16 = 4$

$\underline{\quad -4 \quad -4}$

$2p^2 + 6p - 20 = 0$

$a = 2, b = 6, c = -20$

$\frac{-6 \pm \sqrt{6^2 - 4(2)(-20)}}{2(2)} = \frac{-6 \pm \sqrt{36 + 160}}{4} = \frac{-6 \pm \sqrt{196}}{4} = \frac{-6 \pm 14}{4} = 2, -5$

21) $3n^2 + 3n = -3$

$\underline{\quad +3 \quad +3}$

$3n^2 + 3n + 3 = 0$

$a = 3, b = 3, c = 3$

$\frac{-3 \pm \sqrt{(3)^2 - 4(3)(3)}}{2(3)} = \frac{-3 \pm \sqrt{9 - 36}}{6} = \frac{-3 \pm \sqrt{-9 \cdot 3}}{6} = \frac{-3 \pm 3i\sqrt{3}}{6} = \frac{-1 \pm i\sqrt{3}}{6}$

23) $2x^2 \quad = -7x + 49$

$\underline{+7x - 49 + 7x - 49}$

$2x^2 + 7x - 49 = 0$

$a = 2, b = 7, c = -49$

$\frac{-7 \pm \sqrt{(7)^2 - 4(2)(-49)}}{2(2)} = \frac{-7 \pm \sqrt{49 + 392}}{4} = \frac{-7 \pm \sqrt{441}}{4}$

25) $5x^2 \quad = 7x + 7$

$\underline{-7x - 7 \quad -7x - 7}$

$5x^2 - 7x - 7 = 0$

$a = 5, b = -7, c = -7$

$\frac{7 \pm \sqrt{(-7)^2 - 4(5)(-7)}}{2(5)} = \frac{7 \pm \sqrt{49 + 140)}}{10} = \frac{7 \pm \sqrt{189}}{10} = \frac{7 \pm \sqrt{9 \cdot 21}}{10} = \frac{7 \pm 3\sqrt{21}}{10}$

27) $8n^2 \qquad = -3n - 8$

$\underline{+3n + 8 \qquad + 3n + 8}$

$8n^2 + 3n + 8 = 0$

$a = 8, b = 3, c = 8$

$\frac{-3 \pm \sqrt{3^2 - 4(8)(8)}}{2(8)} = \frac{-3 \pm \sqrt{9 - 256}}{16} = \frac{-3 \pm \sqrt{-247}}{16} = \frac{-3 \pm i\sqrt{247}}{16}$

29) $2x^2 + 5x = -3$

$\underline{\qquad +3 \quad +3}$

$2x^2 + 5x + 3 = 0$

$a = 2, b = 5, c = 3$

$\frac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)} = \frac{-5 \pm \sqrt{25 - 24}}{4} = \frac{-5 \pm \sqrt{1}}{4} = \frac{-5 \pm 1}{4} = -1, -\frac{3}{2}$

31) $4a^2 - 64 = 0$

$a = 4, b = 0, c = -64$

$\frac{-0 \pm \sqrt{o^2 - 4(4)(-64)}}{2(4)} = \frac{\pm\sqrt{1024}}{8} = \frac{\pm 32}{8} = \pm 4$

33) $4p^2 + 5p - 36 = 3p^2$

$\underline{-3p^2 \qquad\qquad - 3p^2}$

$p^2 + 5p - 36 = 0$

$a = 1, b = 5, c = -36$

$\frac{-5 \pm \sqrt{5^2 - 4(1)(-36)}}{2(1)} = \frac{-5 \pm \sqrt{25 + 144}}{2} = \frac{-5 \pm \sqrt{169}}{2} = \frac{-5 \pm 13}{2} = 4, -9$

35) $-5n^2 - 3n - 52 = 2 - 7n^2$

$\underline{+7n^2 \qquad - 2 \qquad - 2 + 7n^2}$

$2n^2 - 3n - 54 = 0$

$a = 2, b = -3, c = -54$

$\frac{3 \pm \sqrt{(-3)^2 - 4(2)(-54)}}{2(2)} = \frac{3 \pm \sqrt{9 + 432}}{4} = \frac{3 \pm \sqrt{441}}{4} = \frac{3 \pm 21}{4} = 6, -\frac{9}{2}$

37) $7r^2 - 12 = -3r$

$\underline{+3r \qquad\quad + 3r}$

$7r^2 + 3r - 12 = 0$

$a = 7, b = 3, c = -12$

$\frac{-3 \pm \sqrt{3^2 - 4(7)(-12)}}{2(7)} = \frac{-3 \pm \sqrt{9 + 336}}{14} = \frac{-3 \pm \sqrt{345}}{14}$

39) $2n^2 - 9 = 4$

$\underline{\quad -4 \quad -4}$

$2n^2 - 13 = 0$

$a = 2, b = 0, c = -13$

$\frac{-0\pm\sqrt{0^2-4(2)(-13)}}{2(2)} = \frac{\pm\sqrt{104}}{4} = \frac{\pm\sqrt{4\cdot 26}}{4} = \frac{\pm 2\sqrt{26}}{4} = \frac{\pm\sqrt{26}}{2}$

9.5

1) $2, 5$

$x = 2, \quad x = 5$

$\underline{-2 \ -2 - 5 \ -5}$

$x - 2 = 0 \ \ x - 5 = 0$

$(x-2)(x-5) = 0$

$x^2 - 5x - 2x + 10 = 0$

$x^2 - 7x + 10 = 0$

3) $20, 2$

$x = 20 \quad x = 2$

$\underline{-20 - 20 - 2 \ -2}$

$x - 20 = 0 \ x - 2 = 0$

$(x-20)(x-2) = 0$

$x^2 - 20x - 2x + 40 = 0$

$x^2 - 22x + 40 = 0$

5) $4, 4$

$x = 4 \quad x = 4$

$\underline{-4 - 4 \ -4 - 4}$

$x - 4 = 0 \ \ x - 4 = 0$

$(x-4)(x-4) = 0$

$x^2 - 4x - 4x + 16 = 0$

$x^2 - 8x + 16 = 0$

7) $0, 0$

$x = 0, x = 0$

$xx = 0$

$x^2 = 0$

9) $-4, 11$

$x = -4 \quad x = 11$

$\underline{+4 \ +4 \quad -11 - 11}$

$x + 4 = 0 \ x - 11 = 0$

$(x+4)(x-11) = 0$

$x^2 + 4x - 11x - 44 = 0$

$x^2 - 7x - 44 = 0$

11) $\frac{3}{4}, \frac{1}{4}$

$4(x) = \left(\frac{3}{4}\right)4 \ , 4(x) = \left(\frac{1}{4}\right)4$

$4x = 3 \qquad 4x = 1$

$\underline{-3 \ -3 \qquad -1 \ -1}$

$4x - 3 = 0 \ \ 4x - 1 = 0$

$(4x-3)(4x-1) = 0$

$16x^2 - 12x - 4x + 3 = 0$

$16x^2 - 8x + 3 = 0$

13) $\frac{1}{2} \ , \ \frac{1}{3}$

$(2)x = \frac{1}{2}(2) \quad (3)x = \frac{1}{3}(3)$

$2x = 1 \qquad 3x = 1$

$\underline{-1 \ -1} \qquad \underline{-1 \ -1}$

$2x - 1 = 0 \ \ 3x - 1 = 0$

$(2x-1)(3x-1) = 0$

$6x^2 - 3x - 2x + 1 = 0$

$6x^2 - 5x + 1 = 0$

15) $\frac{3}{7} \ , \ \ 4$

$(7)x = \frac{3}{7}(x) \quad x = 4$

$7x = 3 \qquad \underline{-4 \ -4}$

$\underline{-3 \ -3} \quad x - 4 = 0$

$7x - 3 = 0$

$(7x-3)(x-4) = 0$

$7x^2 - 3x - 28x + 12 = 0$

$7x^2 - 31x + 12 = 0$

17) $-\frac{1}{3}, \frac{5}{6}$

$(3)x = -\frac{1}{3}(3) \ \ 6(x) = \frac{5}{6}(6)$

$3x = -1 \qquad 6x = 5$

$\underline{+1 \ \ +1} \qquad \underline{-5 \ \ -5}$

$3x + 1 = 0 \quad 6x - 5 = 0$

$(3x+1)(6x-5) = 0$

$18x^2 + 6x - 15x - 5 = 0$

$18x^2 - 9x - 5 = 0$

19) $-6, \quad \frac{1}{9}$

$x = -6 \quad (9)x = \frac{1}{9}(9)$

$\underline{+6 \ +6} \quad 9x = 1$

$x + 6 = 0 \ \ \underline{-1 \ \ -1}$

$9x - 1 = 0$

$(x+6)(9x-1) = 0$

$9x^2 + 54x - x - 6 = 0$

$9x^2 - 53x - 6 = 0$

21) ± 5

$x^2 = (\pm 5)^2$

$x^2 = 25$

$\underline{-25 - 25}$

$x^2 - 25 = 0$

23) $\pm\frac{1}{5}$

$x^2 = \left(\pm\frac{1}{5}\right)^2$

$(25)x^2 = \frac{1}{25}(25)$

$25x^2 = 1$

$\underline{-1 \quad -1}$

$25x^2 - 1 = 0$

25) $\pm\sqrt{11}$

$x^2 = \left(\pm\sqrt{11}\right)^2$

$x^2 = 11$

$\underline{-11 \quad -11}$

$x^2 - 11 = 0$

27) $\pm\frac{\sqrt{3}}{4}$

$4x = \pm\frac{\sqrt{3}}{4}\,(4)$

$(4x)^2 = \left(\pm\sqrt{3}\right)^2$

$16x^2 = 3$

$\underline{-3 \quad -3}$

$16x^2 - 3 = 0$

29) $\pm i\sqrt{13}$

$x^2 = \left(\pm i\sqrt{13}\right)^2$

$x^2 = -13$

$\underline{+13 \quad +13}$

$x^2 + 13 = 0$

31) $2 \pm \sqrt{6}$

$x = 2 \pm \sqrt{6}$

$\underline{-2 - 2}$

$(x-2)^2 = \left(\pm\sqrt{6}\right)^2$

$x^2 - 4x + 4 = 6$

$\underline{-6 \quad -6}$

$x^2 - 4x - 2 = 0$

33) $1 \pm 3i$

$x = 1 \pm 3i$

$\underline{-1 - 1}$

$(x-1)^2 = (\pm 3i)^2$

$x^2 - 2x + 1 = -9$

$\underline{+9 \quad +9}$

$x^2 - 2x + 10 = 0$

35) $6 \pm i\sqrt{3}$

$x = 6 \pm i\sqrt{3}$

$\underline{-6 - 6}$

$(x-6)^2 = \left(\pm i\sqrt{3}\right)^2$

$x^2 - 12x + 36 = -3$

$\underline{+3 \quad +3}$

$x^2 - 12x + 39 = 0$

37) $\frac{-1\pm\sqrt{6}}{2}$

$(2)x = \frac{-1\pm\sqrt{6}}{2}(2)$

$2x = -1 \pm \sqrt{6}$

$\underline{+1 \quad +1}$

$(2x+1)^2 = \left(\pm\sqrt{6}\right)^2$

$4x^2 + 4x + 1 = 6$

$\underline{-6 \quad -6}$

$4x^2 + 4x - 5 = 0$

39) $\frac{6\pm i\sqrt{2}}{8}$

$(8)x = \frac{6\pm i\sqrt{2}}{8}(8)$

$8x = 6 \pm i\sqrt{2}$

$\underline{-6 \quad -6}$

$(8x-6)^2 = \left(\pm i\sqrt{2}\right)^2$

$64x^2 - 96x + 36 = -2$

$\underline{+2 \quad +2}$

$64x^2 - 96x + 38 = 0$

9.6

1) $x^4 - 5x^2 + 4 = 0$
$y = x^2, y^2 = x^4$
$y^2 - 5y + 4 = 0$
$(y-4)(y-1) = 0$
$y - 4 = 0 \quad y - 1 = 0$
$\underline{+4 \;\; +4} \quad \underline{+1 \;\; +1}$
$y = 4 \quad y = 1$
$\sqrt{x^2} = \sqrt{4} \quad x^2 = \sqrt{1}$
$x = \pm 2, \pm 1$

