Precalculus

An Investigation of Functions
MTH 111

Edition 2.0

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This book is also available to read free online at
http://www.opentextbookstore.com/precalc/
If you want a printed copy, buying from the bookstore is cheaper than printing yourself.

How To Be Successful In This Course

This is not a high school math course, although for some of you the content may seem familiar. There are key differences to what you will learn here, how quickly you will be required to learn it and how much work will be required of you.

You will no longer be shown a technique and be asked to mimic it repetitively as the only way to prove learning. Not only will you be required to master the technique, but you will also be required to extend that knowledge to new situations and build bridges between the material at hand and the next topic, making the course highly cumulative.

As a rule of thumb, for each hour you spend in class, you should expect this course will require an average of 2 hours of out-of-class focused study. This means that some of you with a stronger background in mathematics may take less, but if you have a weaker background or any math anxiety it will take you more.

Notice how this is the equivalent of having a part time job, and if you are taking a fulltime load of courses as many college students do, this equates to more than a full time job. If you must work, raise a family and take a full load of courses all at the same time, we recommend that you get a head start & get organized as soon as possible. We also recommend that you spread out your learning into daily chunks and avoid trying to cram or learn material quickly before an exam.

To be prepared, read through the material before it is covered in class and note or highlight the material that is new or confusing. The instructor's lecture and activities should not be the first exposure to the material. As you read, test your understanding with the Try it Now problems in the book. If you can't figure one out, try again after class, and ask for help if you still can't get it.

As soon as possible after the class session recap the day's lecture or activities into a meaningful format to provide a third exposure to the material. You could summarize your notes into a list of key points, or reread your notes and try to work examples done in class without referring back to your notes. Next, begin any assigned homework. The next day, if the instructor provides the opportunity to clarify topics or ask questions, do not be afraid to ask. If you are afraid to ask, then you are not getting your money's worth! If the instructor does not provide this opportunity, be prepared to go to a tutoring center or build a peer study group. Put in quality effort and time and you can get quality results.

Lastly, if you feel like you do not understand a topic. Don't wait, ASK FOR HELP!

ASK: **A**sk a teacher or tutor, **S**earch for ancillaries, **K**eep a detailed list of questions
FOR: **F**ind additional resources, **O**rganize the material, **R**esearch other learning options
HELP: **H**ave a support network, **E**xamine your weaknesses, **L**ist specific examples & **P**ractice

Best of luck learning! We hope you like the course & love the price.
David & Melonie

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Chapter 1: Functions

Section 1.1 Functions and Function Notation

What is a Function?
The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function
Function: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18^{th} birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

One-to-One Function

Sometimes in a relationship each input corresponds to exactly one output, and every output corresponds to exactly one input. We call this kind of relationship a **one-to-one function**.

From Example 3, *if* each unique percentage corresponds to one unique decimal grade point and each unique decimal grade point corresponds to one unique percentage then it is a one-to-one function.

Try it Now

Let's consider bank account information.

1. Is your balance a function of your bank account number?
   (if you input a bank account number does it make sense that the output is your balance?)

2. Is your bank account number a function of your balance?
   (if you input a balance does it make sense that the output is your bank account number?)

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable *h* to represent height and we could use the descriptive variable *a* to represent age.

"height is a function of age"	if we name the function *f* we write
"*h* is *f* of *a*"	or more simply
$h = f(a)$	we could instead name the function *h* and write
$h(a)$	which is read "*h* of *a*"

Remember we can use any variable to name the function; the notation $h(a)$ shows us that *h* depends on *a*. The value "*a*" must be put into the function "*h*" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Function Notation

The notation output = *f*(input) defines a function named *f*. This would be read "output is *f* of input"

Example 4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function *f*, we could write "days = *f*(month)" or $d = f(m)$. If we simply name the function *d*, we could write $d(m)$

For example, *d*(March) = 31, since March has 31 days. The notation $d(m)$ reminds us that the number of days, *d* (the output) is dependent on the name of the month, *m* (the input)

Example 5

A function $N = f(y)$ gives the number of police officers, *N*, in a town in year *y*. What does $f(2005) = 300$ tell us?

When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (*y*). The output value is 300, the number of police officers (*N*), a value for the output quantity. Remember $N=f(y)$. This tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.
In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, m	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, D	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function $Q = g(n)$. Remember this notation tells us g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) a, age in years	5	5	6	7	8	9	10
(output) h, height inches	40	42	44	47	50	52	54

Example 6

Which of these tables define a function (if any)? Are any of them one-to-one?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

Only the first table is one-to-one; it is both a function, and each output corresponds to exactly one input. Although table 2 is a function, because each input corresponds to exactly one output, each output does not correspond to exactly one input so this function is not one-to-one. Table 3 is not even a function and so we don't even need to consider if it is a one-to-one function.

Try it Now

3. If each percentage earned translated to one letter grade, would this be a function? Is it one-to-one?

Solving and Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 7

Using the table shown, where $Q=g(n)$

a) Evaluate $g(3)$

n	1	2	3	4	5
Q	8	6	7	6	8

Evaluating $g(3)$ (read: "g of 3")
means that we need to determine the output value, Q, of the function g given the input value of n=3. Looking at the table, we see the output corresponding to n=3 is Q=7, allowing us to conclude $g(3) = 7$.

b) Solve $g(n) = 6$

Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we input 2 into the function g, our output is $Q = 6$

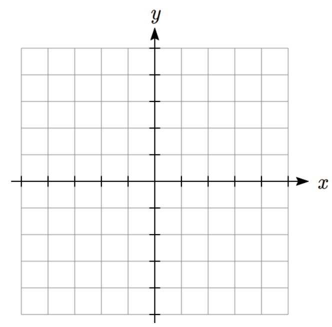
When we input 4 into the function g, our output is also $Q = 6$

Try it Now

4. Using the function in Example 7, evaluate $g(4)$

Graphs as Functions

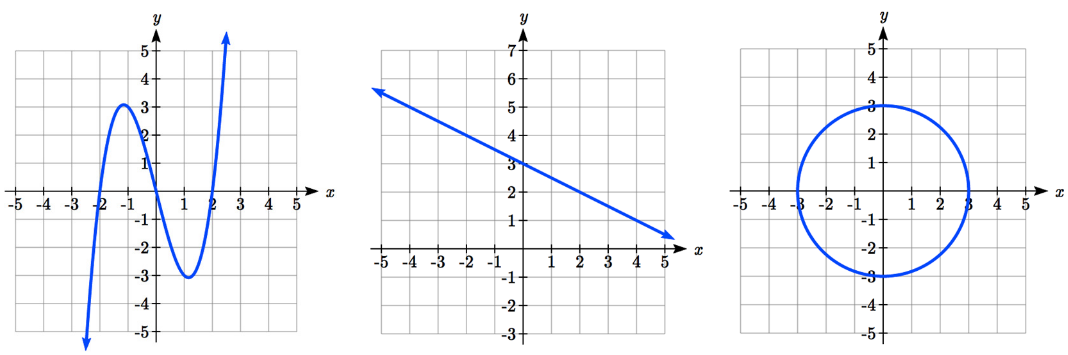
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.



The most common graph has y on the vertical axis and x on the horizontal axis, and we say y is a function of x, or $y = f(x)$ when the function is named f.

Example 8

Which of these graphs defines a function $y=f(x)$? Which of these graphs defines a one-to-one function?



Looking at the three graphs above, the first two define a function $y=f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the y-value of the graph. The 3rd graph does not define a function $y=f(x)$ since some input values, such as x=2, correspond with more than one output value.

Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -1 and 2.3

Graph 2 is a one-to-one function; each input corresponds to exactly one output, and every output corresponds to exactly one input.

Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

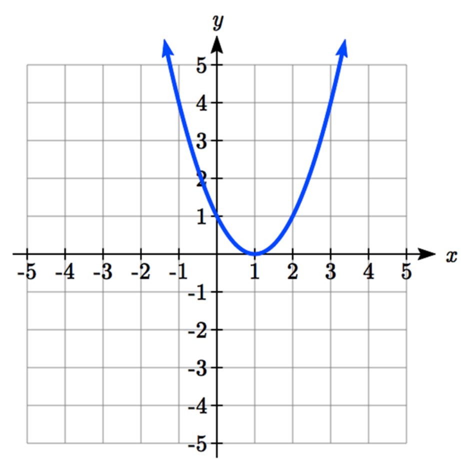
Horizontal Line Test

Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 9

Given the graph of $f(x)$
a) Evaluate $f(2)$
b) Solve $f(x) = 4$



a) To evaluate $f(2)$, we find the input of x=2 on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of y=1. $f(2) = 1$.

b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$
This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4 shows us that this function is not one-to-one.

Try it Now

5. Using the graph from example 9, solve $f(x)=1$.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as p = [something involving n].

$2n + 6p = 12$ subtract $2n$ from both sides
$6p = 12 - 2n$ divide both sides by 6 and simplify

$$p = \frac{12-2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as p=, we can now express p as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 11

Express the relationship $x^2 + y^2 = 1$ as a function $y = f(x)$ if possible.

If we try to solve for y in this equation:

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 12

Given the function $k(t) = t^3 + 2$

a) Evaluate $k(2)$

b) Solve $k(t) = 1$

a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable t, then simplify

$k(2) = 2^3 + 2$

$k(2) = 8 + 2$

So $k(2) = 10$

b) To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output

$k(t) = 1$	substitute the original formula $k(t) = t^3 + 2$
$t^3 + 2 = 1$	subtract 2 from each side
$t^3 = -1$	take the cube root of each side
$t = -1$	

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t) = 1$ is true when $t = -1$.

$k(-1) = (-1)^3 + 2$

$= -1 + 2$

$= 1$ which was the desired result.

Example 13

Given the function $h(p) = p^2 + 2p$

a) Evaluate $h(4)$

b) Solve $h(p) = 3$

To evaluate $h(4)$ we substitute the value 4 for the input variable p in the given function.

a) $h(4) = (4)^2 + 2(4)$

$= 16 + 8$

$= 24$

b) $h(p) = 3$ Substitute the original function $h(p) = p^2 + 2p$

$p^2 + 2p = 3$ This is quadratic, so we can rearrange the equation to get it = 0

$p^2 + 2p - 3 = 0$ subtract 3 from each side

$p^2 + 2p - 3 = 0$ this is factorable, so we factor it

$(p+3)(p-1) = 0$

By the zero factor theorem since $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0) and so we solve both equations for p, finding p = -3 from the first equation and p = 1 from the second equation.

This gives us the solution: $h(p) = 3$ when $p = 1$ or $p = -3$

We found two solutions in this case, which tells us this function is not one-to-one.

Try it Now

6. Given the function $g(m) = \sqrt{m-4}$
   a. Evaluate $g(5)$
   b. Solve $g(m) = 2$

Basic Toolkit Functions

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our "toolkit of functions" – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use x as the input variable and $f(x)$ as the output variable.

Toolkit Functions	
Linear	
Constant:	$f(x) = c$, where c is a constant (number)
Identity:	$f(x) = x$
Absolute Value:	$f(x) = \lvert x \rvert$
Power	
Quadratic:	$f(x) = x^2$
Cubic:	$f(x) = x^3$
Reciprocal:	$f(x) = \dfrac{1}{x}$
Reciprocal squared:	$f(x) = \dfrac{1}{x^2}$
Square root:	$f(x) = \sqrt[2]{x} = \sqrt{x}$
Cube root:	$f(x) = \sqrt[3]{x}$

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

Not every important equation can be written as $y = f(x)$. An example of this is the equation of a circle. Recall the distance formula for the distance between two points:

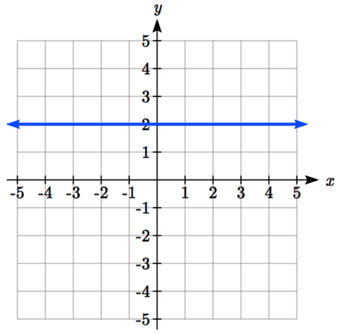
$$dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

A circle with radius r with center at (h, k) can be described as all points (x, y) a distance of r from the center, so using the distance formula, $r = \sqrt{(x-h)^2 + (y-k)^2}$, giving

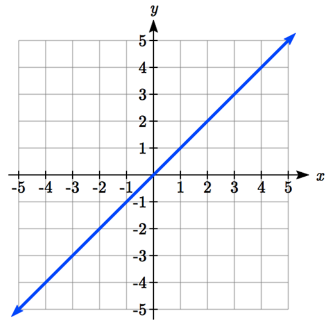
Equation of a circle
A circle with radius r with center (h, k) has equation $r^2 = (x-h)^2 + (y-k)^2$