3) $m^4 - 7m^2 - 8 = 0$
$y = m^2 \;\; y^2 = m^2$
$y^2 - 7y - 8 = 0$
$(y-8)(y+1) = 0$
$y - 8 = 0 \quad y + 1 = 0$
$\underline{+8 \;\; +8} \quad \underline{-1 \;\; -1}$
$y = 8 \quad y = -1$
$\sqrt{m^2} = \sqrt{8} \;\; \sqrt{m^2} = \sqrt{(-1)^2}$
$m = \pm 2\sqrt{2}, \pm i$

5) $a^4 - 50a^2 + 49 = 0$
$y = a^2 \;\; y^2 = a^4$
$y^2 - 50y + 49 = 0$
$(y-49)(y-1) = 0$
$y - 49 = 0 \quad y - 1 = 0$
$\underline{+49 \;\; +49} \quad \underline{+1 \;\; +1}$
$y = 49 \quad y = 1$
$\sqrt{a^2} = \sqrt{49} \quad \sqrt{a^2} = \sqrt{1}$
$a = \pm 7, \pm 1$

7) $x^4 - 25x^2 + 144 = 0$
$y = x^2, \;\; y^2 = x^4$
$y^2 - 25y + 144 = 0$
$(y-9)(y-16) = 0$
$y - 9 = 0 \;\; y - 16 = 0$
$\underline{+9 \;\; +9} \quad \underline{+16 \;\; +16}$
$y = 9 \qquad y = 16$
$\sqrt{x^2} = \sqrt{9} \quad \sqrt{x^2} = \sqrt{16}$
$x = \pm 3, \pm 4$

9) $m^4 - 20m^2 + 64 = 0$
$y = m^2 \;\; y^2 = m^4$
$y^2 - 20y + 64 = 0$
$(y-4)(y-16) = 0$
$y - 4 = 0 \quad y - 16 = 0$
$\underline{+4 \;\; +4} \quad \underline{+16 \;\; +16}$
$y = 4 \qquad y = 16$
$\sqrt{m^2} = \sqrt{4} \quad \sqrt{m^2} = \sqrt{16}$
$m = \pm 2, \pm 4$

11) $z^6 - 216 = 19z^3$

$y = z^3 \; y^2 = z^6$

$y^2 - 216 = 19y$

$\underline{\quad -19y \quad - 19y \quad}$

$y^2 - 19y - 216 = 0$

$(y - 27)(y + 8) = 0$

$y - 27 = 0 \quad y + 8 = 0$

$\underline{+27 \quad + 27} \quad \underline{- 8 \quad - 8}$

$y = 27 \qquad y = -8$

$z^3 = 27 \qquad z^3 = -8$

$\underline{-27 \; - 27} \quad \underline{+ 8 \; + 8}$

$z^3 - 27 = 0 \quad z^3 - 8 = 0$

$(z - 3)(z^2 + 3z + 9) = 0$

$z - 3 = 0 \quad z^2 + 3z + 9 = 0$

$\underline{+3 \; + 3} \quad \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)} = \frac{2 \pm \sqrt{-27}}{2} = \frac{-3 \pm 3i\sqrt{3}}{2}$

$z = 3$

$(z + 2)(z^2 - 2z + 4) = 0$

$z + 2 = 0 \quad z^2 - 2x + 4 = 0$

$\underline{-2 \; - 2} \quad \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$z = -2$

$z = 3, \frac{-3 \pm 3i\sqrt{3}}{2}, -2, 1 \pm i\sqrt{3}$

13) $6z^4 - z^2 = 12$

$y = z^2, y^2 = z^4$

$6y^2 - y = 12$

$\underline{\quad -12 \quad - 12 \quad}$

$6y^2 - y - 12 = 0$

$(3y + 4)(2y - 3) = 0$

$3y + 4 = 0 \quad 2y - 3 = 0$

$\underline{-4 \quad - 4} \quad \underline{+ 3 \quad + 3}$

$\frac{3y}{3} = -\frac{4}{3} \qquad \frac{2y}{2} = \frac{3}{2}$

$y = -\frac{4}{3}, \frac{3}{2}$

$\sqrt{z^2} = \sqrt{-\frac{4}{3}} \left(\frac{\sqrt{3}}{\sqrt{3}}\right) \quad \sqrt{z^2} = \sqrt{\frac{3}{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$z = \frac{\pm 2i\sqrt{3}}{3}, \frac{\pm\sqrt{6}}{2}$

15) $x^{\frac{2}{3}} - 35 = 2x^{\frac{1}{3}}$

$y = x^{\frac{1}{3}}, \; y^2 = x^{\frac{2}{3}}$

$y^2 - 35 = 2y$

$\underline{\quad -2y \quad - 2y \quad}$

$y^2 - 2y - 35 = 0$

$(y - 7)(y + 5) = 0$

$y - 7 = 0 \quad y + 5 = 0$

$\underline{+7 \; + 7} \quad \underline{- 5 \quad - 5}$

$y = 7, \quad y = -5$

$x^{\frac{1}{3}} = -5 \qquad x^{\frac{1}{3}} = 7$

$\left(\sqrt[3]{x}\right)^3 = (-5)^3 \; \left(\sqrt[3]{x}\right)^3 = 7^3$

$x = -125, 343$

17) $y^{-6} + 7y^{-3} = 8$

$z = y^{-3}\ z^2 = y^{-6}$

$z^2 + 7z = 8$

$\underline{\ -8 \quad -8\ }$

$z^2 + 7z - 8 = 0$

$(z+8)(z-1) = 0$

$z + 8 = 0 \quad z - 1 = 0$

$\underline{\ -8\ -8\ } \quad \underline{\ +1\ \ +1\ }$

$z = -8, \quad z = 1$

$y^{-3} = -8, y^{-3} = 1$

$(y^3)\left(\frac{1}{y^3}\right) = -8(y^3) \quad (y^3)\frac{1}{y^3} = 1(y^3)$

$1 = -8y^3 \qquad 1 = y^3$

$\underline{\ +8y^3 \quad +8y^3\ } \qquad \underline{-1 \quad -1}$

$8y^3 + 1 = 0 \qquad 0 = y^3 - 1$

$(2y+1)(4y^2 - 2y + 1) = 0 \qquad 0 = (y-1)(y^2 + y + 1)$

$2y + 1 = 0 \ \ 4y^2 - 2y + 1 = 0 \qquad y - 1 = 0 \quad y^2 + y + 1 = 0$

$\underline{-1 \quad -1} \ \frac{2 \pm \sqrt{(-2)^2 - 4(4)(1)}}{2(4)} \qquad \underline{\ +1\ \ +1\ } \ \frac{-1 \pm \sqrt{1^1 - 4(1)(1)}}{2(1)}$

$\frac{2y}{2} = \frac{-1}{2} \qquad \frac{2 \pm \sqrt{-12}}{8} \qquad y = 1 \qquad \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$y = -\frac{1}{2} \qquad \frac{2 \pm 2i\sqrt{3}}{8} = \frac{1 \pm i\sqrt{3}}{4}$

$y = -\frac{1}{2}, \frac{1 \pm i\sqrt{3}}{4}, 1, \frac{-1 \pm i\sqrt{3}}{2}$

19) $x^4 - 2x^2 - 3 = 0$

$y = x^2 \ \ y^2 = x^4$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y - 3 = 0 \quad y + 1 = 0$

$\underline{\ +3\ +3\ } \quad \underline{-1 \quad -1\ }$

$y = 3 \qquad y = -1$

$\sqrt{x^2} = \sqrt{3} \qquad \sqrt{x^2} = \sqrt{-1}$

$x = \pm\sqrt{3}, \pm i$

21) $2x^4 - 5x^2 + 2 = 0$

$y = x^2, y^2 = x^4$

$2y^2 - 5y + 2 = 0$

$(2y-1)(y-2) = 0$

$2y - 1 = 0 \quad y - 2 = 0$

$\underline{+1 \quad +1} \quad \underline{+2 \quad +2}$

$\frac{2y}{2} = \frac{1}{2} \qquad y = 2$

$y = \frac{1}{2}$

$\sqrt{x^2} = \sqrt{\frac{1}{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \quad \sqrt{x^2} = \sqrt{2}$

$x = \pm\frac{\sqrt{2}}{2}, \pm\sqrt{2}$

23) $x^4 - 9x^2 + 8 = 0$

$y = x^2,\ y^2 = x^4$

$y^2 - 9y + 8 = 0$

$(y-8)(y-1) = 0$

$y - 8 = 0 \quad y - 1 = 0$

$\underline{+8 \ +8} \quad \underline{+1 \quad +1}$

$y = 8 \qquad y = 1$

$\sqrt{x^2} = \sqrt{8} \quad \sqrt{x^2} = \sqrt{1}$

$x = \pm 2\sqrt{2}, \pm 1$

25) $8x^6 - x^3 + 1 = 0$

$y = x^3, y^2 = x^6$

$8y^2 - y + 1 = 0$

$(8y-1)(y-1) = 0$

$8y - 1 = 0 \quad y - 1 = 0$

$\underline{+1 \ +1} \quad \underline{+1 \quad +1}$

$\frac{8y}{8} = \frac{1}{8} \qquad y = 1$

$y = \frac{1}{8}$

$(8)x^3 = \frac{1}{8}(8)$

$8x^3 = 1$

$\underline{-1 \quad -1}$

$8x^3 - 1 = 0$

$(2x-1)(4x^2 + 2x + 1 = 0$

$2x - 1 = 0 \quad 4x^2 + 2x + 1 = 0$

$\underline{+1 \ +1} \quad \frac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)}$

$\frac{2x}{2} = \frac{1}{2} \quad \frac{-2 \pm \sqrt{-12}}{8}$

$x^3 = 1$

$\underline{-1 \ -1}$

$x^3 - 1 = 0$

$(x-1)(x^2 + x + 1) = 0$

$x - 1 = 0 \quad x^2 + x + 1 = 0$

$\underline{+1 \ +1} \quad \frac{-1 \pm \sqrt{1^1 - 4(1)(1)}}{2(1)}$

$x = 1 \quad \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$x = \frac{1}{2} \quad \frac{-2 \pm 2i\sqrt{3}}{8} = \frac{-1 \pm i\sqrt{3}}{4}$

$x = \frac{1}{2}, \frac{-1 \pm i\sqrt{3}}{4}, 1, \frac{-1 \pm i\sqrt{3}}{2}$

27) $x^8 - 17x^4 + 16 = 0$

$y = x^4, y^2 = x^8$

$y^2 - 17y + 16 = 0$

$(y - 16)(y - 1) = 0$

$y - 16 = 0 \quad y - 1 = 0$

$\underline{+16 + 16} \quad \underline{+1 \quad +1}$

$y = 16 \quad y = 1$

$x^4 = 16$

$-16 \quad -16$

$x^4 - 16 = 0$

$(x^2 + 4)(x^2 - 4) = 0$

$x^2 + 4 = 0 \quad x^2 - 4 = 0$

$\underline{-4 \quad -4} \quad \underline{+4 \quad +4}$

$\sqrt{x^2} = \sqrt{-4} \quad \sqrt{x^2} = \sqrt{4}$

$x = \pm 2i \quad x = \pm 2$

$x^4 = 1$

$-1 \quad -1$

$x^4 - 1 = 0$

$(x^2 - 1)(x^2 + 1) = 0$

$x^2 - 1 = 0 \quad x^2 + 1 = 0$

$\underline{+1 \quad +1} \quad \underline{-1 \quad -1}$

$\sqrt{x^2} = \sqrt{1} \quad \sqrt{x^2} = \sqrt{-1}$

$x = \pm 1 \quad x = \pm i$

$x = \pm 2i, \pm 2, \pm i, \pm 1$

29) $(y + b)^2 - 4(y + b) = 21$

$z = (y + b), \; z^2 = (y + b)^2$

$z^2 - 4z = 21$

$\underline{-21 \quad -21}$

$z^2 - 4z - 21 = 0$

$(z - 7)(z + 3) = 0$

$z - 7 = 0 \quad z + 3 = 0$

$\underline{+7 \quad +7} \quad \underline{-3 \quad -3}$

$z = 7 \quad z = -3$

$y + b = 7 \quad y + b = -3$

$\underline{-b \quad -b} \quad \underline{-b \quad -b}$

$y = 7 - b \quad y = -3 - b$

31) $(y+2)^2-6(y+2)=16$

$z=y+2,\quad z^2=(y+2)^2$

$z^2-6z=16$

$\underline{\quad -16 \quad -16}$

$z^2-6z-16=0$

$(z-8)(z+2)=0$

$z-8=0 \quad z+2=0$

$+8 \quad +8 \quad -2 \quad -2$

$z=8 \qquad z=-2$

$y+2=8 \qquad y+2=-2$

$\underline{-2 \quad -2} \qquad \underline{-2 \quad -2}$

$y=6 \qquad y=-4$

33) $(x-3)^2-2(x-3)=35$

$y=(x-3),\ \ y^2=(x-3)^2$

$y^2-2y=35$

$\underline{-35 \quad -35}$

$y^2-2y-35=0$

$(y-7)(y+5)=0$

$y-7=0 \quad y+5=0$

$\underline{+7 \quad +7} \qquad \underline{-5 \quad -5}$

$y=7 \qquad y=-5$

$x-3=7 \quad x-3=-5$

$\underline{+3 \quad +3} \qquad \underline{+3 \quad +3}$

$x=10,-2$

35) $(r-1)^2-8(r-1)=20$

$y=(r-1),\ y^2=(r-1)^2$

$y^2-8y=20$

$\underline{-20 \quad -20}$

$y^2-8y-20=0$

$(y-10)(y+2)=0$

$y-10=0 \quad y+2=0$

$\underline{+10 \quad +10} \quad \underline{-2 \quad -2}$

$y=10 \qquad y=-2$

$r-1=10 \quad r-1=-2$

$\underline{+1 \quad +1} \qquad \underline{+1 \quad +1}$

$r=11,-1$

37) $3(y+1)^2-14(y+1)=5$

$z=(y+1), z^2=(y+1)^2$

$3z^2-14z=5$

$\underline{-5 \quad -5}$

$3z^2-14z-5=0$

$(3z+1)(z-5)=0$

$3z+1=0 \quad z-5=0$

$\underline{-1 \quad -1} \quad \underline{+5 \quad +5}$

$\frac{3z}{3}=\frac{-1}{3} \qquad z=5$

$z=-\frac{1}{3}$

$y+1=-\frac{1}{3} \qquad y+1=5$

$\underline{-1 \quad -1} \qquad \underline{-1 \quad -1}$

$y=-\frac{4}{3},\ 4$

39) $(3x^2-2x)^2+5=6(3x^2-2x)$

$y=(3x^2-2x),\ y^2=(3x^2-2x)^2$

$y^2 + 5 = 6y$
$\underline{-6y \quad -6y}$
$y^2 - 6y + 5 = 0$
$(y-1)(y-5) = 0$
$y - 1 = 0 \quad y - 5 = 0$
$\underline{+1 \quad +1} \quad \underline{+5 \quad +5}$
$y = 1 \qquad y = 5$

$3x^2 - 2x = 1 \qquad 3x^2 - 2x = 5$
$\underline{-1 \quad -1} \qquad \underline{-5 \quad -5}$
$3x^2 - 2x - 1 = 0 \qquad 3x^2 - 2x - 5 = 0$
$(3x-5)(x+1) = 0 \qquad (3x+1)(x-1) = 0$
$3x - 5 = 0 \quad x + 1 = 0 \quad 3x + 1 = 0 \quad x - 1 = 0$
$\underline{+5 \quad +5} \quad \underline{-1 \quad -1} \quad \underline{-1 \quad -1} \quad \underline{+1 \quad +1}$
$\frac{3x}{3} = \frac{5}{3} \qquad x = -1 \qquad \frac{3x}{3} = \frac{-1}{3} \qquad x = 1$
$x = \frac{5}{3}, -1, -\frac{1}{3}, 1$

41) $2(3x+1)^{\frac{2}{3}} - 5(3x+1)^{\frac{1}{3}} = 88$
$y = (3x+1)^{\frac{1}{3}}, y^2 = (3x+1)^{\frac{2}{3}}$
$2y^2 - 5y = 88$
$\underline{-88 - 88}$
$2y^2 - 5y - 88 = 0$
$(2y+11)(y-8) = 0$
$2y + 11 = 0 \quad y - 8 = 0$
$\underline{-11 \quad -11} \quad \underline{+8 \quad +8}$
$\frac{2y}{2} = \frac{-11}{2} \qquad y = 8$
$y = -\frac{11}{2}$
$(3x+1)^{\frac{1}{3}} = -\frac{11}{2} \qquad (3x+1)^{\frac{1}{3}} = 8$
$\left(\sqrt[3]{3x+1}\right)^3 = \left(-\frac{11}{2}\right)^3 \qquad \left(\sqrt[3]{3x+1}\right)^3 = 8^3$
$3x + 1 = -\frac{1331}{8} \qquad 3x + 1 = 512$
$\underline{-1 \quad -1} \qquad \underline{-1 \quad -1}$

$$\frac{3x}{3} = \left(-\frac{\frac{1329}{8}}{3}\right) \qquad \frac{3x}{3} = \frac{511}{3}$$

$$x = \frac{1329}{24}, \frac{511}{3}$$

43) $(x^2+2x)^2 - 2(x^2+2x) = 3$

$y = (x^2+2x),\ y^2 = (x^2+2x)^2$

$y^2 - 2y = 3$

$\underline{\quad -3 \quad -3}$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y - 3 = 0 \quad y + 1 = 0$

$\underline{+3 \quad +3} \quad \underline{-1 \quad -1}$

$y = 3 \qquad y = -1$

$x^2 + 2x = 3 \qquad x^2 + 2x = -1$

$\underline{\quad -3 \quad -3} \qquad \underline{+1 \quad +1}$

$x^2 + 2x - 3 = 0 \qquad x^2 + 2x + 1 = 0$

$(x+3)(x-1) = 0 \qquad (\sqrt{(x+1)^2} = \sqrt{0}$

$x + 3 = 0 \quad x - 1 = 0 \qquad x + 1 = 0$

$\underline{\quad -3 \quad -3} \quad \underline{+1 \quad +1} \qquad \underline{-1 \quad -1}$

$x = -3, 1, -1$

45) $(2x^2 - x)^2 - 4(2x^2 - x) + 3 = 0$

$y = (2x^2 - x),\ y^2 = (2x^2 - x)^2$

$y^2 - 4y + 3 = 0$

$(y-3)(y-1) = 0$

$y - 3 = 0 \quad y - 1 = 0$

$\underline{+3 \quad +3} \quad \underline{+1 \quad +1}$

$y = 3 \qquad y = 1$

$2x^2 - x = 3 \qquad 2x^2 - x = 1$

$\underline{\quad -3 \quad -3} \qquad \underline{\quad -1 \quad -1}$

$2x^2 - x - 3 = 0 \qquad 2x^2 - x - 1 = 0$

$(2x-3)(x+1) = 0 \qquad (2x+1)(x-1) = 0$

$2x - 3 = 0 \quad x + 1 = 0 \qquad 2x + 1 = 0 \quad x - 1 = 0$

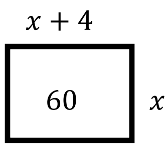
$\underline{+3 \quad +3} \quad \underline{-1 \quad -1} \qquad \underline{-1 \quad -1} \quad \underline{+1 \quad +1}$

$\frac{2x}{2} = \frac{3}{2} \qquad x = -1 \qquad \frac{2x}{2} = \frac{-1}{2} \qquad x = 1$

$x = \frac{3}{2}, -1, -\frac{1}{2}, 1$

9.7

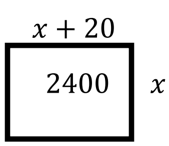
1) In a landscape plan, a rectangular flowerbed is designed to be 4 meters longer than it is wide. If 60 square meters are needed for the plants in the bed, what should the dimensions of the rectangular bed be?



$$x(x+4)=60$$
$$x^2+4x=60$$
$$\quad -60 \quad -60$$
$$x^2+4x-60=0$$
$$(x-6)(x+10)=0$$
$$x-6=0 \quad x+10=0$$
$$\underline{+6 \quad +6} \quad \underline{-10 \quad -10}$$
$$x=6 \quad \cancel{x=-10}$$

$6m\ by\ 10m$

3) A rectangular lot is 20 yards longer than it is wide and its area is 2400 square yards. Find the dimensions of the lot

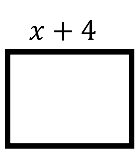


$$x(x+20)=2400$$
$$x^2+20x=2400$$
$$\left(20\cdot\frac{1}{2}\right)^2=10^2=100$$
$$x^2+20x+100=2400+100$$
$$\sqrt{(x+10)^2}=\sqrt{2500}$$
$$x+10=\pm 50$$
$$\underline{-10 \quad -10}$$
$$x=-10\pm 50$$
$$x=40, \cancel{-60}$$

$40yds\ x\ 60\ yds$

5) The length of a rectangular lot is 4 rods greater than its width, and its area is 60 square rods. Find the dimensions of the lot.

$$x(x+4)=60$$
$$x^2+4x=60$$
$$\quad -60 \quad -60$$

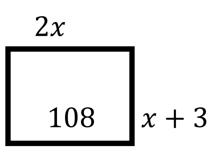


60 $\quad x$

$$x^2+4x-60=0$$
$$(x+10)(x-6)=0$$
$$x+10=0 \quad x-6=0$$
$$\underline{-10-10} \quad \underline{+6\ \ +6}$$
$$\cancel{x=-10} \quad x=6$$

$6\,rods\ x\ 10\,rods$

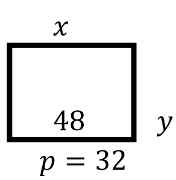
7) A rectangular piece of paper is twice as long as a square piece and 3 inches wider. The area of the rectangular piece is 108 in^2. Find the dimensions of the square piece.



$$2x(x+3)=108$$
$$2x^2+6x=108$$
$$-108-108$$
$$\frac{2x^2}{2}+\frac{6x}{2}-\frac{108}{2}=\frac{0}{2}$$
$$x^2+3x-54=0$$
$$(x+9)(x-6)=0$$
$$x+9=0 \quad x-6=0$$
$$\underline{-9-9} \quad \underline{+6\ +6}$$
$$\cancel{x=-9} \quad x=6$$

$6in\ x\ 6in$

9) The area of a rectangle is 48 ft^2 and its perimeter is 32 ft. Find its length and width.