Graphs of the Toolkit Functions

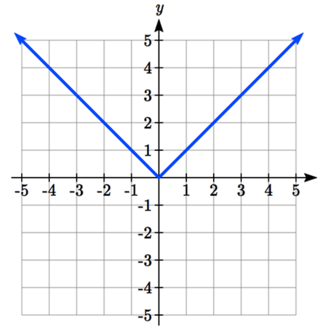
Constant Function: $f(x) = 2$



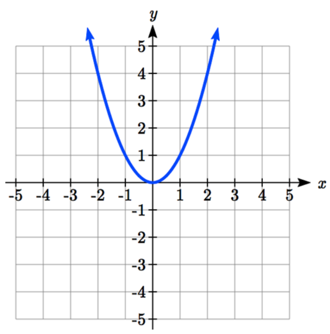
Identity: $f(x) = x$



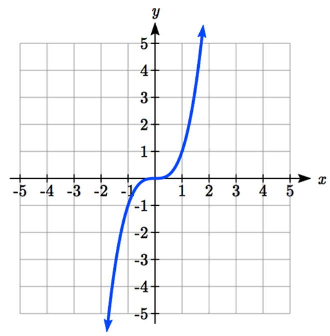
Absolute Value: $f(x) = |x|$



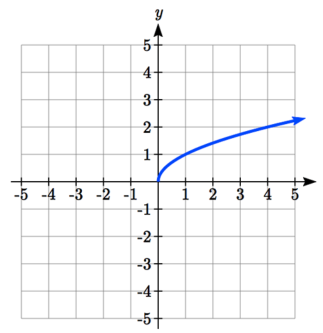
Quadratic: $f(x) = x^2$



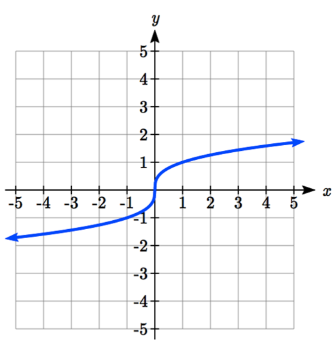
Cubic: $f(x) = x^3$



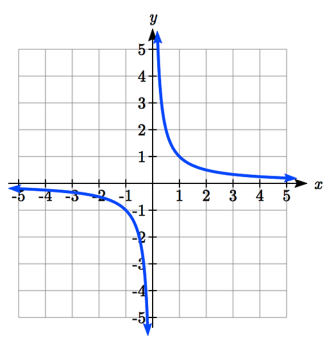
Square root: $f(x) = \sqrt{x}$



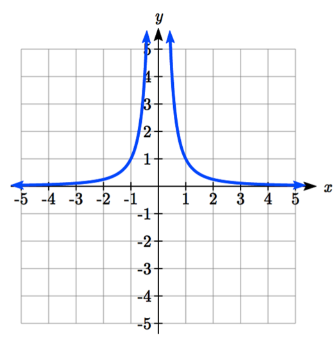
Cube root: $f(x) = \sqrt[3]{x}$



Reciprocal: $f(x) = \dfrac{1}{x}$



Reciprocal squared: $f(x) = \dfrac{1}{x^2}$



Important Topics of this Section
Definition of a function
Input (independent variable)
Output (dependent variable)
Definition of a one-to-one function
Function notation
Descriptive variables
Functions in words, tables, graphs & formulas
Vertical line test
Horizontal line test
Evaluating a function at a specific input value
Solving a function given a specific output value
Toolkit Functions

Try it Now Answers

1. Yes: for each bank account, there would be one balance associated

2. No: there could be several bank accounts with the same balance

3. Yes it's a function; No, it's not one-to-one (several percents give the same letter grade)

4. When n=4, Q=g(4)=6

5. There are two points where the output is 1: $x = 0$ or $x = 2$

6. a. $g(5) = \sqrt{5-4} = 1$

   b. $\sqrt{m-4} = 2$. Square both sides to get $m-4 = 4$. $m = 8$

Section 1.1 Exercises

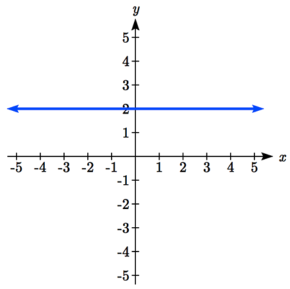
1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
   a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
   b. Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
   a. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.
   b. Explain the meaning of the statement $g(100) = 1$.

3. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
   a. $f(5) = 30$ b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
   a. $h(1) = 200$ b. $h(2) = 350$

5. Select all of the following graphs which represent y as a function of x.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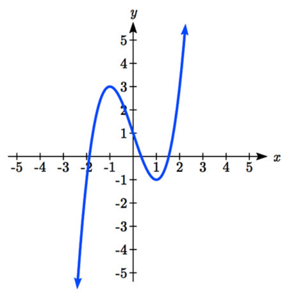
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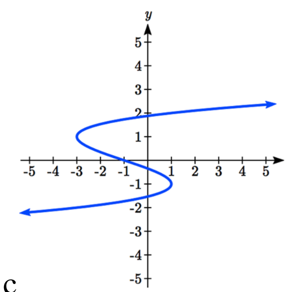
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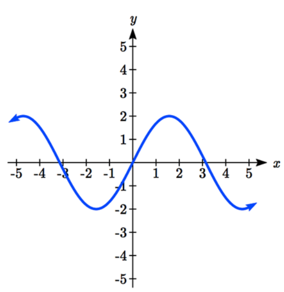
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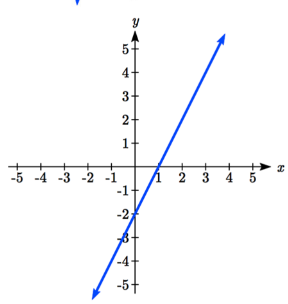
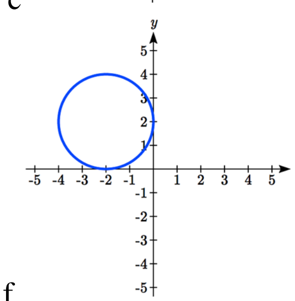
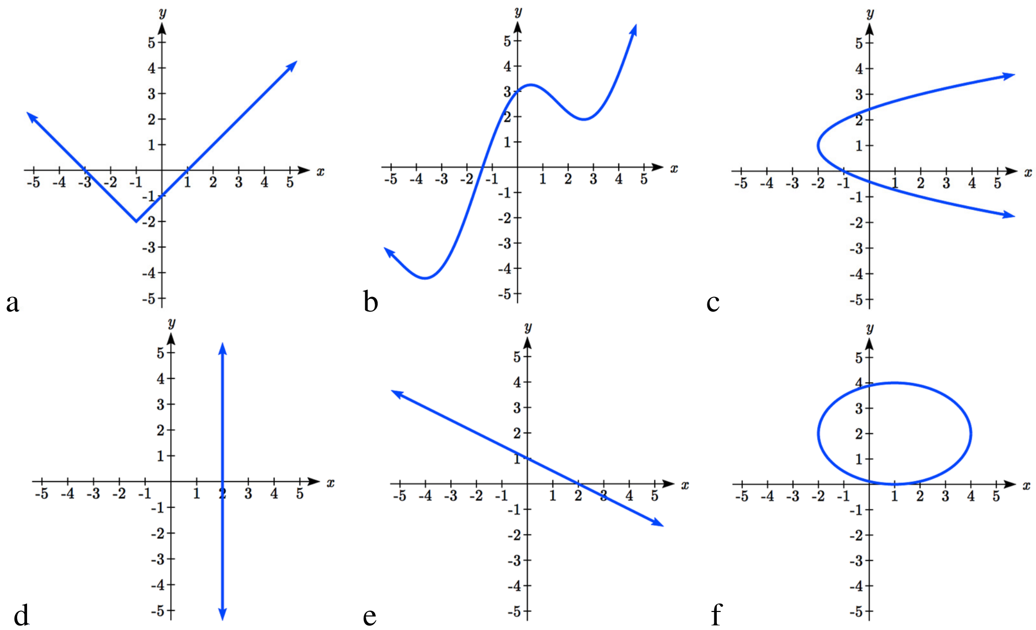

e


f

6. Select all of the following graphs which represent y as a function of x.



7. Select all of the following tables which represent y as a function of x.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent y as a function of x.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent y as a function of x.

a.

x	***y***
0	-2
3	1
4	6
8	9
3	1

b.

x	***y***
-1	-4
2	3
5	4
8	7
12	11

c.

x	***y***
0	-5
3	1
3	4
9	8
16	13

d.

x	***y***
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent y as a function of x.

a.

x	***y***
-4	-2
3	2
6	4
9	7
12	16

b.

x	***y***
-5	-3
2	1
2	4
7	9
11	10

c.

x	***y***
-1	-3
1	2
5	4
9	8
1	2

d.

x	***y***
-1	-5
3	1
5	1
8	7
14	12

11. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	3	8	12
y	4	7	7

b.

x	3	8	12
y	4	7	13

c.

x	3	8	8
y	4	7	13

12. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	2	8	8
y	5	6	13

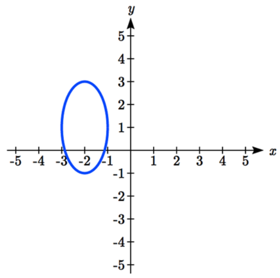
b.

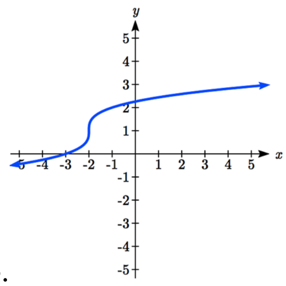
x	2	8	14
y	5	6	6

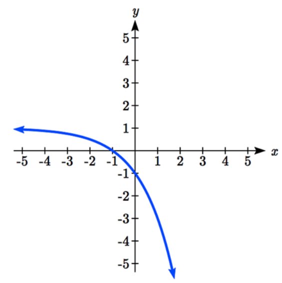
c.

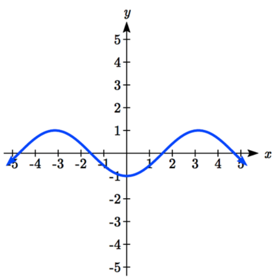
x	2	8	14
y	5	6	13

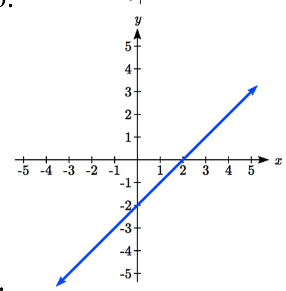
13. Select all of the following graphs which are **one-to-one functions**.

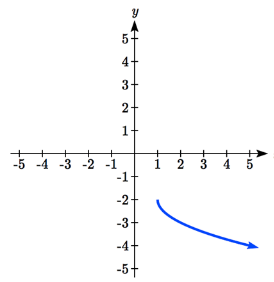

a.


b.

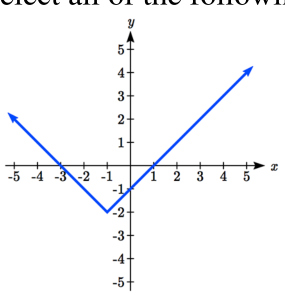

c.

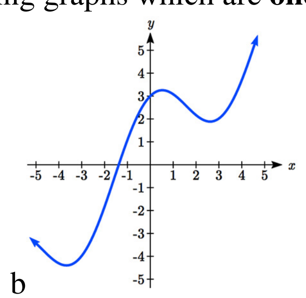

d.

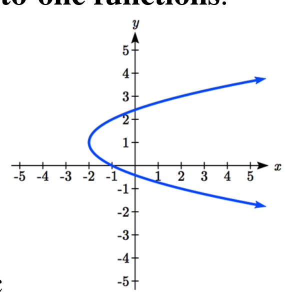

e.

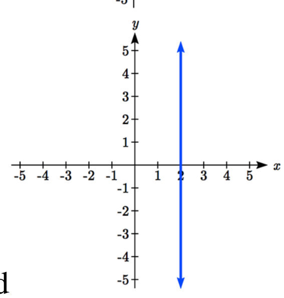

f.

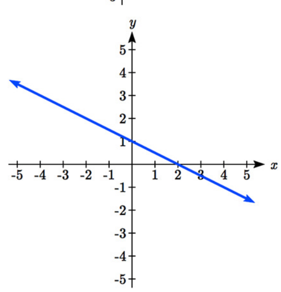
14. Select all of the following graphs which are **one-to-one functions**.

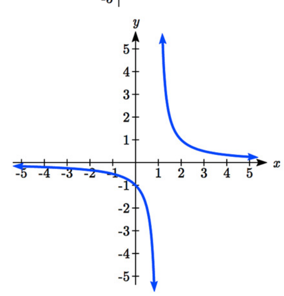

a


b

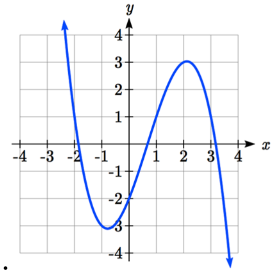

c

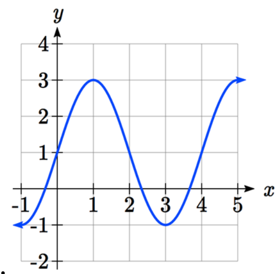

d


e


f

Given each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

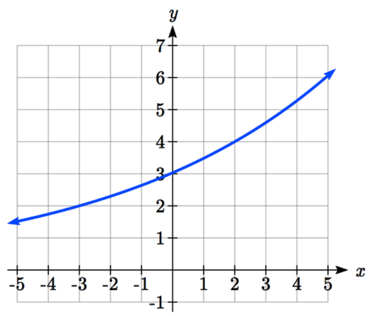
15. 

16. 

17. Given the function $g(x)$ graphed here,

a. Evaluate $g(2)$

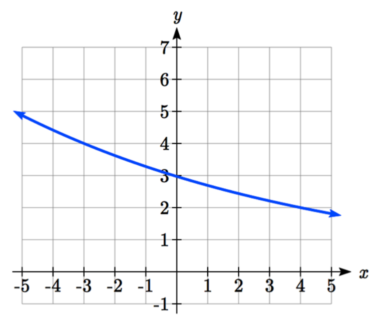
b. Solve $g(x)=2$



18. Given the function $f(x)$ graphed here.

a. Evaluate $f(4)$

b. Solve $f(x)=4$



19. Based on the table below,

a. Evaluate $f(3)$ b. Solve $f(x)=1$

$\boldsymbol{x}$	0	1	2	3	4	5	6	7	8	9
$\boldsymbol{f(x)}$	74	28	1	53	56	3	36	45	14	47

20. Based on the table below,

a. Evaluate $f(8)$ b. Solve $f(x)=7$

$\boldsymbol{x}$	0	1	2	3	4	5	6	7	8	9
$\boldsymbol{f(x)}$	62	8	7	38	86	73	70	39	75	34

For each of the following functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

21. $f(x)=4-2x$

22. $f(x)=8-3x$

23. $f(x)=8x^2-7x+3$

24. $f(x)=6x^2-7x+4$

25. $f(x)=-x^3+2x$

26. $f(x)=5x^4+x^2$

27. $f(x)=3+\sqrt{x+3}$

28. $f(x)=4-\sqrt[3]{x-2}$

29. $f(x)=(x-2)(x+3)$

30. $f(x)=(x+3)(x-1)^2$

31. $f(x)=\dfrac{x-3}{x+1}$

32. $f(x)=\dfrac{x-2}{x+2}$

33. $f(x)=2^x$

34. $f(x)=3^x$

35. Suppose $f\left(x\right)=x^2+8x-4$. Compute the following:

a. $f(-1)+f(1)$ b. $f(-1)-f(1)$

36. Suppose $f\left(x\right)=x^2+x+3$. Compute the following:

a. $f(-2)+f(4)$ b. $f(-2)-f(4)$

37. Let $f\left(t\right)=3t+5$

a. Evaluate $f(0)$ b. Solve $f\left(t\right)=0$

38. Let $g\left(p\right)=6-2p$

a. Evaluate $g(0)$ b. Solve $g\left(p\right)=0$

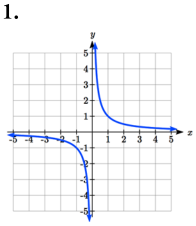
39. Match each function name with its equation.

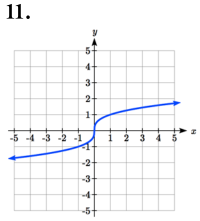
a. $y=x$
b. $y=x^3$
c. $y=\sqrt[3]{x}$
d. $y=\dfrac{1}{x}$
e. $y=x^2$
f. $y=\sqrt{x}$
g. $y=\left|x\right|$
h. $y=\dfrac{1}{x^2}$

i. Cube root
ii. Reciprocal
iii. Linear
iv. Square Root
v. Absolute Value
vi. Quadratic
vii. Reciprocal Squared
viii. Cubic

40. Match each graph with its equation.

a. $y=x$
b. $y=x^3$
c. $y=\sqrt[3]{x}$
d. $y=\dfrac{1}{x}$
e. $y=x^2$
f. $y=\sqrt{x}$
g. $y=\left|x\right|$
h. $y=\dfrac{1}{x^2}$

i.


ii.


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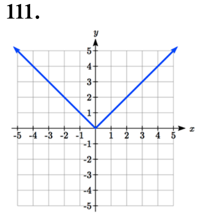

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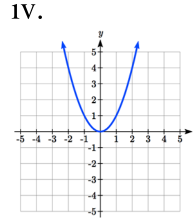
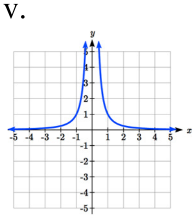


v.


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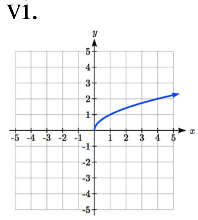

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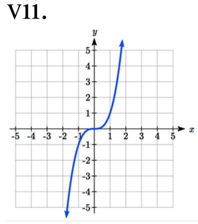
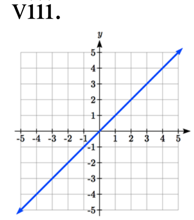


viii.


41. Match each table with its equation.

a. $y = x^2$
b. $y = x$
c. $y = \sqrt{x}$
d. $y = 1/x$
e. $y = |x|$
f. $y = x^3$

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

42. Match each equation with its table

a. Quadratic
b. Absolute Value
c. Square Root
d. Linear
e. Cubic
f. Reciprocal

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

43. Write the equation of the circle centered at $(3, -9)$ with radius 6.

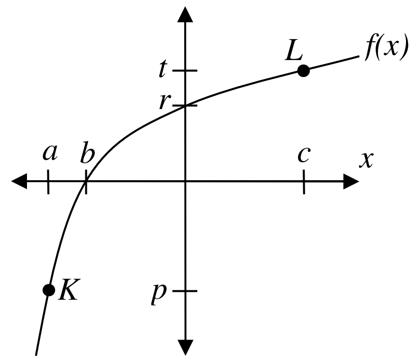
44. Write the equation of the circle centered at $(9, -8)$ with radius 11.

45. Sketch a reasonable graph for each of the following functions. [UW]

a. Height of a person depending on age.
b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
c. The amount of postage you must put on a first class letter, depending on the weight of the letter.

46. Sketch a reasonable graph for each of the following functions. [UW]
    a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
    b. Your height above the water level in a swimming pool after you dive off the high board.
    c. The percentage of dates and names you'll remember for a history test, depending on the time you study.

47. Using the graph shown,
    a. Evaluate $f(c)$
    b. Solve $f(x) = p$
    c. Suppose $f(b) = z$. Find $f(z)$
    d. What are the coordinates of points *L* and *K*?



48. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]

    a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.

    b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

    c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of “distance from Padelford” vs. “time” for the both Angela and Dave.

f. Suppose you want to sketch the graph of a new function $s = g(t)$ that keeps track of Dave’s distance s from Gould Hall at time t. How would your graphs change in (a)-(e)?

Section 1.2 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, *c*	1.7	2.5	5.5	8.2	13.7
Height, *h*	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

Domain: The set of possible input values to a function

Range: The set of possible output values of a function

Example 1

Using the tree table above, determine a reasonable domain and range.

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet[1]. With this information, we would say a reasonable domain is $0 < c \le 119$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \le 379$ feet.

[1] http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010

Example 2

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter[2], as shown in the table below. Determine the domain and range.

Letters	
Weight not Over	Price
1 ounce	$0.44
2 ounces	$0.61
3 ounces	$0.78
3.5 ounces	$0.95

Suppose we notate Weight by w and Price by p, and set up a function named P, where Price, p is a function of Weight, w. $p = P(w)$.

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be $0 < w \leq 3.5$. Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is p = $0.44, $0.61, $0.78, or $0.95.

Try it Now

1. The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$, $0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, p = $0.44, $0.61, $0.78, or $0.95. These numbers represent a set of specific values: {0.44, 0.61, 0.78, 0.95}

[2] http://www.usps.com/prices/first-class-mail-prices.htm, retrieved July 19, 2010

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable *x* that are 10 or greater, but less than 30. In inequalities, we would write $10 \le x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x \mid 10 \le x < 30\}$. The curly brackets { } are read as "the set of", and the vertical bar | is read as "such that", so altogether we would read $\{x \mid 10 \le x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to *x* and *x* is less than 30."

When describing ranges in set-builder notation, we could similarly write something like $\{f(x) \mid 0 < f(x) < 100\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h \mid 0 < h \le 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t \mid t \text{ is a real number}\}$, or $\{t \mid t \in \mathbb{R}\}$, read as "the set of *t*-values such that *t* is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than," and square brackets are used for "less than or equal to." Since infinity is not a number, we can't include it in the interval, so we always use curved parentheses with ∞ and -∞. The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	**Set Builder Notation**	**Interval notation**
$5 < h \le 10$	$\{h \mid 5 < h \le 10\}$	(5, 10]
$5 \le h < 10$	$\{h \mid 5 \le h < 10\}$	[5, 10)
$5 < h < 10$	$\{h \mid 5 < h < 10\}$	(5, 10)
$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
$h \ge 10$	$\{h \mid h \ge 10\}$	$[10, \infty)$
all real numbers	$\{h \mid h \in \mathbb{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Example 3

Describe the intervals of values shown on the line graph below using set builder and interval notations.

To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is: $1 \le x \le 3$ or $x > 5$

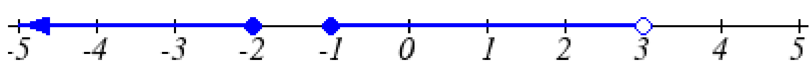
In set builder notation: $\{x \mid 1 \le x \le 3 \text{ or } x > 5\}$

In interval notation: $[1,3] \cup (5,\infty)$

Remember when writing or reading interval notation:
Using a square bracket [means the start value is included in the set
Using a parenthesis (means the start value is not included in the set

Try it Now

2. Given the following interval, write its meaning in words, set builder notation, and interval notation.



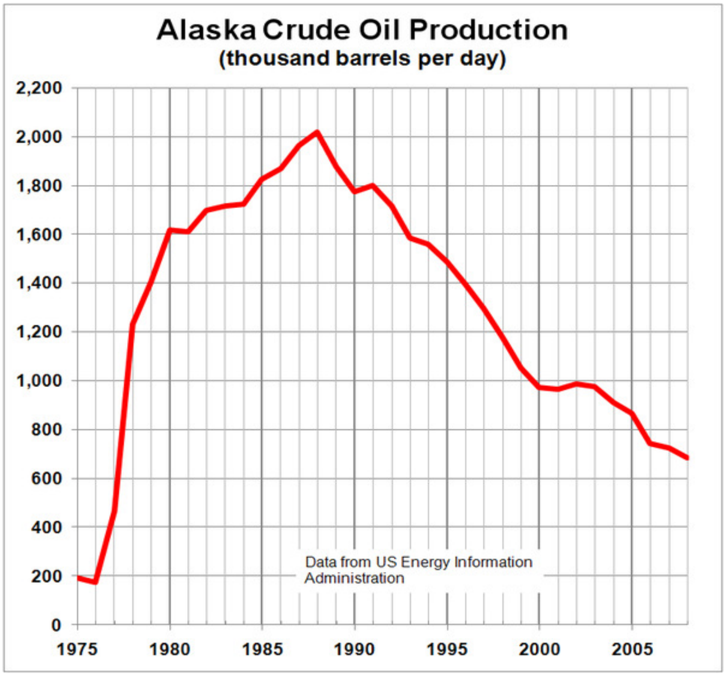
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 4

Determine the domain and range of the graph below.



In the graph above[3], the input quantity along the horizontal axis appears to be "year", which we could notate with the variable *y*. The output is "thousands of barrels of oil per day", which we might notate with the variable *b*, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \le y \le 2008$, and the range is approximately $180 \le b \le 2010$.

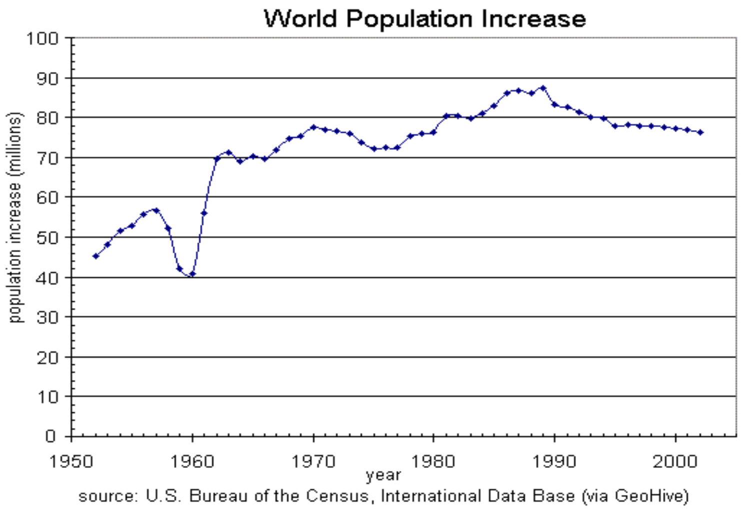
In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

Remember that, as in the previous example, *x* and *y* are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

[3] http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG, CC-BY-SA, July 19, 2010

Try it Now

3. Given the graph below write the domain and range in interval notation



Domains and Ranges of the Toolkit functions

We will now return to our set of toolkit functions to note the domain and range of each.

<u>Constant Function</u>: $f(x) = c$

The domain here is not restricted; x can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.

Domain: $(-\infty, \infty)$

Range: $[c]$

Since there is only one output value, we list it by itself in square brackets.

<u>Identity Function</u>: $f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

<u>Quadratic Function</u>: $f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Multiplying a negative or positive number by itself can only yield a positive output.

<u>Cubic Function</u>: $f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Reciprocal: $f(x)=\dfrac{1}{x}$

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,0)\cup(0,\infty)$

We cannot divide by 0 so we must exclude 0 from the domain.
One divide by any value can never be 0, so the range will not include 0.

Reciprocal squared: $f(x)=\dfrac{1}{x^2}$

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(0,\infty)$

We cannot divide by 0 so we must exclude 0 from the domain.

Cube Root: $f(x)=\sqrt[3]{x}$

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

Square Root: $f(x)=\sqrt[2]{x}$, commonly just written as, $f(x)=\sqrt{x}$

Domain: $[0,\infty)$

Range: $[0,\infty)$

When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.

Absolute Value Function: $f(x)=|x|$

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

Since absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

Example 5

Find the domain of each function: a) $f(x)=2\sqrt{x+4}$ b) $g(x)=\dfrac{3}{6-3x}$

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$x+4\ge 0$ when $x\ge -4$.

The domain of *f(x)* is $[-4,\infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.

$6-3x=0$ when $x=2$, so we must exclude 2 from the domain.

The domain of *g(x)* is $(-\infty,2)\cup(2,\infty)$.

Piecewise Functions

In the toolkit functions we introduced the absolute value function $f(x) = |x|$.
With a domain of all real numbers and a range of values greater than or equal to 0, the absolute value can be defined as the magnitude or modulus of a number, a real number value regardless of sign, the size of the number, or the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value the output is unchanged
$f(x) = x \quad \text{if} \quad x \geq 0$

If we input a negative value the sign must change from negative to positive.
$f(x) = -x \quad \text{if} \quad x < 0$, since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1} & \text{if} \quad \text{domain to use formula 1} \\ \text{formula 2} & \text{if} \quad \text{domain to use formula 2} \\ \text{formula 3} & \text{if} \quad \text{domain to use formula 3} \end{cases}$$

Example 6

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, *n*, to the cost, *C*.

To set up this function, two different formulas would be needed. $C = 5n$ would work for *n* values under 10, and $C = 50$ would work for values of *n* ten or greater. Notating this:

$$C(n) = \begin{cases} 5n & if \quad 0 < n < 10 \\ 50 & if \quad n \geq 10 \end{cases}$$

Example 7

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g)=\begin{cases} 25 & if \quad 0<g<2 \\ 25+10(g-2) & if \quad g\geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

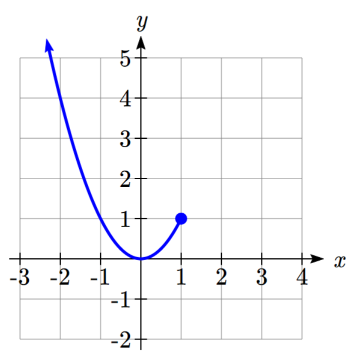
To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5)$ = \$25.