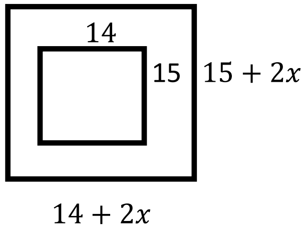
$$2x+2y=32$$
$$-2x \qquad -2x$$
$$\frac{2y}{2}=\frac{32}{2}-\frac{2x}{2}$$
$$y=16-x$$
$$xy=48$$
$$x(16-x)=48$$
$$16x-x^2=48$$
$$\underline{-16x+x^2-16x+x^2}$$
$$0=x^2-16x+48$$
$$0=(x-12)(x-4)$$
$$x-12=0 \quad x-4=0$$
$$\underline{+12\ \ +12} \quad \underline{+4\ +4}$$
$$x=12 \quad x=4$$

$12ft\ x\ 4\ ft$

$$y=16-12=4 \quad y=16-4=12$$

11) A mirror 14 inches by 15 inches has a frame of uniform width. If the area of the frame equals that of the mirror, what is the width of the frame?

$$(15+2x)(14+2x)=420$$

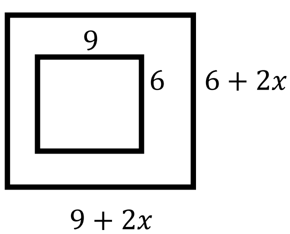


$$210 + 30x + 28 + 4x^2 = 420$$
$$4x^2 + 58x + 210 = 420$$
$$\underline{\quad -420 \;\; -420}$$
$$\frac{4x^2}{2} + \frac{58x}{2} - -\frac{210}{2} = \frac{0}{2}$$
$$2x^2 + 29x - 105 = 0$$
$$(2x + 35)(x - 3) = 0$$
$$2x + 35 = 0 \qquad\qquad x - 3 = 0$$
$$\underline{-35 \quad -35} \qquad\qquad \underline{+3 \quad +3}$$
$$\frac{2x}{2} = \frac{-35}{2} = \cancel{x = \frac{35}{2}} \qquad\qquad x = 3$$

$A = 2(14 \cdot 15) = 420$

$3\ in$

13) A grass plot 9 yards long and 6 yards wide has a path of uniform width around it. If the area of the path is equal to the area of the plot, determine the width of the path.



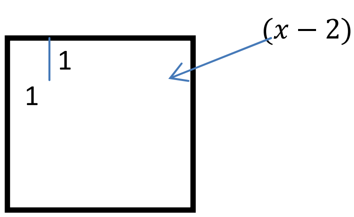
$$(6 + 2x)(9 + 2x) = 108$$
$$54 + 12x + 18x + 4x^2 = 108$$
$$4x^2 + 30x + 54 = 108$$
$$\underline{\quad -108 \;\; -108}$$
$$\frac{4x^2}{2} + \frac{30x}{2} - \frac{54}{2} = \frac{0}{2}$$
$$2x^2 + 15x - 27 = 0$$
$$(2x - 3)(x + 9) = 0$$
$$2x - 3 = 0 \quad x + 9 = 0$$
$$\underline{+3 \quad +3} \quad \underline{-9 \quad -9}$$
$$\frac{2x}{2} = \frac{3}{2} \qquad \cancel{x = -9}$$
$$x = \frac{3}{2} = 1.5$$

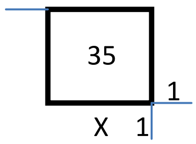
$A = 2(9 \cdot 6) = 108$

$1.5\ yds$

15) A page is to have a margin of 1 inch, and is to contain 35 in^2 of painting. How large must the page be if the length is to exceed the width by 2 inches?

$$x(x - 2) = 35$$
$$x^2 - 2x = 35$$

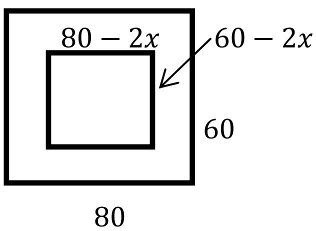




$$\underline{\quad -35 \quad -35}$$
$$x^2 - 2x - 35 = 0$$
$$(x-7)(x+5) = 0$$
$$x - 7 = 0 \quad x + 5 = 0$$
$$\underline{+7 \ +7} \quad \underline{-5 \ -5}$$
$$x = 7 \quad \cancel{x = -5}$$

$7in\ x\ 9in$

17) A rectangular wheat field is 80 rods long by 60 rods wide. A strip of uniform width is cut around the field, so that half the grain is left standing in the form of a rectangular plot. How wide is the strip that is cut?

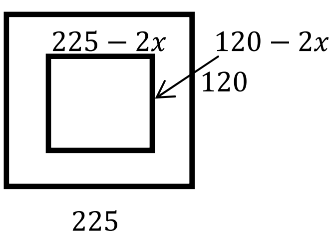


$$(80-2x)(60-2x) = 2400$$
$$48000 - 160x - 120x + 4x^2 = 2400$$
$$4x^2 - 280x + 4800 = 2400$$
$$\underline{-2400 \quad -2400}$$
$$\frac{4x^2}{4} - \frac{280x}{4} + \frac{2400}{4} = \frac{0}{4}$$
$$x^2 - 70x + 600 = 0$$
$$(x-10)(x-60) = 0$$
$$x - 10 = 0 \quad x - 60 = 0$$
$$\underline{+10 \ +10} \quad \underline{+60 \ +60}$$
$$x = 10 \quad \cancel{x = 60}$$

$$A = \frac{1}{2}(60 \cdot 80) = 2400$$

$10\ rods$

19) A rectangular field 225 ft by 120 ft has a ring of uniform width cut around the outside edge. The ring leaves 65% of the field uncut in the center. What is the width of the ring?



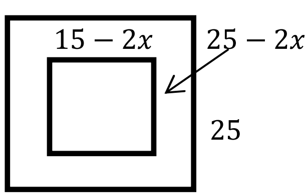
$$(225-2x)(120-2x) = 17500$$
$$27000 - 450x - 240x + 4x^2 = 17500$$
$$4x^2 - 690x + 27000 = 17500$$
$$\underline{-17500 \quad -17500}$$
$$\frac{4x^2}{2} - \frac{690x}{2} + \frac{9450}{2} = \frac{0}{2}$$
$$2x^2 - 345x + 4725 = 0$$

$$A = 0.65(120 \cdot 225)17500$$

$$\frac{345 \pm \sqrt{345^2 - 4(2)(4725)}}{2(2)} = \frac{345 \pm \sqrt{81225}}{4} = \frac{345 \pm 285}{4} = \cancel{\frac{315}{2}}\ \&\ 15$$

$15ft$

21) A frame is 15 in by 25 in and is of uniform width. The inside of the frame leaves 75% of the total area available for the picture. What is the width of the frame?



$$(15-2x)(25-2x) = 281.25$$
$$375 - 30x - 50x + 4x^2 = 281.25$$
$$4x^2 - 80x + 375 = 281.25$$
$$\underline{-281.25 - 281.25}$$
$$\frac{4x^2}{4} - \frac{80x}{4} + \frac{93.75}{4} = \frac{0}{4}$$

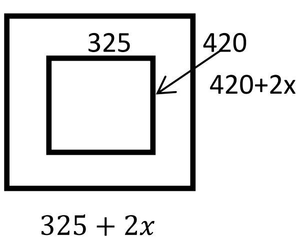
15

$A = .75(25 \cdot 15) = 281.25$

$$x^2 - 20x + 23.4375 = 0$$
$$\underline{\quad -23.4375 \quad -23.4375}$$
$$x^2 - 20x \qquad = -23.4375$$
$$\left(-20 \cdot \frac{1}{2}\right)^2 = (-10)^2 = 100$$
$$x^2 - 20x + 100 = -23.4375 + 100$$
$$\sqrt{(x-10)^2} = \sqrt{(76.5625)}$$
$$x - 10 = \pm 8.75$$
$$\underline{+10 \quad +10}$$
$$x = 10 \pm 8.75$$
$$x = \cancel{18.75},\ 1.25$$

$1.25\ in$

23) The farmer in the previous problem has a neighbor who has a field 325 ft by 420 ft. His neighbor wants to increase the size of his field by 20% by cultivating a band of uniform width around the outside of his lot. How wide a band should his neighbor cultivate?

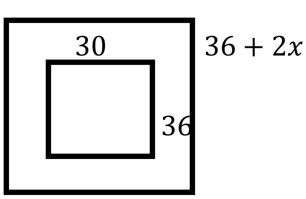


$A = 1.2(325)(420) = 163800$

$$(420 + 2x)(325 + 2x) = 163800$$
$$136500 + 840x + 650x + 4x^2 = 163800$$
$$4x^2 + 1490x + 136500 = 163800$$
$$\underline{\quad -163800 \quad -163800}$$
$$\frac{4x^2}{4} + \frac{1490x}{4} - \frac{27300}{4} = \frac{0}{4}$$
$$x^2 + 745x - 13650 = 0$$
$$\frac{-745 \pm \sqrt{(745^2 - 4(2)(-13650)}}{2(2)} = \frac{-745 \pm \sqrt{664225}}{4} \cancel{=}$$
$$\frac{-745 \pm 815}{4} = 17.5, -390$$

$17.5\ ft$

25) Donna has a garden that is 30 ft by 36 ft. She wants to increase the size of the garden by 40%. How wide a ring around the outside should she cultivate?



$$(36 + 2x)(30 + 2x) = 1512$$
$$1080 + 72x + 60x + 4x^2 = 1512$$
$$4x^2 + 132x + 1080 = 1512$$
$$\underline{\quad -1512 \quad -1512}$$

$$\frac{4x^2}{4}+\frac{132x}{4}-\frac{432}{4}=\frac{0}{4}$$

$30+2x$ $\qquad$ $x^2+33x-108=0$

$A=1.4(30\cdot 36)=1512$ $\qquad$ $\frac{-33\pm\sqrt{33^2-4(1)(-108)}}{2(1)}=\frac{-33\pm\sqrt{1521}}{2}=\frac{-33\pm 39}{2}=3,\cancel{-36}$

$3\ ft.$

9.8

1) Bills father can paint a room in two hours less than Bill can paint it. Working together they can complete the job in two hours and 24 minutes. How much time would each require working alone?

$Father: x-2$

$Bill: x$

$Team: 2\frac{24}{60}=2\frac{2}{5}=\frac{12}{5}$

$$\frac{1}{x-2}\,12x(x-2)+\frac{1}{x}\,12x(x-2)=\frac{5}{12}\,12x(x-2)$$

$LCD: 12x(x-2)$

$$12x+12(x-2)=5x(x-2)$$

$$12x+12x-24=5x^2-10x$$

$$24x-24=5x^2-10$$

$$\underline{-24x+24\ \ -24x+24}$$

$$0=5x^2-34x+24$$

$$0=(5x-4)(x-6)$$

$$5x-4=0 \quad x-6=0$$

$$\underline{+4\quad +4} \quad \underline{+6\quad +6}$$

$$\frac{5x}{5}=\frac{4}{5} \quad x=6$$

$$\cancel{x=.8}$$

$\boxed{Bill: 6hr, Father: 4}$

3) Jack can wash and wax the family car in one hour less than Bob can. The two working together can complete the job in $1\frac{1}{5}$ hours. How much time would each require if they worked alone?