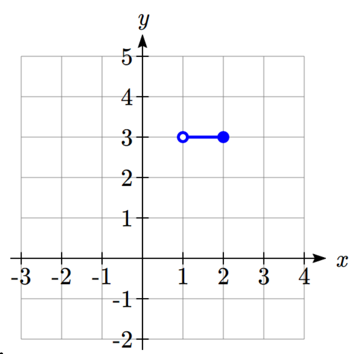
The find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4) = 25 + 10(4-2)$ = \$45.

Example 8

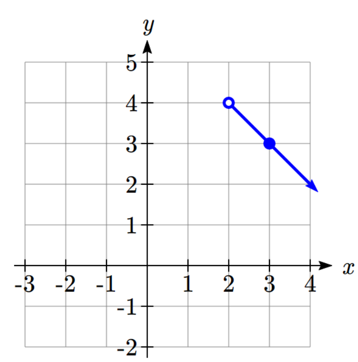
Sketch a graph of the function $f(x)=\begin{cases} x^2 & if \quad x\leq 1 \\ 3 & if \quad 1<x\leq 2 \\ 6-x & if \quad x>2 \end{cases}$

The first two component functions are from our library of Toolkit functions, so we know their shapes. We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality.



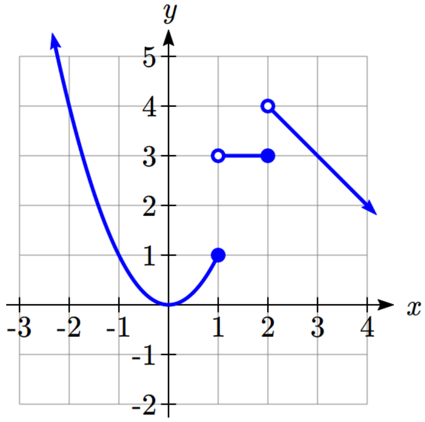


For the third function, you should recognize this as a linear equation from your previous coursework. If you remember how to graph a line using slope and intercept, you can do that. Otherwise, we could calculate a couple values, plot points, and connect them with a line.



At $x = 2$, $f(2) = 6 - 2 = 4$. We place an open circle here.
At $x = 3$, $f(3) = 6 - 3 = 3$. Connect these points with a line.

Now that we have each piece individually, we combine them onto the same graph:



Try it Now

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit[4]. Write a piecewise defined function for the total tuition, *T*, at Pierce College during 2009-2010 as a function of the number of credits taken, *c*. Be sure to consider a reasonable domain and range.

Important Topics of this Section
Definition of domain
Definition of range
Inequalities
Interval notation
Set builder notation
Domain and Range from graphs
Domain and Range of toolkit functions
Piecewise defined functions

[4] https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf, retrieved August 6, 2010

Try it Now Answers

1. Domain; y = years $[1960,2010]$; Range, p = population, $[100,1400]$

2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3
   b. $\{x \mid x \le -2 \ or \ -1 \le x < 3\}$
   c. $(-\infty,-2] \cup [-1,3)$

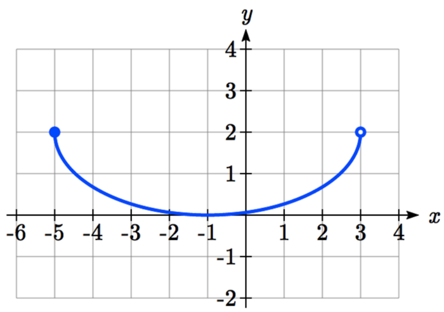
3. Domain; y=years, $[1952,2002]$; Range, p=population in millions, $[40,88]$

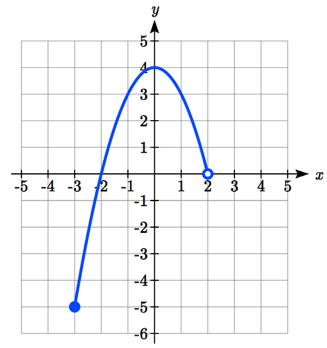
4. $T(c) = \begin{cases} 89.5c & if \quad c \le 10 \\ 895 + 33(c-10) & if \quad 10 < c \le 18 \\ 1159 + 73(c-18) & if \quad c > 18 \end{cases}$ Tuition, T, as a function of credits, c.

   Reasonable domain should be whole numbers 0 to (answers may vary), e.g. $[0,23]$
   Reasonable range should be \$0 – (answers may vary), e.g. $[0,1524]$

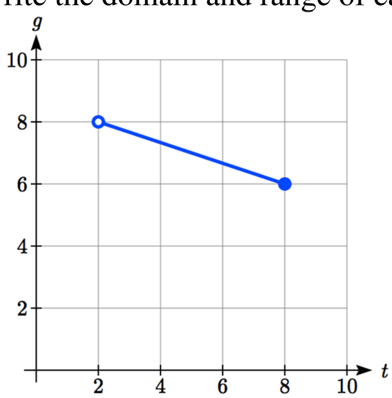
Section 1.2 Exercises

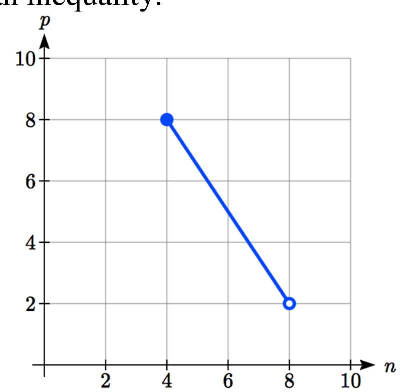
Write the domain and range of the function using interval notation.

1. 

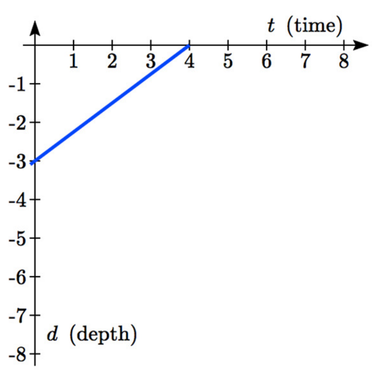
2. 

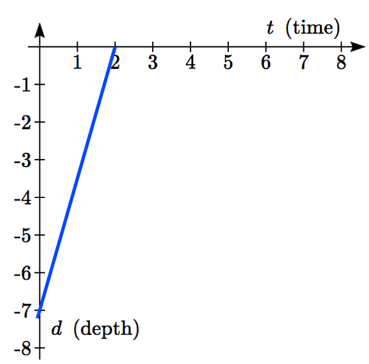
Write the domain and range of each graph as an inequality.

3. 

4. 

Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time, stopping once the sub surfaces. What is the domain and range of the function in the graph?

5. 

6. 

Find the domain of each function

7. $f(x) = 3\sqrt{x-2}$

8. $f(x) = 5\sqrt{x+3}$

9. $f(x) = 3 - \sqrt{6-2x}$

10. $f(x) = 5 - \sqrt{10-2x}$

11. $f(x) = \dfrac{9}{x-6}$

12. $f(x) = \dfrac{6}{x-8}$

13. $f(x) = \dfrac{3x+1}{4x+2}$

14. $f(x) = \dfrac{5x+3}{4x-1}$

15. $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

16. $f(x) = \dfrac{\sqrt{x+5}}{x-6}$

17. $f(x) = \dfrac{x-3}{x^2+9x-22}$

18. $f(x) = \dfrac{x-8}{x^2+8x-9}$

Given each function, evaluate: $f(-1)$, $f(0)$, $f(2)$, $f(4)$

19. $f(x) = \begin{cases} 7x+3 & \text{if} \quad x < 0 \\ 7x+6 & \text{if} \quad x \geq 0 \end{cases}$

20. $f(x) = \begin{cases} 4x-9 & \text{if} \quad x < 0 \\ 4x-18 & \text{if} \quad x \geq 0 \end{cases}$

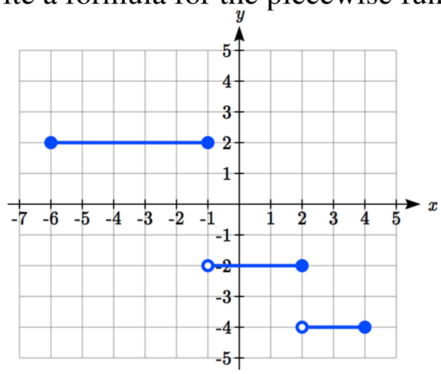
21. $f(x) = \begin{cases} x^2-2 & \text{if} \quad x < 2 \\ 4+|x-5| & \text{if} \quad x \geq 2 \end{cases}$

22. $f(x) = \begin{cases} 4-x^3 & \text{if} \quad x < 1 \\ \sqrt{x+1} & \text{if} \quad x \geq 1 \end{cases}$

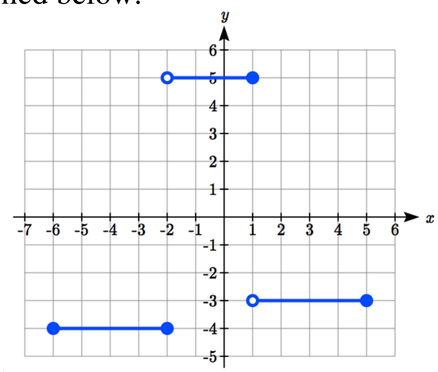
23. $f(x) = \begin{cases} 5x & \text{if} \quad x < 0 \\ 3 & \text{if} \quad 0 \leq x \leq 3 \\ x^2 & \text{if} \quad x > 3 \end{cases}$

24. $f(x) = \begin{cases} x^3+1 & \text{if} \quad x < 0 \\ 4 & \text{if} \quad 0 \leq x \leq 3 \\ 3x+1 & \text{if} \quad x > 3 \end{cases}$

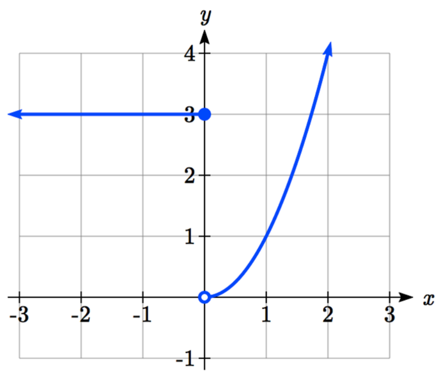
Write a formula for the piecewise function graphed below.



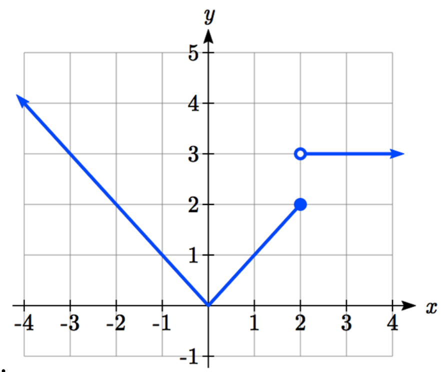
25.



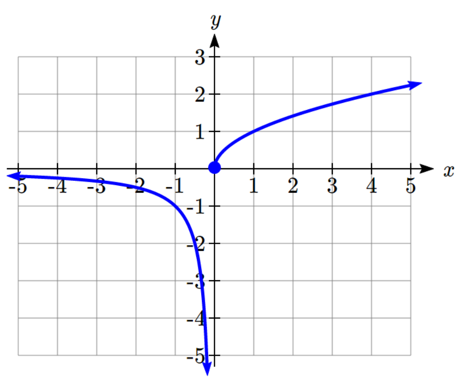
26.



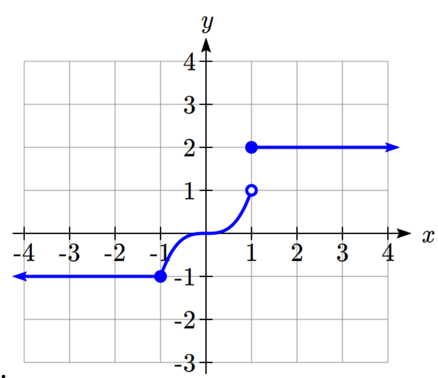
27.



28.



29.



30.

Sketch a graph of each piecewise function

31. $f(x)=\begin{cases}|x| & if \quad x<2\\ 5 & if \quad x\ge 2\end{cases}$

32. $f(x)=\begin{cases}4 & if \quad x<0\\ \sqrt{x} & if \quad x\ge 0\end{cases}$

33. $f(x)=\begin{cases}x^2 & if \quad x<0\\ x+2 & if \quad x\ge 0\end{cases}$

34. $f(x)=\begin{cases}x+1 & if \quad x<1\\ x^3 & if \quad x\ge 1\end{cases}$

35. $f(x)=\begin{cases}3 & if \quad x\le -2\\ -x+1 & if \quad -2<x\le 1\\ 3 & if \quad x>1\end{cases}$

36. $f(x)=\begin{cases}-3 & if \quad x\le -2\\ x-1 & if \quad -2<x\le 2\\ 0 & if \quad x>2\end{cases}$

Section 1.3 Rates of Change and Behavior of Graphs

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing.

For example, the function *C(t)* below gives the average cost, in dollars, of a gallon of gasoline *t* years after 2000.

t	2	3	4	5	6	7	8	9
C(t)	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from \$1.47 to \$2.14, an increase of \$0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by \$0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7\,years} \approx 0.096$ dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Rate of Change
A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are "<u>output units</u> per <u>input units</u>"

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns \$9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by \$4,000 per quarter

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50.

The average rate of change is then $\frac{-\$0.50}{2\,years}$ = -0.25 dollars per year

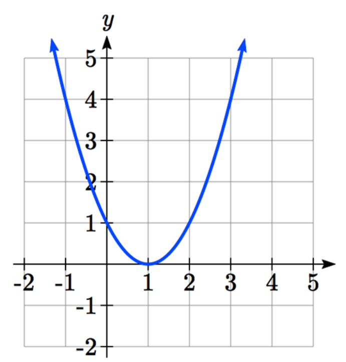
Try it Now

1. Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 2

Given the function *g(t)* shown here, find the average rate of change on the interval [0, 3].



At $t = 0$, the graph shows $g(0) = 1$

At $t = 3$, the graph shows $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example 3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
D(t) (miles)	10	55	90	153	214	240	292	300

Here, your average speed is the average rate of change.
You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation.

Average Rate of Change using Function Notation

Given a function $f(x)$, the average rate of change on the interval $[a, b]$ is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b)-f(a)}{b-a}$$

Example 4

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4]

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4)-f(2)}{4-2} = \frac{\frac{63}{4}-\frac{7}{2}}{4-2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

Try it Now

2. Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9]

Example 5

The magnetic force F, measured in Newtons, between two magnets is related to the distance between the magnets d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6].

Average rate of change = $\frac{F(6) - F(2)}{6-2}$ Evaluating the function

$$\frac{F(6) - F(2)}{6-2} =$$

$$\frac{\frac{2}{6^2} - \frac{2}{2^2}}{6-2}$$ Simplifying

$$\frac{\frac{2}{36} - \frac{2}{4}}{4}$$ Combining the numerator terms

$$\frac{\frac{-16}{36}}{4}$$ Simplifying further

$\frac{-1}{9}$ Newtons per centimeter

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter over this interval.

Example 6

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. Your answer will be an expression involving a.

Using the average rate of change formula

$$\frac{g(a) - g(0)}{a - 0}$$ Evaluating the function

$$\frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0}$$ Simplifying

$\dfrac{a^2+3a+1-1}{a}$ Simplifying further, and factoring

$\dfrac{a(a+3)}{a}$ Cancelling the common factor a

$a+3$

This result tells us the average rate of change between $t = 0$ and any other point $t = a$. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

Try it Now

3. Find the average rate of change of $f(x) = x^3 + 2$ on the interval $[a, a+h]$.

Graphical Behavior of Functions

As part of exploring how functions change, it is interesting to explore the graphical behavior of functions.

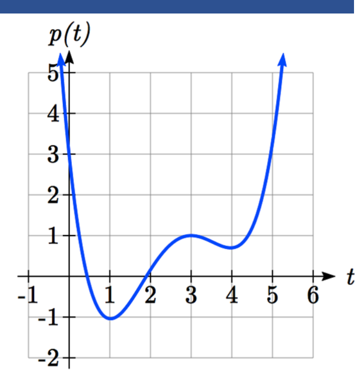
Increasing/Decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase. More formally, a function is increasing if $f(b) > f(a)$ for any two input values a and b in the interval with $b>a$. The average rate of change of an increasing function is **positive.**

A function is **decreasing** on an interval if the function values decrease as the inputs increase. More formally, a function is decreasing if $f(b) < f(a)$ for any two input values a and b in the interval with $b>a$. The average rate of change of a decreasing function is **negative.**

Example 7

Given the function $p(t)$ graphed here, on what intervals does the function appear to be increasing?



The function appears to be increasing from $t = 1$ to $t = 3$, and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1,3)$ and the interval $(4,\infty)$.

Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is neither increasing nor decreasing at t = 1, 3, or 4.

Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**.

A point where a function changes from decreasing to increasing is called a **local minimum**.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

Example 8

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

It appears that the cost of gas increased from t = 2 to t = 8. It appears the cost of gas decreased from t = 8 to t = 9, so the function appears to be decreasing on the interval (8, 9).

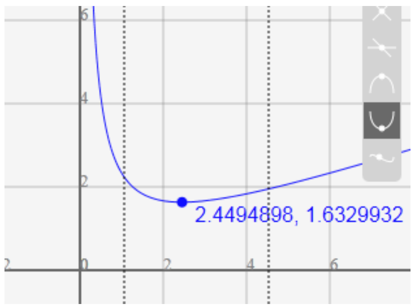
Since the function appears to change from increasing to decreasing at t = 8, there is local maximum at t = 8.

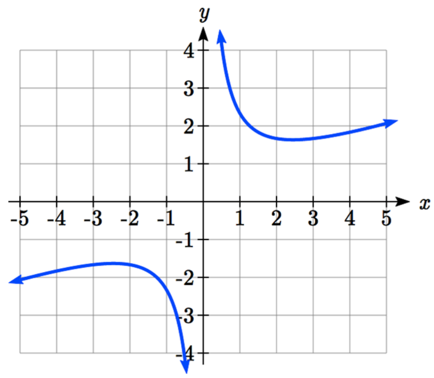
Example 9

Use a graph to estimate the local extrema of the function $f(x)=\frac{2}{x}+\frac{x}{3}$. Use these to determine the intervals on which the function is increasing.

Using technology to graph the function, it appears there is a local minimum somewhere between x = 2 and x =3, and a symmetric local maximum somewhere between x = -3 and x = -2.

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.





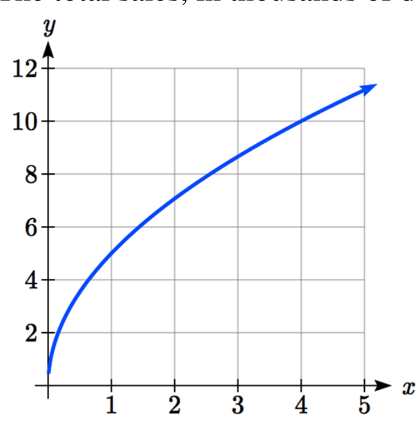
Based on these estimates, the function is increasing on the intervals $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that while we expect the extrema to be symmetric, the two different technologies agree only up to 4 decimals due to the differing approximation algorithms used by each.

Try it Now

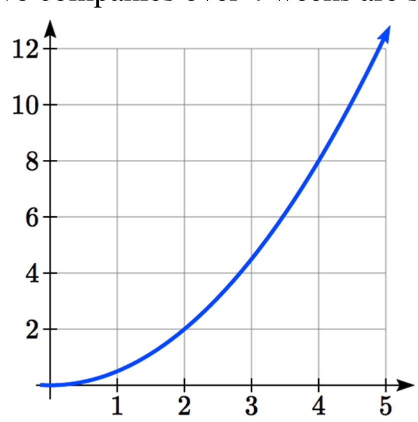
4. Use a graph of the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Concavity

The total sales, in thousands of dollars, for two companies over 4 weeks are shown.



Company A



Company B

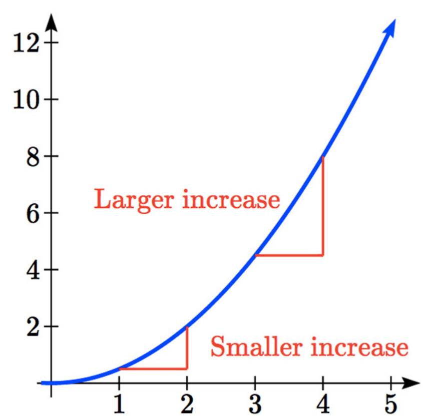
As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can investigate how the average rate of change varies over different intervals. Using tables of values, Company A

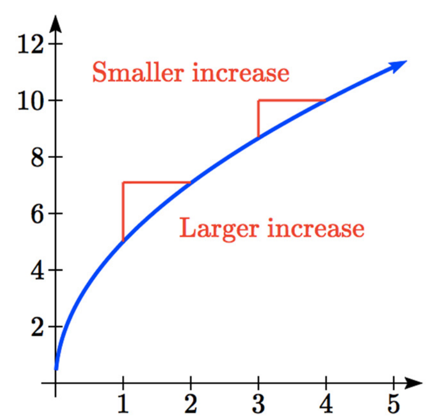
Week	Sales	Rate of Change
0	0	
		5
1	5	
		2.1
2	7.1	
		1.6
3	8.7	
		1.3
4	10	

Company B

Week	Sales	Rate of Change
0	0	
		0.5
1	0.5	
		1.5
2	2	
		2.5
3	4.5	
		3.5
4	8	

From the tables, we can see that the rate of change for company A is *decreasing*, while the rate of change for company B is *increasing*.





When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.

Concavity

A function is **concave up** if the rate of change is increasing.

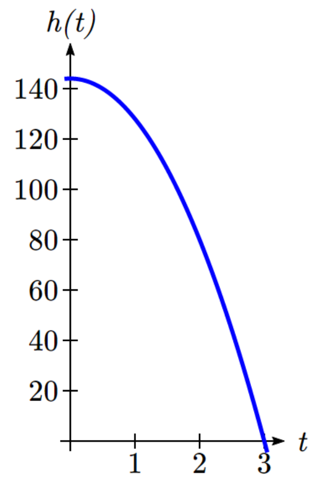
A function is **concave down** if the rate of change is decreasing.

A point where a function changes from concave up to concave down or vice versa is called an **inflection point**.

Example 10

An object is thrown from the top of a building. The object's height in feet above ground after *t* seconds is given by the function $h(t) = 144 - 16t^2$ for $0 \le t \le 3$. Describe the concavity of the graph.

Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior.



t	*h(t)*	Rate of Change
0	144	
		-16
1	128	
		-48
2	80	
		-80
3	0	

Notice that the rates of change are becoming more negative, so the rates of change are *decreasing*. This means the function is concave down.

Example 11

The value, *V*, of a car after *t* years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down?

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994

can compute rates of change to determine concavity.

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994
Rate of change	-1829	-1590	-1382.5	-1201.5	

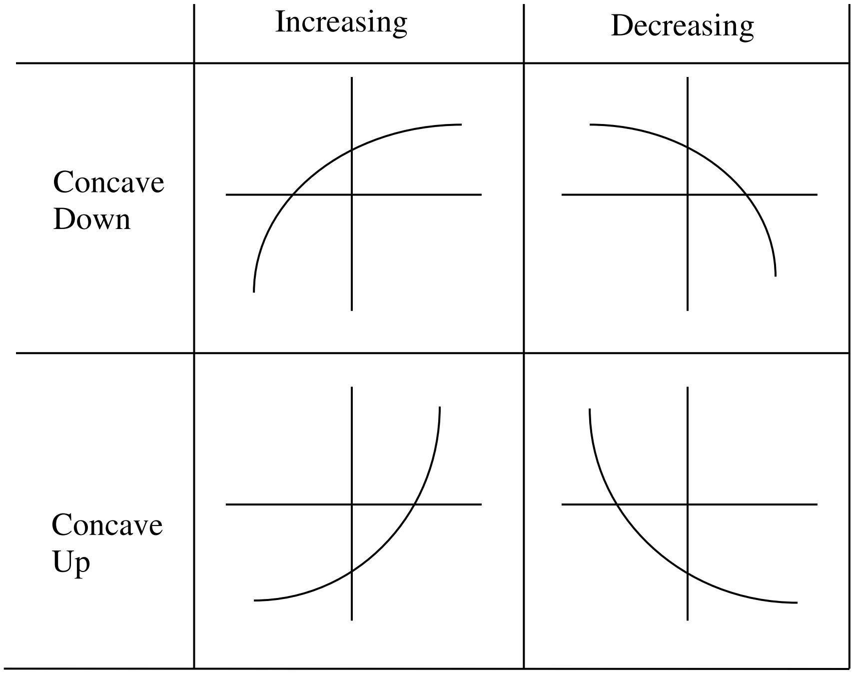
Since these values are becoming less negative, the rates of change are *increasing*, so this function is concave up.

Try it Now

5. Is the function described in the table below concave up or concave down?

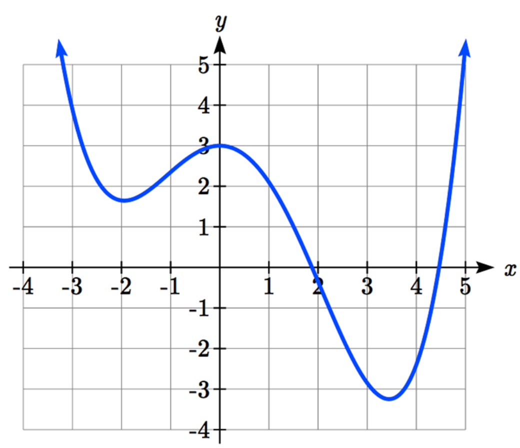
x	0	5	10	15	20
g(x)	10000	9000	7000	4000	0

Graphically, concave down functions bend downwards like a frown, and concave up function bend upwards like a smile.



Example 12

Estimate from the graph shown the intervals on which the function is concave down and concave up.



On the far left, the graph is decreasing but concave up, since it is bending upwards. It begins increasing at x = -2, but it continues to bend upwards until about x = -1.

From x = -1 the graph starts to bend downward, and continues to do so until about x = 2. The graph then begins curving upwards for the remainder of the graph shown.

From this, we can estimate that the graph is concave up on the intervals $(-\infty,-1)$ and $(2,\infty)$, and is concave down on the interval $(-1,2)$. The graph has inflection points at x = -1 and x = 2.

Try it Now

6. Using the graph from Try it Now 4, $f(x)=x^3-6x^2-15x+20$, estimate the intervals on which the function is concave up and concave down.