$Jack: x-1$

$Bob: x$

$Team: 1\frac{1}{5} = \frac{6}{5}$

$$\frac{1}{x-1}\big(6x(x-1)\big) + \frac{1}{x}\big(6x(x-1)\big) = \frac{5}{6}\big(6x(x-1)\big)$$

$LCD: 6x(x-1)$

$$6x + 6(x-1) = 5x(x-1)$$
$$6x + 6x - 6 = 5x^2 - 5x$$
$$12x - 6 = 5x^2 - 5x$$
$$\underline{-12x + 6 \quad -12x + 6}$$
$$0 = 5x^2 - 17x + 6$$
$$0 = (5x-2)(x-3)$$
$$5x - 2 = 0 \quad x - 3 = 0$$
$$\underline{+2 \quad +2} \quad \underline{+3 \quad +3}$$
$$\frac{5x}{5} = \frac{2}{5} \qquad x = 3$$
$$\cancel{x = .4}$$

$\boxed{Bob: 3hr, Jack: 2hr}$

5) Working alone it takes John 8 hours longer than Carlos to do a job. Working together they can do the job in 3 hours. How long will it take each to do the job working alone

$John: x+8$

$Carlos: x$

$Team: 3$

$$\frac{1}{x+8}\,3x(x+8) + \frac{1}{x}\,3x(x+8) = \frac{1}{3}\,3x(x+8)$$

$LCD: 3x(x+8)$

$$3x + 3(x+8) = x(x+8)$$
$$3x + 3x + 24 = x^2 + 8x$$
$$6x + 24 = x^2 + 8x$$
$$\underline{-6x - 24 \qquad -6x - 24}$$
$$0 = x^2 + 2x - 24$$
$$0 = (x+6)(x-4)$$
$$x + 6 = 0 \quad x - 4 = 0$$
$$\underline{-6 \quad -6} \quad \underline{+4 \quad +4}$$
$$\cancel{x = -6} \qquad x = 4$$

$\boxed{Carlos: 4hr, John: 12hr}$

7) A can do a piece of work in 4 days and B can do it in half the time. How long will it take them to do the work together?

$A: 4$

$B: 2$

$Team: x$

$$\frac{1}{4}(4x) + \frac{1}{2}(4x) = \frac{1}{x}(4x)$$

$LCD: 4x$

$$x + 2x = 4$$

$$\frac{3x}{3} = \frac{4}{3}$$
$$x = .133$$
$$\boxed{(1\ hr., 20\ \text{min.})}$$

9) If A can do a piece of work in 24 days and A and B together can do it in 6 days, how long would it take B to do the work alone?

$A: 24$ | $\frac{1}{24}(24x) + \frac{1}{x}(24x) = \frac{1}{6}(24x)$

$B: x$ | $LCD: 24x$

$Team: 6$ | $x + 24 = 4x$

$$\underline{-x \qquad -x}$$
$$\frac{24}{3} = \frac{3x}{3}$$
$$8 = x \qquad \boxed{8\ days}$$

11) If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

$Sam: 3$ | $\frac{1}{3}(6x) + \frac{1}{6}(6x) = \frac{1}{x}(6x)$

$Fred: 6$ | $LCD: 6x$

$Team: x$ | $2x + x = 6$

$$\frac{3x}{3} = \frac{6}{3}$$
$$x = \boxed{2\ days}$$

13) Two people working together can complete a job in 6 hours. If one of them works twice as fast as the other, how long would it take the faster person, working alone, to do the job?

$A = x$ | $\frac{1}{x}(6x) + \frac{1}{2x}(6x) = \frac{1}{6}(6x)$

$B = 2x$ | $6 + 3 = x$

$Team: 6$ | $9 = x$

$\boxed{A = 9hr}$, $B = 18hr$

15) A water tank can be filled by an inlet pipe in 8 hours. It takes twice that long for the outlet pipe to empty the tank. How long will it take to fill the tank if both pipes are open?

$In: 8$ | $\frac{1}{8}(16x) - \frac{1}{16}(16x) = \frac{1}{x}(16x)$

$Out: -16$ | $2x - x = 16$

$Team: x$ | $x = \boxed{16hr}$

17) It takes 10 hours to fill a pool with the inlet pipe. It can be emptied in 15 hrs. with the outlet pipe. If the pool is half full to begin with, how long will it take to fill it from there if both pipes are open?

$In: 10$ | $\frac{1}{10}(30x) - \frac{1}{15}(30x) = \frac{1}{x}(30x)$

$Out: -15$ | $3x - 2x = 30$

$Team: x$ | $x = 30$

$$\frac{1}{2}(30) = \boxed{15\ hr}$$

19) A sink has two faucets, one for hot water and one for cold water. The sink can be filled by a cold-water faucet in 3.5 minutes. If both faucets are open, the sink is filled in 2.1 minutes. How long does it take to fill the sink with just the hot-water faucet open?

$Hot: x$

$Cold: 3.5 = \frac{35}{10} = \frac{7}{2}$

$Team: 2.1 = \frac{21}{10}$

$$\frac{1}{x}(21x) + \frac{2}{7}(21x) = \frac{10}{21}(21x)$$

$LCD: 21x$

$$21 + 6x = 10x$$
$$-6x \quad -6x$$
$$\frac{21}{4} = \frac{4x}{4}$$
$$\boxed{5.25\ hr} = x$$

21) A tank can be emptied by any one of three caps. The first can empty the tank in 20 minutes while the second takes 32 minutes. If all three working together could empty the tank in $8\frac{8}{59}$ minutes, how long would the third take to empty the tank?

$First: 20$

$Second: 32$

$Third: x$

$Team: 8\frac{8}{59} = \frac{480}{59}$

$$\frac{1}{20}(480x) + \frac{1}{32}(480x) + \frac{1}{x}(480x) = \frac{59}{480}(480x)$$

$LCD: 480x$

$$24x + 15x + 480 = 59x$$
$$39x + 480 = 59x$$
$$\underline{-39x \qquad\quad -39x}$$
$$\frac{480}{20} = \frac{20x}{20}$$
$$24 = x \qquad \boxed{24\ min.}$$

23) Sam takes 6 hours longer than Susan to wax a floor. Working together they can wax the floor in 4 hours. How long will it take each of them working alone to wax the floor?

$Sam: x + 6$

$Susan: x$

$Team: 4$

$$\frac{1}{x+6}\big(4x(x+6)\big) + \frac{1}{x}\big(4x(x+6)\big) = \frac{1}{4}\big(4x(x+6)\big)$$

$LCD: \big(4x(x+6)\big)$

$$4x + 4(x+6) = x(x+6)$$
$$4x + 4x + 24 = x^2 + 6x$$
$$8x + 24 = x^2 + 6x$$
$$\underline{-8x - 24 \qquad -8x - 24}$$

$$0 = x^2 - 2x - 24$$
$$0 = (x-6)(x+4)$$
$$x - 6 = 0 \quad x + 4 = 0$$
$$\underline{+6 \quad +6} \quad \underline{-4 \quad -4}$$
$$\cancel{x = 6} \quad x = -4$$

$\boxed{Susan = 6hr, Sam = 12hr}$

25) It takes Sally $10\frac{1}{2}$ minutes longer than Patricia to clean up their dorm room. If they work together they can clean it in 5 minutes. How long will it take each of them if they work alone?

$Sally: x + 10.5 = x + \frac{21}{2} = \frac{2x+21}{2}$

$Patricia: x$

$Team: 5$

$$\frac{2}{2x+21}\left(5x(2x+21)\right) + \frac{1}{x}\left(5x(2x+21)\right) = \frac{1}{5}\left(5x(2x+21)\right)$$
$$LCD: \left(5x(2x+21)\right)$$
$$10x + 5(2x+21) = x(2x+21)$$
$$10x + 10x + 105 = 2x^2 + 21x$$
$$20x + 105 = 2x^2 + 21x$$
$$\underline{-20x - 105 \qquad -20x - 105}$$
$$0 = 2x^2 + x - 105$$
$$0 = (2x+15)(x-7)$$
$$2x + 15 = 0 \quad x - 7 = 0$$
$$\underline{-15 \quad -15} \quad \underline{+7 \quad +7}$$
$$\frac{2x}{2} = -\frac{15}{2} \qquad x = 7$$
$$\cancel{x = -\frac{15}{2}}$$

$\boxed{Patricia = 7min., Sally = 17.5\ min.}$

27) Secretary A takes 6 minutes longer than Secretary B to type 10 pages of manuscript. If they divide the job and work together it will take them $8\frac{3}{4}$ minutes to type 10 pages. How long will it take each working alone to type the 10 pages?

$A: x + 6$

$B: x$

$Team: 8\frac{3}{4} = \frac{35}{4}$

$$\frac{1}{x+6}\left(35x(x+6)\right) + \frac{1}{x}\left(35x(x+6)\right) = \frac{4}{35}\left(35x(x+6)\right)$$
$$LCD: \left(35x(x+6)\right)$$
$$35x + 35(x+6) = 4x(x+6)$$
$$35x + 35x + 210 = 4x^2 + 24x$$
$$70x + 210 = 4x^2 + 24x$$
$$\underline{-70x - 210 \qquad -70x - 210}$$
$$\frac{0}{2} = \frac{4x^2}{2} - \frac{46x}{2} - \frac{210}{2}$$
$$0 = 2x^2 - 23x - 105$$
$$0 = (2x+7)(x-15)$$
$$2x + 7 = 0 \quad x - 15 = 0$$
$$\underline{-7 \quad -7} \quad \underline{+15 \quad +15}$$
$$\frac{2x}{2} = -\frac{7}{2} \qquad x = 15$$
$$\cancel{x = -\frac{7}{2}}$$

$\boxed{B: 15hr,\ A: 21hr}$

9.9

1) $\frac{xy}{y} = \frac{72}{y} \qquad x = \frac{72}{y}$

$\frac{(x+2)(y-4)}{y-4} = \frac{128}{y-4}$

$x + 2 = \frac{128}{y-4}$

$y(y-4)\frac{72}{y} + 2\,y(y-4) = \frac{128}{y-4}y(y-4)$

$LCD: y(y-4)$

$72(y-4) + 2y(y-4) = 128y$

$72y - 288 + 2y^2 - 8y = 128y$

$2y^2 + 64y - 288 = 128y$

$\underline{\quad -128y \qquad\qquad -128y}$

$\frac{2y^2}{2} - \frac{64y}{2} - \frac{288}{2} = 0$

$y^2 - 32y - 144 = 0$

$\underline{\quad +144 \quad +144}$

$y^2 - 32y \qquad = 144$

$\left(32 \cdot \frac{1}{2}\right)^2 = 16^2 = 256$

$y^2 - 32y + 256 = 144 + 256$

$\sqrt{(y-16)^2} = \sqrt{400}$

$y - 16 = \pm 20$

$\underline{\quad +16 \quad +16}$

$y = 36, -4$

$x = \frac{72}{36} = 2 \quad x = \frac{72}{-4} = -18$

$(2, 36), (48, -4)$

3) $\frac{xy}{y} = \frac{150}{y} \qquad x = \frac{150}{y}$

$\frac{(x-6)(y+1)}{y+1} = \frac{64}{y+1}$

$x - 6 = \frac{64}{y+1}$

$\frac{150y}{y}\ \big(y(y+1)\big) - 6\big(y(y+1)\big) = \frac{64}{y+1}\big(y(y+1)\big)$