Behaviors of the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

Function	Increasing/Decreasing	Concavity
Constant Function $f(x)=c$	Neither increasing nor decreasing	Neither concave up nor down
Identity Function $f(x)=x$	Increasing	Neither concave up nor down
Quadratic Function $f(x)=x^2$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$ Minimum at $x=0$	Concave up $(-\infty,\infty)$
Cubic Function $f(x)=x^3$	Increasing	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$ Inflection point at (0,0)
Reciprocal $f(x)=\frac{1}{x}$	Decreasing $(-\infty,0)\cup(0,\infty)$	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$
Function	Increasing/Decreasing	Concavity
Reciprocal squared $f(x)=\frac{1}{x^2}$	Increasing on $(-\infty,0)$ Decreasing on $(0,\infty)$	Concave up on $(-\infty,0)\cup(0,\infty)$
Cube Root $f(x)=\sqrt[3]{x}$	Increasing	Concave down on $(0,\infty)$ Concave up on $(-\infty,0)$ Inflection point at (0,0)
Square Root $f(x)=\sqrt{x}$	Increasing on $(0,\infty)$	Concave down on $(0,\infty)$
Absolute Value $f(x)=\lvert x\rvert$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$	Neither concave up or down

Important Topics of This Section
Rate of Change
Average Rate of Change
Calculating Average Rate of Change using Function Notation
Increasing/Decreasing
Local Maxima and Minima (Extrema)
Inflection points
Concavity

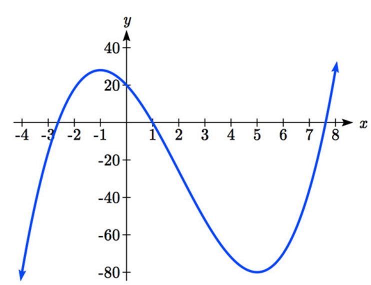
Try it Now Answers

1. $\dfrac{\$3.01 - \$1.69}{5\,years} = \dfrac{\$1.32}{5\,years} = 0.264$ dollars per year.

2. Average rate of change = $\dfrac{f(9) - f(1)}{9-1} = \dfrac{\left(9 - 2\sqrt{9}\right) - \left(1 - 2\sqrt{1}\right)}{9-1} = \dfrac{(3) - (-1)}{9-1} = \dfrac{4}{8} = \dfrac{1}{2}$

3. $\dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{\left((a+h)^3 + 2\right) - \left(a^3 + 2\right)}{h} = \dfrac{a^3 + 3a^2h + 3ah^2 + h^3 + 2 - a^3 - 2}{h} =$
$\dfrac{3a^2h + 3ah^2 + h^3}{h} = \dfrac{h\left(3a^2 + 3ah + h^2\right)}{h} = 3a^2 + 3ah + h^2$

4. Based on the graph, the local maximum appears to occur at (-1, 28), and the local minimum occurs at (5,-80). The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



5. Calculating the rates of change, we see the rates of change become *more* negative, so the rates of change are *decreasing*. This function is concave down.

x	0	5	10	15	20
g(x)	10000	9000	7000	4000	0
Rate of change	-1000	-2000	-3000	-4000	

6. Looking at the graph, it appears the function is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

Section 1.3 Exercises

1. The table below gives the annual sales (in millions of dollars) of a product. What was the average rate of change of annual sales…
a) Between 2001 and 2002? b) Between 2001 and 2004?

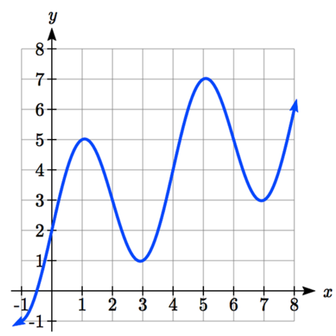
year	1998	1999	2000	2001	2002	2003	2004	2005	2006
sales	201	219	233	243	249	251	249	243	233

2. The table below gives the population of a town, in thousands. What was the average rate of change of population…
a) Between 2002 and 2004? b) Between 2002 and 2006?

year	2000	2001	2002	2003	2004	2005	2006	2007	2008
population	87	84	83	80	77	76	75	78	81

3. Based on the graph shown, estimate the average rate of change from $x = 1$ to $x = 4$.

4. Based on the graph shown, estimate the average rate of change from $x = 2$ to $x = 5$.



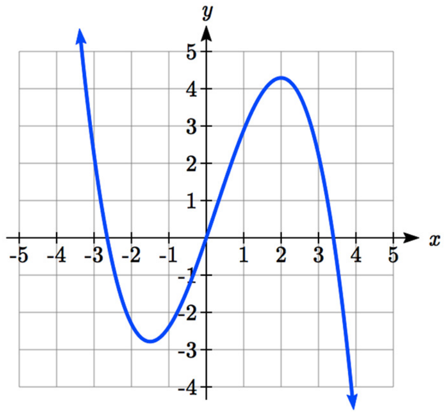
Find the average rate of change of each function on the interval specified.

5. $f(x) = x^2$ on [1, 5]
6. $q(x) = x^3$ on [-4, 2]
7. $g(x) = 3x^3 - 1$ on [-3, 3]
8. $h(x) = 5 - 2x^2$ on [-2, 4]
9. $k(t) = 6t^2 + \frac{4}{t^3}$ on [-1, 3]
10. $p(t) = \frac{t^2 - 4t + 1}{t^2 + 3}$ on [-3, 1]

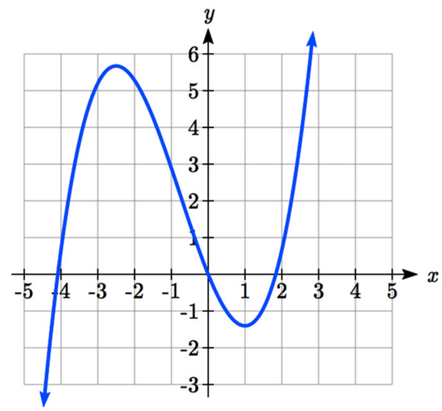
Find the average rate of change of each function on the interval specified. Your answers will be expressions involving a parameter (*b* or *h*).

11. $f(x) = 4x^2 - 7$ on [1, *b*]
12. $g(x) = 2x^2 - 9$ on [4, *b*]
13. $h(x) = 3x + 4$ on [2, 2+*h*]
14. $k(x) = 4x - 2$ on [3, 3+*h*]
15. $a(t) = \frac{1}{t + 4}$ on [9, 9+*h*]
16. $b(x) = \frac{1}{x + 3}$ on [1, 1+*h*]
17. $j(x) = 3x^3$ on [1, 1+*h*]
18. $r(t) = 4t^3$ on [2, 2+*h*]
19. $f(x) = 2x^2 + 1$ on [*x*, *x*+*h*]
20. $g(x) = 3x^2 - 2$ on [*x*, *x*+*h*]

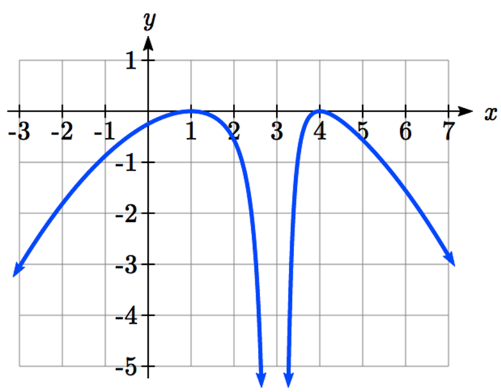
For each function graphed, estimate the intervals on which the function is increasing and decreasing.



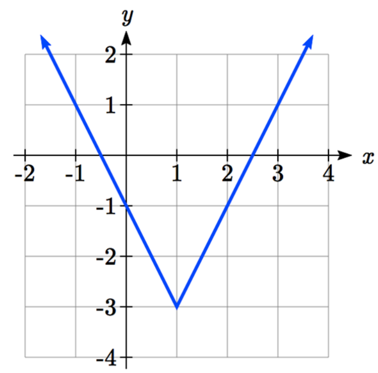
21.



22.



23.



24.

For each table below, select whether the table represents a function that is increasing or decreasing, and whether the function is concave up or concave down.

25.

x	*f*(*x*)
1	2
2	4
3	8
4	16
5	32

26.

x	*g*(*x*)
1	90
2	80
3	75
4	72
5	70

27.

x	*h*(*x*)
1	300
2	290
3	270
4	240
5	200

28.

x	*k*(*x*)
1	0
2	15
3	25
4	32
5	35

29.

x	*f*(*x*)
1	-10
2	-25
3	-37
4	-47
5	-54

30.

x	*g*(*x*)
1	-200
2	-190
3	-160
4	-100
5	0

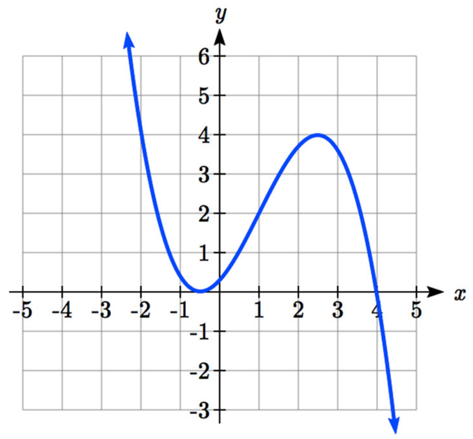
31.

x	*h*(*x*)
1	-100
2	-50
3	-25
4	-10
5	0

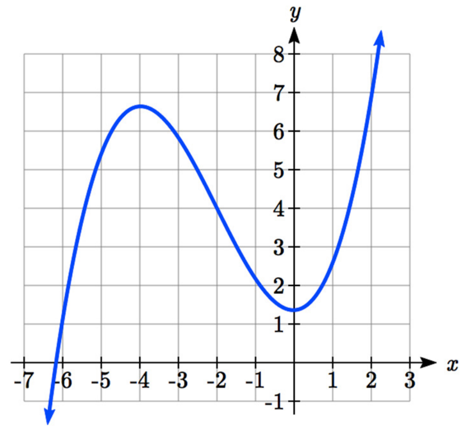
32.

x	*k*(*x*)
1	-50
2	-100
3	-200
4	-400
5	-900

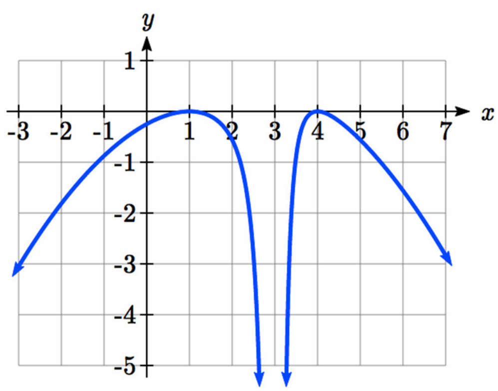
For each function graphed, estimate the intervals on which the function is concave up and concave down, and the location of any inflection points.



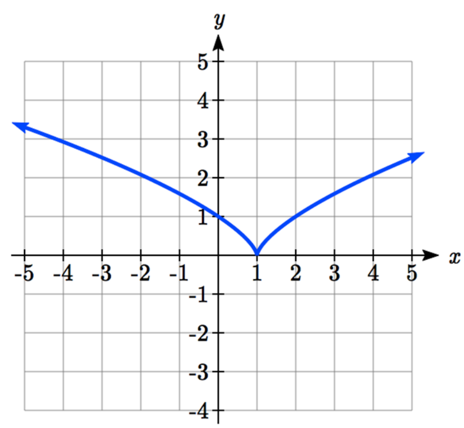
33.



34.



35.



36.

Use a graph to estimate the local extrema and inflection points of each function, and to estimate the intervals on which the function is increasing, decreasing, concave up, and concave down.

37. $f(x) = x^4 - 4x^3 + 5$

38. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

39. $g(t) = t\sqrt{t+3}$

40. $k(t) = 3t^{2/3} - t$

41. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

42. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Section 1.4 Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, $C(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Celsius, and the second, $T(d)$, gives the average daily temperature on day d of the year in some city. If we wanted to determine the cost of heating the house on the 5^{th} day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function $T(d)$, we could evaluate $T(5)$ to determine the average daily temperature on the 5^{th} day of the year. We could then use that temperature as the input to the $C(T)$ function to find the cost to heat the house on the 5^{th} day of the year: $C(T(5))$.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write $f(g(x))$, and read this as "f of g of x" or "f composed with g at x".

An alternate notation for composition uses the composition operator: $\circ$
$(f \circ g)(x)$ is read "f of g of x" or "f composed with g at x", just like $f(g(x))$.

Example 1

Suppose $c(s)$ gives the number of calories burned doing s sit-ups, and $s(t)$ gives the number of sit-ups a person can do in t minutes. Interpret $c(s(3))$.

When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside expression in the composition is $s(3)$. Since the input to the s function is time, the 3 is representing 3 minutes, and $s(3)$ is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the $c(s)$ function will gives us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it *is* essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 2

Suppose $f(x)$ gives miles that can be driven in x hours, and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

The expression $g(y)$ takes miles as the input and outputs a number of gallons. The function $f(x)$ is expecting a number of hours as the input; trying to give it a number of gallons as input does not make sense. Remember the units must match, and number of gallons does not match number of hours, so the expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and outputs a number of miles driven. The function $g(y)$ is expecting a number of miles as the input, so giving the output of the $f(x)$ function (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will give the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Try it Now

1. In a department store you see a sign that says 50% off clearance merchandise, so final cost C depends on the clearance price, p, according to the function $C(p)$. Clearance price, p, depends on the original discount, d, given to the clearance item, $p(d)$. Interpret $C(p(d))$.

Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph, as discussed in section 1.1. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 3

Using the tables below, evaluate $f(g(3))$ and $g(f(4))$

x	$f(x)$
1	6
2	8
3	3
4	1

x	$g(x)$
1	3
2	5
3	2
4	7

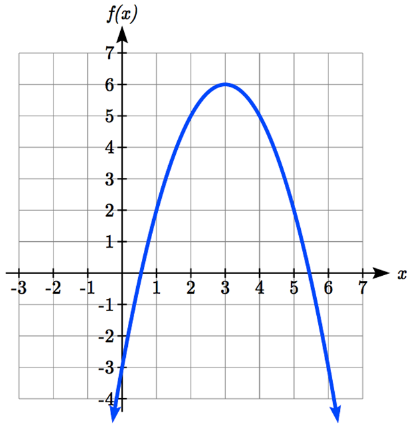
To evaluate $f(g(3))$, we start from the inside with the value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3) = 2$.