$LCD: \big(y(y+1)\big)$

$150(y+1) - 6y(y+1) = 64y$

$150y + 150 - 6y^2 + 6y = 64y$

$-6y^2 + 156y + 150 = 64y$

$\underline{+6y^2 - 156y - 150 \quad +6y^2 - 156y - 150}$

$\frac{0}{2} = \frac{6y^2}{2} - \frac{80y}{2} - \frac{150}{2}$

$0 = 3y^2 - 40y - 75$

$0 = (3y+5)(y-15)$

$3y + 5 = 0 \quad y - 15 = 0$

$\underline{\quad -5 \quad -5} \quad \underline{+15 \quad +15}$

$\frac{3y}{3} = -\frac{5}{3} \qquad y = 15$

$y = -\frac{5}{3} \quad x = \frac{150}{15} = 10$

$x = \frac{150}{-\frac{5}{3}} = 150\left(-\frac{3}{5}\right) = -90$

$\left(-90, -\frac{5}{3}\right),\ (10, 15)$

5) $\frac{xy}{y} = \frac{45}{y} \qquad x = \frac{45}{y}$

$\frac{(x+2)(y+1)}{y+1} = \frac{70}{y+1}$

$x + 2 = \frac{70}{y+1}$

$\frac{45}{y}\left(y(y+1)\right) + 2\left(y(y+1)\right) = \frac{70}{y+1}\left(y(y+1)\right)$

$LCD: \left(y(y+1)\right)$

$45(y+1) + 2y(y+1) = 70y$

$45y + 45 + 2y^2 + 2y = 70y$

$2y^2 + 47y + 2y = 70y$

$\underline{\quad -70y \qquad -70y}$

$2y^2 - 23y + 45 = 0$

$(2y-5)(y-9) = 0$

$2y - 5 = 0 \quad y - 9 = 0$

$\underline{+5 \quad +5} \quad \underline{+9 \quad +9}$

$\frac{2y}{2} = \frac{5}{2} \qquad y = 9$

$y = \frac{5}{2} \qquad x = \frac{45}{9} = 5$

$x = \frac{45}{\frac{5}{2}} = 45 \cdot \frac{2}{5} = 18$

$\left(18, \frac{5}{2}\right),\ (5, 9)$

7) $\frac{xy}{y} = \frac{90}{y} \qquad x = \frac{90}{y}$

$\frac{(x-5)(y+1)}{y+1} = \frac{120}{y+1}$

$x - 5 = \frac{120}{y+1}$

$\frac{90}{y}\left(y(y+1)\right) - 5\left(y(y+1)\right) = \frac{120}{y+1}\left(y(y+1)\right)$

$LCD: \left(y(y+1)\right)$

$90(y+1) - 5y(y+1) = 120y$

$90y + 90 - 5y^2 - 5y = 120y$

$-5y^2 + 85y + 90 = 120y$

$\underline{+5y^2 - 85y - 90 \quad +5y^2 - 85y - 90}$

$0 = \frac{5y^2}{5} + \frac{35y}{5} - \frac{90}{5}$

$0 = y^2 + 7y - 18$

$0 = (y+9)(y-2)$

$y + 9 = 0 \quad y - 2 = 0$

$\underline{-9 \quad -9} \quad \underline{+2 \quad +2}$

$y = -9 \qquad y = 2$

$x = \frac{90}{-9} = -10 \qquad x = \frac{90}{2} = 45$

9) $\frac{xy}{y} = \frac{12}{y} \qquad x = \frac{12}{y}$

$\frac{(x+1)(y-4)}{y-4} = \frac{16}{y-4}$

$x + 1 = \frac{16}{y-4}$

$\frac{12}{y}\left(y(y-4)\right) + 1\left(y(y-4)\right) = \frac{16}{y-4}\left(y(y-4)\right)$

$LCD: \left(y(y-4)\right)$

$12(y-4) + \left(y(y-4)\right) = 16y$

$12y - 48 + y^2 - 4y = 16y$

$y^2 + 8y - 48 = 16y$

$\underline{-16y \qquad -16y}$

$y^2 - 8y - 48 = 0$

$(y-12)(y+4) = 0$

$y - 12 = 0 \quad y + 4 = 0$

$\underline{+12 \quad +12 \quad -4 \quad -4}$

$y = 12 \qquad y = -4$

$x = \frac{12}{12} = 1 \qquad x = \frac{12}{-4} = -3$

$(1, 12), (-3, -4)$

11) $\frac{xy}{y} = \frac{45}{y} \qquad x = \frac{45}{y}$

$\frac{(x-5)(y+3)}{y+3} = \frac{160}{y+3}$

$x - 5 = \frac{160}{y+3}$

$\frac{45}{y}\left(y(y+3)\right) - 5\left(y(y+3)\right) = \frac{160}{y+3}\left(y(y+3)\right)$

$45(y+3) - 5y(y+3) = 160y$

$45y + 135 - 5y^2 - 15y = 160y$

$-5y^2 + 30y + 135 = 160y$

$\underline{+5y^2 - 30y - 135 \quad +5y^2 - 30y - 135}$

$0 = \frac{5y^2}{5} + \frac{130y}{5} - \frac{135}{5}$

$0 = y^2 + 26y - 27$

$0 = (y+27)(y-1)$

$y + 27 = 0 \quad y - 1 = 0$

$\underline{-27 \quad -27} \quad \underline{+1 \quad +1}$

$y = -27 \quad y = 1$

$x = \frac{45}{-27} = -\frac{5}{3} \qquad x = \frac{45}{1}$

$\left(-\frac{5}{3}, -27\right),\ (45, 1)$

9.10

1) A merchant bought some pieces of silk for $900. Had he bought 3 pieces more for the same money, he would have paid $15 less for each piece. Find the number of pieces purchased.

N	P	T
N	P	900
$n+3$	$p-15$	900

$\frac{np}{n}=\frac{900}{n} \qquad p=900n$

$\frac{(n+3)(p-15)}{n+3}=\frac{900}{n+3}$

$n+3=\frac{900}{n+3}$

$\frac{900}{n}\big(n(n+3)\big)-15\big(n(n+3)\big)=\frac{900}{n+3}n(n+3)$

$LCD\colon n(n+3)$

$900(n+3)-15n(n+3)=900n$

$900n+2700-15n^2-45n=900n$

$-15n^2+855n+2700=900n$

$\underline{+15n^2-855n-2700 \quad +15n^2-855n-2700}$

$0=\frac{15n^2}{15}+\frac{45n}{15}-\frac{2700}{15}$

$0=n^2+3n-180$

$0=(n+15)(n-12)$

$n+15=0 \quad n-12=0$

$\underline{\quad -15 \quad -15 \quad +12 \quad +12}$

$\cancel{n=-15} \qquad n=12$

3) A merchant bought a number of barrels of apples for S120. He kept two barrels and sold the remainder at a profit of $2 per barrel making a total profit of $34. How many barrels did he originally buy?

N	P	T
N	P	120
$n-2$	$p+2$	900

$\frac{np}{n}=\frac{120}{n} \qquad p=\frac{120}{n}$

$\frac{(n-2)(p+2)}{n-2}=\frac{154}{n-2}$

$p+2=\frac{154}{n-2}$

$\frac{120}{n}\big(n(n-2)\big)+2\big(n(n-2)\big)=\frac{154}{n-2}\big(n(n-2)\big)$

$LCD\colon \big(n(n-2)\big)$

$120(n-2)+2n(n-2)=154n$

$120n-240+2n^2-4n=154n$

$2n^2+116n-240=154n$

$\underline{\quad -154n \qquad\qquad -154n}$

$\frac{2n^2}{2}-\frac{38n}{2}-\frac{240}{2}=\frac{0}{2}$

$n^2-17n-120=0$

$(n-24)(n+5)=0$

$n-24=0 \quad n+5=0$

$\underline{\quad +24 \quad +24 \quad -5 \quad -5}$

$n=24 \qquad \cancel{n=-5}$

5) A man bought a number of articles at equal cost for \$500. He sold all but two for \$540 at a profit of S5 for each item. How many articles did he buy?

N	P	T
N	P	500
$n-2$	$p+5$	540

$$\frac{np}{n}=\frac{500}{n} \quad p=\frac{500}{n}$$
$$\frac{(n-2)(p+5)}{n-2}=\frac{540}{n-2}$$
$$p+5=\frac{540}{n-2}$$
$$\frac{500}{n}\left(n(n+2)\right)+5\left(n(n+2)\right)=\frac{540}{n-2}\left(n(n+2)\right)$$
$$LCD:\left(n(n+2)\right)$$
$$500(n-2)+5n(n-2)=540n$$
$$500n-1000+5n^2-10n=540n$$
$$5n^2+490n-1000=540n$$
$$\underline{-540n \qquad\qquad -540n}$$
$$\frac{5n^2}{5}-\frac{50n}{5}-\frac{1000}{5}=\frac{0}{5}$$
$$n^2-10n-200=0$$
$$(n-20)(n+40)=0$$
$$n-20=0 \quad n+40=0$$
$$\underline{+20 \ \ +20 \quad -40 \ \ -40}$$
$$n=20 \qquad \cancel{n=-40}$$

7) A group of boys bought a boat for \$450. Five boys failed to pay their share, hence each remaining boys were compelled to pay \$4.50 more. How many boys were in the original group and how much had each agreed to pay?

N	P	T
N	P	450
$n-5$	$p+4.5$	450

$$\frac{np}{n}=\frac{450}{n} \quad p=\frac{450}{n}$$
$$\frac{(n-5)(p+4.5)}{n-5}=\frac{450}{n-5}$$
$$p+4.5=\frac{450}{n-5}$$
$$\frac{450}{n}\left(2n(n-5)\right)+\frac{9}{2}\left(2n(n-5)\right)=\frac{450}{n-5}\left(2n(n-5)\right)$$
$$LCD:\left(2n(n-5)\right)$$
$$900(n-5)+9n(n-5)=900n$$
$$900n-4500+9n^2-45n=900n$$
$$9n^2+855n-4500=900n$$
$$\underline{-900n \qquad\qquad -900n}$$
$$\frac{9n^2}{9}-\frac{45n}{9}-\frac{4500}{9}=\frac{0}{9}$$
$$n^2-5n-500=0$$
$$(n-25)(n+20)=0$$
$$n-25=0 \quad n+20=0$$
$$\underline{+25 \ \ +25 \quad -20 \ \ -20}$$
$$n=25 \qquad \cancel{n=-20}$$

9) A factory tests the road performance of new model cars by driving them at two different rates of speed for at least 100 kilometers at each rate. The speed rates range from 50 to 70 km/hr in the lower range and from 70 to 90 km/hr in the higher range. A driver plans to test a car on an available speedway by driving it for 120 kilometers at a speed in the lower range and then driving 120 kilometers at a rate that is 20 km/hr faster. At what rates should he drive if he plans to complete the test in 3 $\frac{1}{2}$ hours?

r	t	d
r	t	120
$r+20$	$35-t$	120

$\frac{rt}{r} = \frac{120}{r} \quad t = \frac{120}{r}$

$\frac{(r+20)(3.5-t)}{r+20} = \frac{120}{r+20}$

$\frac{7}{2}\left(2r(r+20)\right) - \frac{120}{r}\left(2r(r+20)\right) = \frac{120}{r+20}\left(2r(r+20)\right)$

$LCD: \left(2r(r+20)\right)$

$7r(r+20) - 240(r+20) = 240r$

$7r^2 + 140r - 240r - 480 = 240r$

$7r^2 - 100r - 480 = 240r$

$\underline{\quad -240r \qquad\qquad -240r}$

$7r^2 - 340r - 4800 = 0$

$\frac{340 \pm \sqrt{(340)^2 - 4(7)(-4800)}}{2(7)} = \frac{340 \pm \sqrt{25000}}{14} = \frac{340 \pm 500}{14} = 60, \cancel{-\frac{80}{7}}$

$60mph \;\&\; 80mph$

11) The rate of the current in a stream is 3 km/hr. A man rowed upstream for 3 kilometers and then returned. The round trip required 1 hour and 20 minutes. How fast was he rowing?

$1\frac{1}{3} = \frac{4}{3}$

r	t	d
r	t	3
$r-3$	$\frac{4}{3}-t$	3

$\frac{(r+3)t}{r+3} = \frac{3}{r+3} \quad t = \frac{3}{r+3}$

$\frac{(r-3)\left(\frac{4}{3}-t\right)}{r-3} = \frac{3}{r-3}$

$\frac{4}{3} - t = \frac{3}{r-3}$

$\frac{4}{3}\, 3(r+3)(r-3) - \frac{3}{r+3} 3(r+3)(r-3) = \frac{3}{r-3} 3(r+3)(r-3)$

$LCD: 3(r+3)(r-3)$

$4(r^2 - 9) - 9(r-3) = 9(r+3)$

$4r^2 - 36 - 9r + 27 = 9r + 27$

$4r^2 - 9r - 9 = 9r + 27$

$\underline{\quad -9r - 27 \quad -9r - 27}$

$\frac{4r^2}{2} - \frac{18r}{2} - \frac{36}{2} = 0$

$2r^2 - 9r - 18 = 0$

$(2r+3)(r-6) = 0$

$2r + 3 = 0 \quad r - 6 = 0$

$\underline{\quad -3 \;\; -3 \quad +6 \;\; +6}$

$\frac{2r}{2} = -\frac{3}{2} \qquad r = 6$

$\cancel{r = -\frac{3}{2}}$

13) Two drivers are testing the same model car at speeds that differ by 20 km/hr. The one driving at the slower rate drives 70 kilometers down a speedway and returns by the same route. The one driving at the faster rate drives 76 kilometers down the speedway and returns by the same route. Both drivers leave at the same time, and the faster car returns $\frac{1}{2}$ hour earlier than the slower car. At what rates were the cars driven?

r	t	d
$r+20$	$t-\frac{1}{2}$	152
r	t	140

$$\frac{rt}{r}=\frac{140}{r} \quad t=\frac{140}{r}$$
$$\frac{(r+20)\left(t-\frac{1}{2}\right)}{r+20}=\frac{152}{r+20}$$
$$t-\frac{1}{2}=\frac{152}{r+20}$$
$$\frac{140}{r}\big(2r(r+20)\big)-\frac{1}{2}\big(2r(r+20)\big)=\frac{152}{r+20}\big(2r(r+20)\big)$$
$$LCD: \big(2r(r+20)\big)$$
$$280(r+20)-r(r+20)=304r$$
$$280r+560-r^2-20r=304r$$
$$-r^2+260r+5600=304r$$
$$\underline{+r^2-260r-5600 \quad +r^2-260r-5600}$$
$$0=r^2+44r-5600$$
$$0=(r-56)(r+100)$$
$$r-56=0 \quad r+100=0$$
$$\underline{+56 \;\; +56 \quad -100 \;\; -100}$$
$$r=56 \qquad \cancel{r=-100}$$

$56km/hr \;\&\; 76km/hr$

15) An automobile goes to a place 72 miles away and then returns, the round trip occupying 9 hours. His speed in returning is 12 miles per hour faster than his speed in going. Find the rate of speed in both going and returning.

r	t	d
r	t	72
$r+12$	$9-t$	72

$$\frac{rt}{r}=\frac{72}{r} \quad t=\frac{72}{r}$$
$$\frac{(r+12)(9-t)}{r+12}=\frac{72}{r+12}$$
$$9-t=\frac{72}{r+12}$$
$$9\big(r(r+12)\big)-\frac{72}{r}\big(r(r+12)\big)=\frac{72}{r+12}\big(r(r+12)\big)$$
$$LCD: r(r+12)$$
$$9r(r+12)-72(r+12)=72r$$
$$9r^2+108r-72r-864=72r$$
$$9r^2+36r-864=72r$$
$$\underline{-72r \qquad\qquad -72r}$$
$$\frac{9r^2}{9}-\frac{36r}{9}-\frac{864}{9}=\frac{0}{9}$$
$$r^2-4r-96=0$$
$$(r-12)(r+8)=0$$
$$r-12=0 \quad r+8=0$$
$$\underline{+12 \;\; +12 \quad -8 \;\; -8}$$
$$r=12 \qquad \cancel{r=-8}$$

$12\ mph \;\&\; 24mph$

17) The rate of a stream is 3 miles an hour. If a crew rows downstream for a distance of 8 miles and then back again, the round trip occupying 5 hours, what is the rate of the crew in still water?

r	t	d
$r+3$	t	8
$r-3$	$5-t$	8

$\frac{(r+3)t}{r+3} = \frac{8}{r+3} \quad t = \frac{8}{r+3}$

$\frac{(r-3)(5-t)}{r-3} = \frac{8}{r-3}$

$5 - t = \frac{8}{r-3}$

$5\,(r+3)(r-3) - \frac{8}{r+3}(r+3)(r-3) = \frac{8}{r-3}(r+3)(r-3)$

$5(r^2-9) - 8(r-3) = 8(r+3)$

$5r^2 - 45 - 8r + 24 = 8r + 24$

$5r^2 - 8r - 21 = 8r + 24$

$\underline{-8r - 24 \;\; -8r - 24}$

$5r^2 - 16r - 46 = 0$

$(5r+9)(r-5) = 0$

$5r + 9 = 0 \quad r - 5 = 0$

$\underline{-9 \;\; -9 \quad +5 \;\; +5}$

$\frac{5r}{5} = -\frac{9}{5} \quad r = 5mph$

$\cancel{r = -\frac{9}{5}}$

19) By going 15 miles per hour faster, a train would have required 1 hour less to travel 180 miles. How fast did it travel?

r	t	d
r	t	180
$r+15$	$t-1$	180

$\frac{rt}{r} = \frac{180}{r} \; t = \frac{180}{r}$

$\frac{(r+15)(t-1)}{r+15} = \frac{180}{r+15}$

$t - 1 = \frac{180}{r+15}$

$\frac{180}{r}\big(r(r+15)\big) - 1\big(r(r+15)\big) = \frac{180}{r+15}\big(r(r+15)\big)$

$LCD: r(r+15)$

$180(r+15) - r(r+15) = 180r$

$180r + 2700 - r^2 - 15r = 180r$

$-r^2 + 165r + 2700 = 180r$

$\underline{+r^2 - 165 - 2700 \quad +r^2 - 165 - 2700}$

$0 = r^2 + 15r - 2700$

$0 = (r+60)(r-45)$

$r + 60 = 0 \quad r - 45 = 0$

$\underline{-60 \;\; -60 \quad +45 \;\; +45}$

$\cancel{r = -60} \quad r = 45mph$

21) If a train had traveled 5 miles an hour faster, it would have needed $1\frac{1}{2}$ hours less time to travel 150 miles. Find the rate of the train.

r	t	d
r	t	150
$r+5$	$t-\frac{3}{2}$	150

$\frac{rt}{r} = \frac{150}{r}\ t = \frac{150}{r}$

$\frac{(r+5)\left(t-\frac{3}{2}\right)}{r+5} = \frac{150}{r+5}$

$t - \frac{3}{2} = \frac{150}{r+5}$

$\frac{150}{r}\big(2r(r+5)\big) - \frac{3}{2}\big(2r(r+5)\big) = \frac{150}{r+5}\big(2r(r+5)\big)$

$LCD: \big(2r(r+5)\big)$

$300(r+5) - 3r(r+5) = 300r$

$300r + 1500 - 3r^2 - 15r = 300r$

$-3r^2 + 285r + 1500 = 300r$

$\underline{+3r^2 - 285 - 1500 \qquad + 3r^2 - 285r - 1500}$

$0 = \frac{3r^2}{3} + \frac{15r}{3} - \frac{1500}{3}$

$0 = (r+25)(r-20)$

$r + 25 = 0 \quad r - 20 = 0$

$\underline{-25\ -25 \quad +20\ +20}$

$\cancel{r = -25} \qquad r = 20mph$

9.11

1) $y = x^2 - 2x - 8$

$y - inter: (0,-8)$

$x - inter: 0 = x^2 - 2x - 8$

$0 = (x-4)(x+2)$

$x - 4 = 0 \quad x + 2 = 0$

$\underline{+4\ +4 \quad -2\ -2}$

$x = 4 \qquad x = -2$

$(4,0),\ (-2,0)$

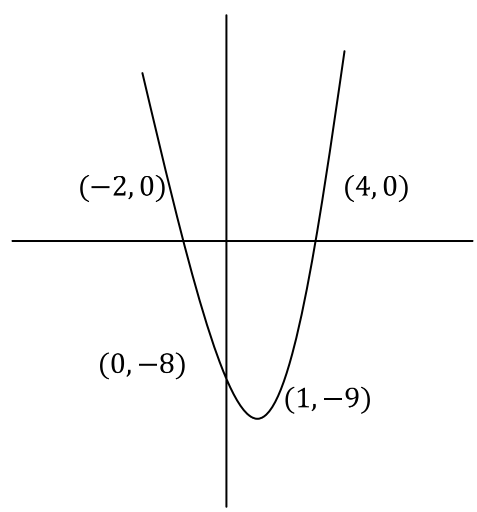
$vertex: x = \frac{2}{2(1)} = \frac{2}{2} = 1$