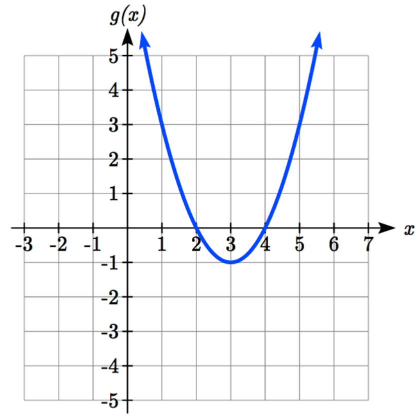
We can then use that result as the input to the *f* function, so $g(3)$ is replaced by the equivalent value 2 and we can evaluate $f(2)$. Then using the table that defines the function *f*, we find that $f(2)=8$.
$f(g(3)) = f(2) = 8$.

To evaluate $g(f(4))$, we first evaluate the inside expression $f(4)$ using the first table: $f(4)=1$. Then using the table for g we can evaluate:
$g(f(4)) = g(1) = 3$.

Try it Now
2. Using the tables from the example above, evaluate $f(g(1))$ and $g(f(3))$.

Example 4
Using the graphs below, evaluate $f(g(1))$.





To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the *g(x)* function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1)=3$.

Using this value as the input to the *f* function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the *f(x)* function, finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input.
$f(3)=6$, so $f(g(1))=6$.

Try it Now
3. Using the graphs from the previous example, evaluate $g(f(2))$.

Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First, we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Since the inside evaluation is $h(1)$ we start by evaluating the *h(x)* function at 1:
$h(1) = 3(1) + 2 = 5$

Then $f(h(1)) = f(5)$, so we evaluate the *f(t)* function at an input of 5:
$f(h(1)) = f(5) = 5^2 - 5 = 20$

Try it Now

4. Using the functions from the example above, evaluate $h(f(-2))$.

While we can compose the functions as above for each individual input value, sometimes it would be really helpful to find a single formula which will calculate the result of a composition *f(g(x))*. To do this, we will extend our idea of function evaluation. Recall that when we evaluate a function like $f(t) = t^2 - t$, we put whatever value is inside the parentheses after the function name into the formula wherever we see the input variable.

Example 6

Given $f(t) = t^2 - t$, evaluate $f(3)$ and $f(-2)$.

$f(3) = 3^2 - 3$
$f(-2) = (-2)^2 - (-2)$

We could simplify the results above if we wanted to
$f(3) = 3^2 - 3 = 9 - 3 = 6$
$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

We are not limited, however, to using a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an algebraic expression, provided we use the input expression everywhere we see the input variable.

Example 7

Using the function from the previous example, evaluate *f(a)*.

This means that the input value for *t* is some unknown quantity *a*. As before, we evaluate by replacing the input variable *t* with the input quantity, in this case *a*.

$f(a) = a^2 - a$

The same idea can then be applied to expressions more complicated than a single letter.

Example 8

Using the same *f(t)* function from above, evaluate $f(x+2)$.

Everywhere in the formula for *f* where there was a *t*, we would replace it with the input $(x+2)$. Since in the original formula the input *t* was squared in the first term, the entire input $x+2$ needs to be squared when we substitute, so we need to use grouping parentheses. To avoid problems, it is advisable to always use parentheses around inputs.

$f(x+2) = (x+2)^2 - (x+2)$

We could simplify this expression further to $f(x+2) = x^2 + 3x + 2$ if we wanted to:

$f(x+2) = (x+2)(x+2) - (x+2)$ Use the "FOIL" technique (first, outside, inside, last)

$f(x+2) = x^2 + 2x + 2x + 4 - (x+2)$ distribute the negative sign

$f(x+2) = x^2 + 2x + 2x + 4 - x - 2$ combine like terms

$f(x+2) = x^2 + 3x + 2$

Example 9

Using the same function, evaluate $f(t^3)$.

Note that in this example, the same variable is used in the input expression and as the input variable of the function. This doesn't matter – we still replace the original input *t* in the formula with the new input expression, t^3.

$f(t^3) = (t^3)^2 - (t^3) = t^6 - t^3$

Try it Now

5. Given $g(x) = 3x - \sqrt{x}$, evaluate $g(t-2)$.

This now allows us to find an expression for a composition of functions. If we want to find a formula for *f(g(x))*, we can start by writing out the formula for *g(x)*. We can then evaluate the function *f(x)* at that expression, as in the examples above.

Example 10

Let $f(x) = x^2$ and $g(x) = \dfrac{1}{x} - 2x$, find *f(g(x))* and *g(f(x))*.

To find *f(g(x))*, we start by evaluating the inside, writing out the formula for *g(x)*.

$$g(x) = \frac{1}{x} - 2x$$

We then use the expression $\left(\dfrac{1}{x} - 2x\right)$ as input for the function *f*.

$$f(g(x)) = f\left(\frac{1}{x} - 2x\right)$$

We then evaluate the function *f(x)* using the formula for *g(x)* as the input.

Since $f(x) = x^2$, $f\left(\dfrac{1}{x} - 2x\right) = \left(\dfrac{1}{x} - 2x\right)^2$

This gives us the formula for the composition: $f(g(x)) = \left(\dfrac{1}{x} - 2x\right)^2$.

Likewise, to find *g(f(x))*, we evaluate the inside, writing out the formula for *f(x)*

$$g(f(x)) = g\left(x^2\right)$$

Now we evaluate the function *g(x)* using x^2 as the input.

$$g(f(x)) = \frac{1}{x^2} - 2x^2$$

Try it Now

6. Let $f(x) = x^3 + 3x$ and $g(x) = \sqrt{x}$, find *f(g(x))* and *g(f(x))*.

Example 11

A city manager determines that the tax revenue, R, in millions of dollars collected on a population of p thousand people is given by the formula $R(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t) = 60 + 2t + 0.3t^2$, where t is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have to predict the city population, and then use that result as the input to the tax function. So we need to find $R(p(t))$. Evaluating this,

$$R(p(t)) = R\left(60 + 2t + 0.3t^2\right) = 0.03\left(60 + 2t + 0.3t^2\right) + \sqrt{60 + 2t + 0.3t^2}$$

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when $t = 7$ (because t is measured in years after 2010),

$$R(p(7)) = 0.03\left(60 + 2(7) + 0.3(7)^2\right) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars}$$

Domain of Compositions

When we think about the domain of a composition $h(x) = f(g(x))$, we must consider both the domain of the inner function and the domain of the composition itself. While it is tempting to only look at the resulting composite function, if the inner function were undefined at a value of x, the composition would not be possible.

Example 12

Let $f(x) = \dfrac{1}{x^2 - 1}$ and $g(x) = \sqrt{x-2}$. Find the domain of $f\left(g(x)\right)$.

Since we want to avoid the square root of negative numbers, the domain of $g(x)$ is the set of values where $x - 2 \geq 0$. The domain is $x \geq 2$.

The composition is $f\left(g(x)\right) = \dfrac{1}{\left(\sqrt{x-2}\right)^2 - 1} = \dfrac{1}{(x-2) - 1} = \dfrac{1}{x-3}$.

The composition is undefined when $x = 3$, so that value must also be excluded from the domain. Notice that the composition doesn't involve a square root, but we still have to consider the domain limitation from the inside function.

Combining the two restrictions, the domain is all values of x greater than or equal to 2, except $x = 3$.

In inequalities, the domain is: $2 \le x < 3$ or $x > 3$.
In interval notation, the domain is: $[2,3) \cup (3,\infty)$.

Try it Now

7. Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{x}$. Find the domain of $f\left(g(x)\right)$.

Decomposing Functions

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions.

Example 13

Write $f(x) = 3 + \sqrt{5 - x^2}$ as the composition of two functions.

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for *f(x)*. As one possibility, we might notice that $5 - x^2$ is the inside of the square root. We could then decompose the function as:
$h(x) = 5 - x^2$
$g(x) = 3 + \sqrt{x}$

We can check our answer by recomposing the functions:
$g(h(x)) = g\left(5 - x^2\right) = 3 + \sqrt{5 - x^2}$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be $h(x) = x^2$ and $g(x) = 3 + \sqrt{5 - x}$

Important Topics of this Section
Definition of Composition of Functions
Compositions using:
Words
Tables
Graphs
Equations
Domain of Compositions
Decomposition of Functions

Try it Now Answers

1. The final cost, *C*, depends on the clearance price, *p*, which is based on the original discount, *d*. (Or the original discount *d*, determines the clearance price and the final cost is half of the clearance price.)

2. $f(g(1)) = f(3) = 3$ and $g(f(3)) = g(3) = 2$

3. $g(f(2)) = g(5) = 3$

4. $h(f(-2)) = h(6) = 20$ *did you remember to insert your input values using parentheses?*

5. $g(t-2) = 3(t-2) - \sqrt{(t-2)}$

6. $f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^3 + 3\left(\sqrt{x}\right)$

   $g(f(x)) = g\left(x^3 + 3x\right) = \sqrt{\left(x^3 + 3x\right)}$

7. $g(x) = \dfrac{1}{x}$ is undefined at $x = 0$.

   The composition, $f\left(g(x)\right) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{\frac{1}{x} - 2} = \dfrac{1}{\frac{1}{x} - \frac{2x}{x}} = \dfrac{1}{\frac{1-2x}{x}} = \dfrac{x}{1-2x}$ is undefined

   when $1 - 2x = 0$, when $x = \dfrac{1}{2}$.

   Restricting these two values, the domain is $(-\infty, 0) \cup \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$.

Section 1.4 Exercises

Given each pair of functions, calculate $f\left(g(0)\right)$ and $g\left(f(0)\right)$.

1. $f(x)=4x+8$, $g(x)=7-x^2$
2. $f(x)=5x+7$, $g(x)=4-2x^2$
3. $f(x)=\sqrt{x+4}$, $g(x)=12-x^3$
4. $f(x)=\dfrac{1}{x+2}$, $g(x)=4x+3$

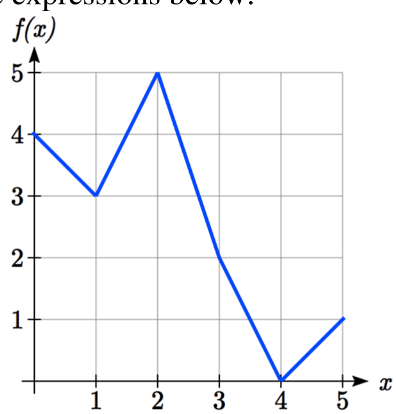
Use the table of values to evaluate each expression

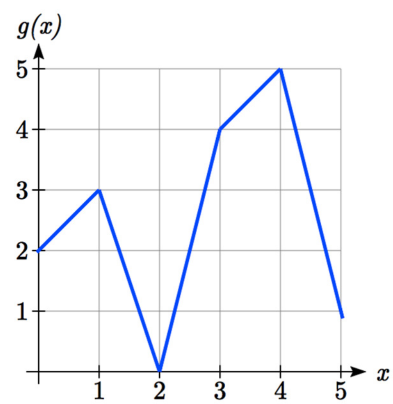
5. $f(g(8))$
6. $f\left(g(5)\right)$
7. $g(f(5))$
8. $g\left(f(3)\right)$
9. $f(f(4))$
10. $f\left(f(1)\right)$
11. $g(g(2))$
12. $g\left(g(6)\right)$

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Use the graphs to evaluate the expressions below.

13. $f(g(3))$
14. $f\left(g(1)\right)$
15. $g(f(1))$
16. $g\left(f(0)\right)$
17. $f(f(5))$
18. $f\left(f(4)\right)$
19. $g(g(2))$
20. $g\left(g(0)\right)$





For each pair of functions, find $f\left(g(x)\right)$ and $g\left(f(x)\right)$. Simplify your answers.

21. $f(x)=\dfrac{1}{x-6}$, $g(x)=\dfrac{7}{x}+6$
22. $f(x)=\dfrac{1}{x-4}$, $g(x)=\dfrac{2}{x}+4$
23. $f(x)=x^2+1$, $g(x)=\sqrt{x+2}$
24. $f(x)=\sqrt{x}+2$, $g(x)=x^2+3$
25. $f(x)=|x|$, $g(x)=5x+1$
26. $f(x)=\sqrt[3]{x}$, $g(x)=\dfrac{x+1}{x^3}$

27. If $f(x) = x^4 + 6$, $g(x) = x - 6$ and $h(x) = \sqrt{x}$, find $f(g(h(x)))$

28. If $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$ and $h(x) = x + 3$, find $f(g(h(x)))$

29. The function $D(p)$ gives the number of items that will be demanded when the price is *p*. The production cost, $C(x)$ is the cost of producing *x* items. To determine the cost of production when the price is \$6, you would do which of the following:
a. Evaluate $D(C(6))$ b. Evaluate $C(D(6))$
c. Solve $D(C(x)) = 6$ d. Solve $C(D(p)) = 6$

30. The function $A(d)$ gives the pain level on a scale of 0-10 experienced by a patient with *d* milligrams of a pain reduction drug in their system. The milligrams of drug in the patient's system after *t* minutes is modeled by $m(t)$. To determine when the patient will be at a pain level of 4, you would need to:
a. Evaluate $A(m(4))$ b. Evaluate $m(A(4))$
c. Solve $A(m(t)) = 4$ d. Solve $m(A(d)) = 4$

31. The radius *r*, in inches, of a spherical balloon is related to the volume, *V*, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after *t* seconds is given by $V(t) = 10 + 20t$.
    a. Find the composite function $r(V(t))$
    b. Find the radius after 20 seconds

32. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where *T* is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where *t* is the time in hours.
    a. Find the composite function $N(T(t))$
    b. Find the bacteria count after 4 hours

33. Given $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, find the domain of $m(p(x))$.

34. Given $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = 9 - x^2$, find the domain of $m(p(x))$.

35. Given $f(x) = \frac{1}{x+3}$ and $g(x) = \frac{2}{x-1}$, find the domain of $f(g(x))$.

36. Given $f(x)=\dfrac{x}{x+1}$ and $g(x)=\dfrac{4}{x}$, find the domain of $f\left(g\left(x\right)\right)$.

37. Given $f(x)=\sqrt{x-2}$ and $g(x)=\dfrac{2}{x^2-3}$, find the domain of $g(f(x))$.

38. Given $f(x)=\sqrt{4-x}$ and $g(x)=\dfrac{1}{x^2-2}$, find the domain of $g(f(x))$.

Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f\left(g\left(x\right)\right)$.