$y = (1)^2 - 2(1) - 8$

$y = 1 - 2 - 8$

$y = -9$

$(1,-9)$



3) $y = 2x^2 - 12x + 10$

$y - inter: (0, 10)$

$x - inter: 0 = 2x^2 - 12x + 10$

$0 = 2(x^2 - 6x + 5)$

$0 = 2(x - 5)(x - 1)$

$x - 5 = 0 \quad x - 1 = 0$

$\underline{\ \ +5 \quad +5 \quad +1 \quad +1\ \ }$

$x = 5 \qquad x = 1$

$(5, 0) \ \ (1, 0)$

$vertex: x = \frac{12}{2(2)} = \frac{12}{4} = 3$

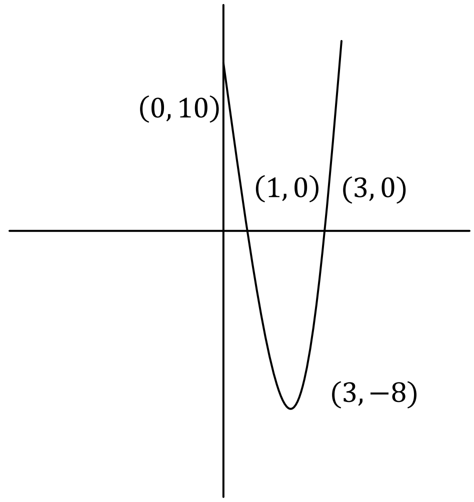
$y = 2(3)^2 - 12(3) + 10$

$y = 2(9) - 36 + 10$

$y = 18 - 36 + 10$

$y = -8$

$(3, -8)$



5) $y = -2x^2 + 12x - 18$

$y - inter: (0, 10)$

$x - inter: 0 = -2x^2 + 12x - 18$

$0 = -2(x^2 - 6x + 9)$

$0 = -2(x - 3)^2$

$x - 3 = 0$

$\overline{\ +3 \quad +3\ }$

$x = 3$

$(3, 0)$

$vertex: x = \frac{-12}{2(-2)} = \frac{-12}{-4} = 3$

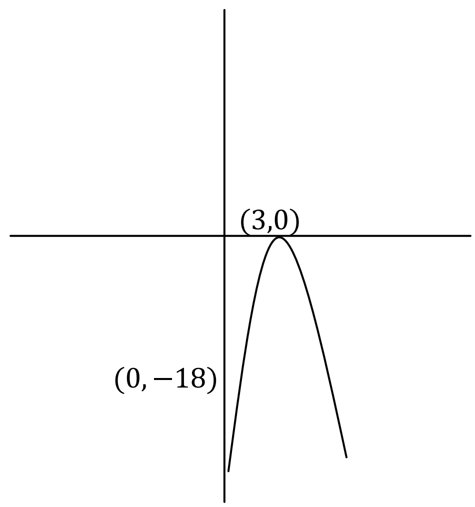
$y = -2(3)^2 + 12(3) - 18$

$y = -2(9) + 36 - 18$

$y = -18 + 36 - 18$

$y = 0$

$(3, 0)$



7) $y = -3x^2 + 24x - 45$

$y - inter: (0, -45)$

$x - inter: 0 = -3x^2 + 24x - 45$

$0 = -3(x^2 - 8x + 15)$

$0 = -3(x - 5)(x - 3)$

$x - 5 = 0 \quad x - 3 = 0$

$\underline{\;+5 \quad +5 \qquad +3 \quad +3\;}$

$x = 5 \qquad x = 3$

$(5, 0) \;\; (3,0)$

$vertex: x = \frac{-24}{2(-3)} = \frac{-24}{-6} = 4$

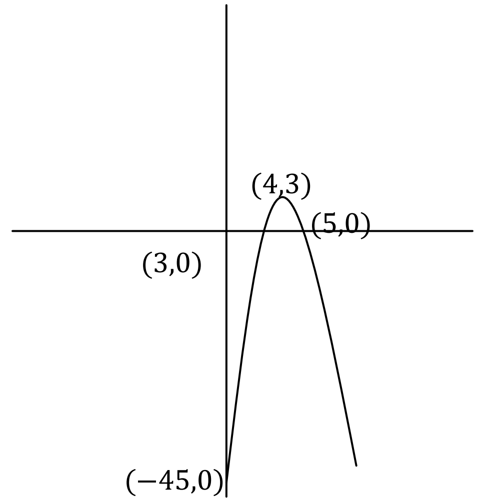
$y = -3(4)^2 + 24(4) - 45$

$y = -3(16) + 96 - 45$

$y = -48 + 96 - 45$

$y = 3$

$(4, 3)$



9) $y = -x^2 + 4x + 5$

$y - inter: (0, 9)$

$x - inter: 0 = -x^2 + 4x + 5$

$0 = -1(x^2 - 4x - 5)$

$0 = -1(x - 5)(x + 1)$

$x - 5 = 0 \quad x + 1 = 0$

$\underline{\;+5 \quad +5 \quad -1 \quad -1\;}$

$x = 5 \qquad x = -1$

$(5, 0) \qquad (-1, 0)$

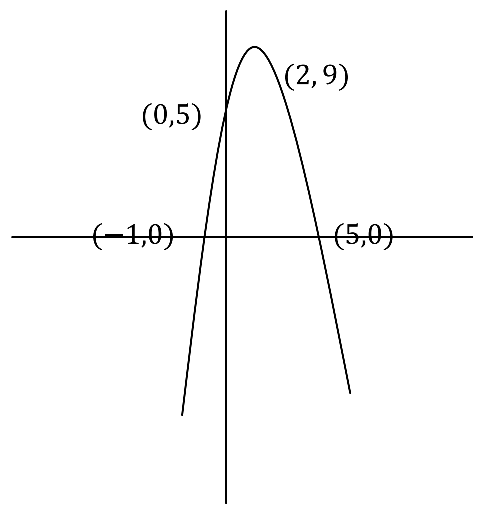
$vertex: x = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2$

$y = -(2)^2 + 4(2) + 5$

$y = -4 + 8 + 5$

$y = 9$

$(2, 9)$



11) $y = -x^2 + 6x - 5$

$y - inter: (0, -5)$

$x - inter: 0 = -x^2 + 6x - 5$

$0 = -1(x^2 - 6x + 5)$

$0 = -1(x - 1)(x - 5)$

$$\begin{array}{cc} x - 1 = 0 & x - 5 = 0 \\ \underline{+1 \quad +1} & \underline{+5 \quad +5} \\ x = 1 & x = 5 \\ (1, 0) & (5, 0) \end{array}$$

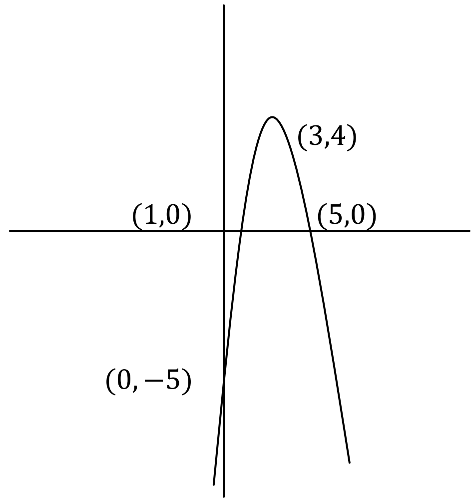
$vertex: x = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3$

$y = -(3)^2 + 6(3) - 5$

$y = -9 + 18 - 5$

$y = 4$

$(3, 4)$



13) $y = -2x^2 + 16x - 24$

$y - inter: (0, -24)$

$x - inter: 0 = -2x^2 + 16x - 24$

$0 = -2(x^2 - 8x + 12)$

$0 = -2(x - 2)(x - 6)$

$$\begin{array}{cc} x - 2 = 0 & x - 6 = 0 \\ \underline{+2 \quad +2} & \underline{+6 \quad +6} \\ x = 2 & x = 6 \\ (2, 0) & (6, 0) \end{array}$$

$vertex: x = \frac{-16}{2(-2)} = \frac{-16}{-4} = 4$

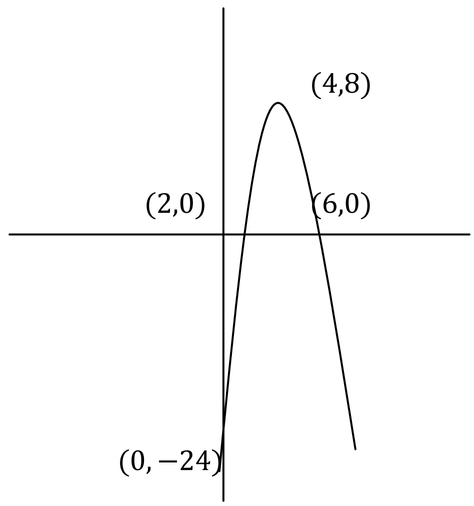
$y = -2(4)^2 + 16(4) - 24$

$y = -2(16) + 64 - 24$

$y = -32 + 64 - 24$

$y = 8$

$(4, 8)$



15) $y = 3x^2 + 12x + 9$

$y - inter: (0, 9)$

$x - inter: 0 = 3x^2 + 12x + 9$

$0 = 3(x^2 + 4x + 3)$

$0 = 3(x + 1)(x + 3)$

$$\begin{array}{cc} x + 1 = 0 & x + 3 = 0 \\ -1 \quad -1 & -3 \quad -3 \\ \hline x = -1 & x = -3 \end{array}$$

$(-1, 0) \qquad (-3, 0)$

$vertex: x = \frac{-12}{2(3)} = \frac{-12}{6} = -2$

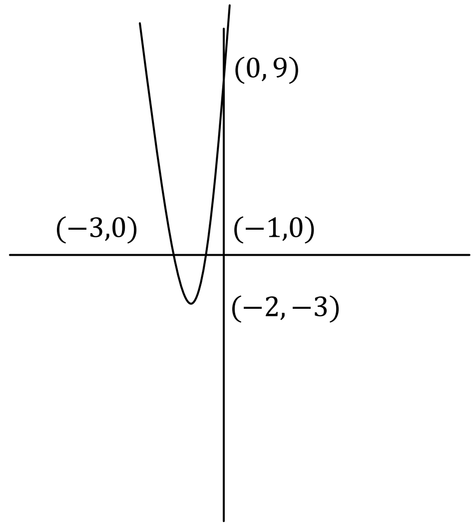
$y = 3(-2)^2 + 12(-2) + 9$

$y = 3(4) - 24 + 9$

$y = 12 - 24 + 9$

$y = -3$

$(-2, -3)$



17) $y = 5x^2 - 40x + 75$

$y - inter: (0, 75)$

$x - inter: 0 = 5x^2 - 40x + 75$

$0 = 5(x^2 - 8x + 15$

$0 = 5(x - 3)(x - 5)$

$$\begin{array}{cc} x - 3 = 0 & x - 5 = 0 \\ +3 \quad +3 & +5 \quad +5 \\ \hline x = 3 & x = 5 \end{array}$$

$vertex: \frac{40}{2(5)} = \frac{40}{10} = 4$

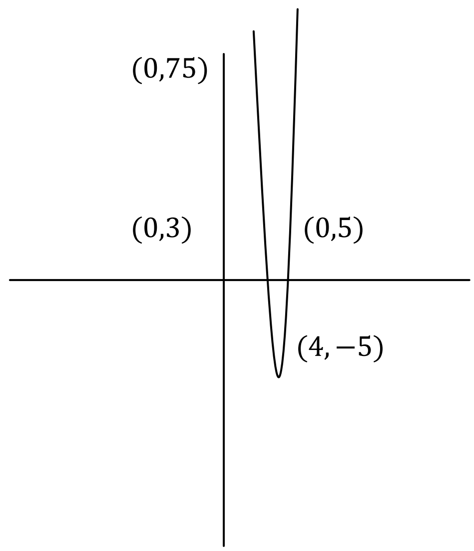
$y = 5(4)^2 - 40(4) + 75$

$y = 5(16) - 160 + 75$

$y = 80 - 160 + 75$

$y = -5$

$(4, -5)$



19) $y = -5x^2 - 60x - 175$

$y - inter: (0, -175)$

$x - inter: 0 = -5x^2 - 60x - 175$

$0 = -5(x^2 + 12x + 35)$

$0 = -5(x + 5)(x + 7)$

$x + 5 = 0 \quad x + 7 = 0$

$\underline{\quad -5 \quad -5 \quad -7 \quad -7 \quad}$

$x = -5 \quad x = -7$

$(-5, 0) \quad (-7, 0)$

$vertex: x = \frac{60}{2(-5)} = \frac{60}{-10} = -6$

$y = -5(-6)^2 - 60(-6) - 175$

$y = -5(36) + 360 - 175$

$y = -180 + 360 - 175$

$y = 5$

$(-6, 5)$