39. $h(x)=(x+2)^2$

40. $h(x)=(x-5)^3$

41. $h(x)=\dfrac{3}{x-5}$

42. $h(x)=\dfrac{4}{(x+2)^2}$

43. $h(x)=3+\sqrt{x-2}$

44. $h(x)=4+\sqrt[3]{x}$

45. Let $f(x)$ be a linear function, with form $f(x)=ax+b$ for constants *a* and *b*. [UW]

a. Show that $f\left(f\left(x\right)\right)$ is a linear function

b. Find a function $g(x)$ such that $g\left(g\left(x\right)\right)=6x-8$

46. Let $f(x)=\dfrac{1}{2}x+3$ [UW]

a. Sketch the graphs of $f\left(x\right), f\left(f\left(x\right)\right), f\left(f\left(f\left(x\right)\right)\right)$ on the interval $-2 \le x \le 10$.

b. Your graphs should all intersect at the point (6, 6). The value x = 6 is called a fixed point of the function *f(x)*since $f(6)=6$; that is, 6 is fixed - it doesn't move when *f* is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(...f(x)...)))$.

c. Linear functions (with the exception of $f(x)=x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g\left(x\right)=x^2-2$.

d. Give a quadratic function whose fixed points are $x = -2$ and $x = 3$.

47. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function $C(m) = \dfrac{70m^2}{10+m^2}$. [UW]

a. Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?

b. Find a function $m = g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C(g(h))$; what does this function calculate?

c. Find a function $z = v(s)$ that converts mph s into ft/sec z. Write out the formula for the new function $v(C(m)$; what does this function calculate?

Section 1.5 Transformation of Functions

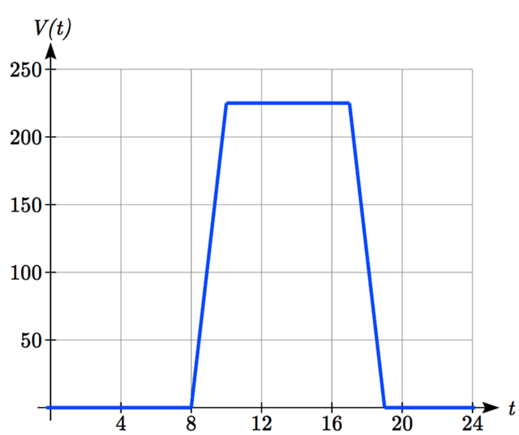
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs and equations in order to explain or solve it. When building models, it is often helpful to build off of existing formulas or models. Knowing the basic graphs of your tool-kit functions can help you solve problems by being able to model new behavior by adapting something you already know. Unfortunately, the models and existing formulas we know are not always exactly the same as the ones presented in the problems we face.

Fortunately, there are systematic ways to shift, stretch, compress, flip and combine functions to help them become better models for the problems we are trying to solve. We can transform what we already know into what we need, hence the name, "Transformation of functions." When we have a story problem, formula, graph, or table, we can then transform that function in a variety of ways to form new functions.

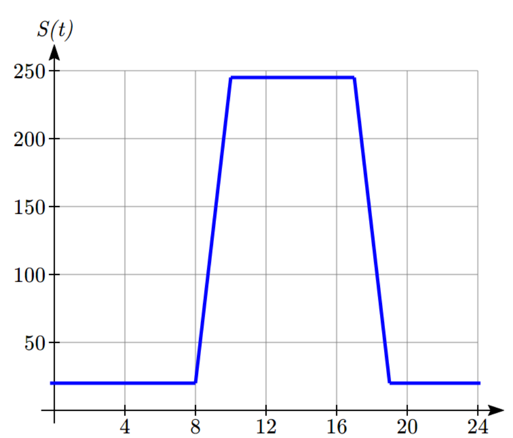
Shifts

Example 1

To regulate temperature in a green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents *V* (in square feet) throughout the day, *t* in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.



We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up.



Notice that in the second graph, for each input value, the output value has increased by twenty, so if we call the new function *S(t)*, we could write $S(t) = V(t) + 20$.

Note that this notation tells us that for any value of *t*, *S(t)* can be found by evaluating the *V* function at the same input, then adding twenty to the result.
This defines *S* as a transformation of the function *V*, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

Vertical Shift

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = f(x) + k$, where *k* is a constant

then *g(x)* is a **vertical shift** of the function *f(x)*, where all the output values have been increased by *k*.

If *k* is positive, then the graph will shift up

If *k* is negative, then the graph will shift down

Example 2

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x) - 3$

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the *g* function by subtracting 3 from the output values of the *f* function. For example,

$f(2) = 1$ is found from the given table

$g(x) = f(x) - 3$ is our given transformation

$g(2) = f(2) - 3 = 1 - 3 = -2$

Subtracting 3 from each *f(x)* value, we can complete a table of values for *g(x)*

x	2	4	6	8
g(x)	-2	0	4	8

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

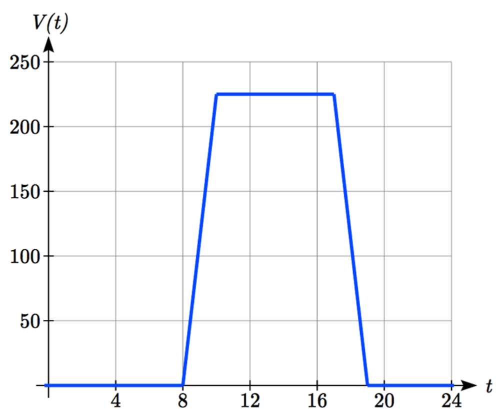
Try it Now

1. The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upwards from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function $b(t)$ to $h(t)$, then find a formula for $b(t)$.

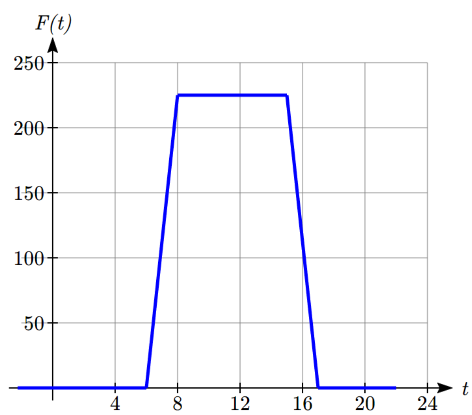
The vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning.

Example 3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.



$V(t)$ = the original venting plan



$F(t)$ = starting 2 hours sooner

In the new graph, which we can call $F(t)$, at each time, the air flow is the same as the original function $V(t)$ was two hours later. For example, in the original function V, the air flow starts to change at 8am, while for the function $F(t)$ the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$.

Notice also that the vents first opened to 220 sq. ft. at 10 a.m. under the original plan, while under the new plan the vents reach 220 sq. ft. at 8 a.m., so $V(10) = F(8)$.

In both cases we see that since $F(t)$ starts 2 hours sooner, the same output values are reached when, $F(t) = V(t+2)$

Note that $V(t+2)$ had the effect of shifting the graph to the *left*.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function *F(t)* uses the same outputs as *V(t)*, but matches those outputs to inputs two hours earlier than those of *V(t)*. Said another way, we must add 2 hours to the input of *V* to find the corresponding output for *F*: $F(t) = V(t+2)$.

Horizontal Shift

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = f(x+k)$, where *k* is a constant

then *g(x)* is a **horizontal shift** of the function *f(x)*

If *k* is positive, then the graph will shift left

If *k* is negative, then the graph will shift right

Example 4

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x-3)$

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = f(x-3)$ tells us that the output values of *g* are the same as the output value of *f* with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the *g* function, we will need an input value that is 3 *larger*: We input a value that is three larger for *g(x)* because the function takes three away before evaluating the function *f*.

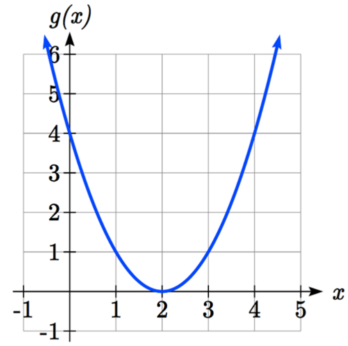
$g(5) = f(5-3) = f(2) = 1$

x	5	7	9	11
g(x)	1	3	7	11

The result is that the function *g(x)* has been shifted to the right by 3. Notice the output values for *g(x)* remain the same as the output values for *f(x)* in the chart, but the corresponding input values, *x*, have shifted to the right by 3: 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

Example 5

The graph shown is a transformation of the toolkit function $f(x) = x^2$. Relate this new function *g(x)* to *f(x)*, and then find a formula for *g(x)*.



Notice that the graph looks almost identical in shape to the $f(x) = x^2$ function, but the *x* values are shifted to the right two units. The vertex used to be at (0, 0) but now the vertex is at (2, 0). The graph is the basic quadratic function shifted two to the right, so
$g(x) = f(x-2)$

Notice how we must input the value *x* = 2, to get the output value *y* = 0; the *x* values must be two units larger, because of the shift to the right by 2 units.

We can then use the definition of the *f(x)* function to write a formula for *g(x)* by evaluating $f(x-2)$:
Since $f(x) = x^2$ and $g(x) = f(x-2)$
$g(x) = f(x-2) = (x-2)^2$

If you find yourself having trouble determining whether the shift is +2 or -2, it might help to consider a single point on the graph. For a quadratic, looking at the bottom-most point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the *f* function, we need to decide whether a +2 or -2 will work to satisfy $g(2) = f(2 ? 2) = f(0) = 0$. For this to work, we will need to subtract 2 from our input values.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

Example 6

The function *G(m)* gives the number of gallons of gas required to drive *m* miles. Interpret $G(m)+10$ and $G(m+10)$.

$G(m)+10$ is adding 10 to the output, gallons. This is 10 gallons of gas more than is required to drive *m* miles. So, this is the gas required to drive *m* miles, plus another 10 gallons of gas.

$G(m+10)$ is adding 10 to the input, miles. This is the number of gallons of gas required to drive 10 miles more than *m* miles.

Try it Now

2. Given the function $f(x) = \sqrt{x}$ graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$.
   a. Is this a horizontal or a vertical change?
   b. Which way is the graph shifted and by how many units?
   c. Graph *f(x)* and *g(x)* on the same axes.

Now that we have two transformations, we can combine them together.

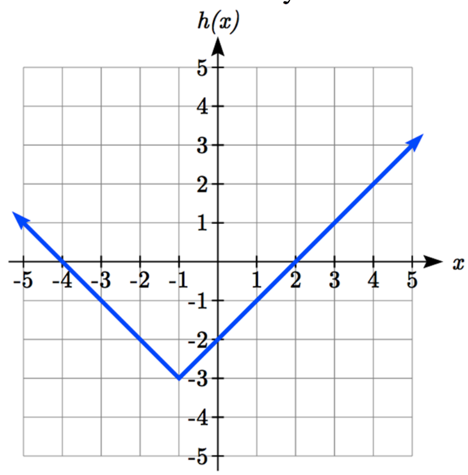
Remember:
Vertical Shifts are outside changes that affect the output (vertical) axis values shifting the transformed function up or down.

Horizontal Shifts are inside changes that affect the input (horizontal) axis values shifting the transformed function left or right.

Example 7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x+1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives



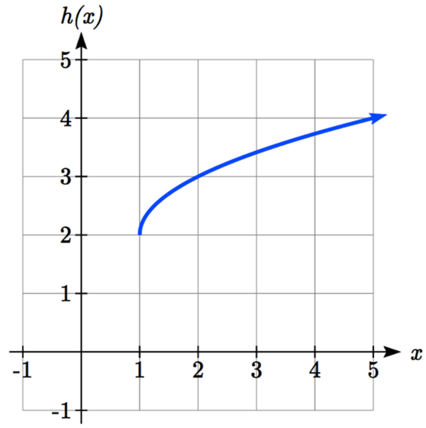
We could also find a formula for this transformation by evaluating the expression for *h(x)*:

$h(x) = f(x+1) - 3$

$h(x) = |x+1| - 3$

Example 8

Write a formula for the graph shown, a transformation of the toolkit square root function.



The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as
$h(x) = f(x-1)+2$. Using the formula for the square root function we can write
$h(x) = \sqrt{x-1}+2$

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1,\infty)$ and range $[2,\infty)$.

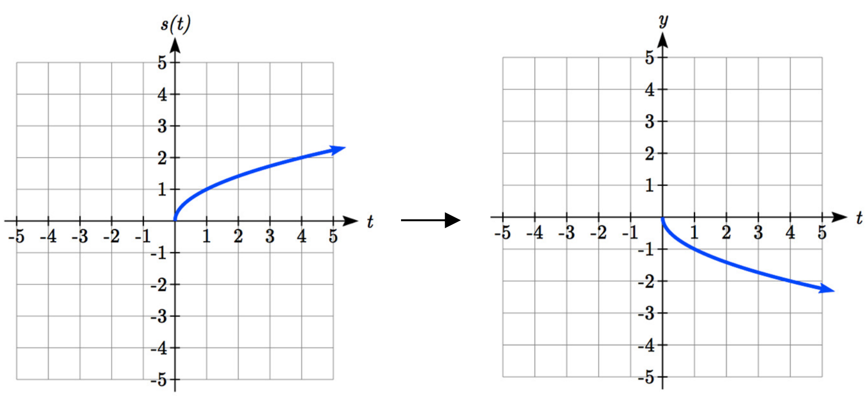
Reflections

Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

Example 9

Reflect the graph of $s(t)=\sqrt{t}$ both vertically and horizontally.

Reflecting the graph vertically, each output value will be reflected over the horizontal *t* axis:



Since each output value is the opposite of the original output value, we can write
$V(t) = -s(t)$
$V(t) = -\sqrt{t}$

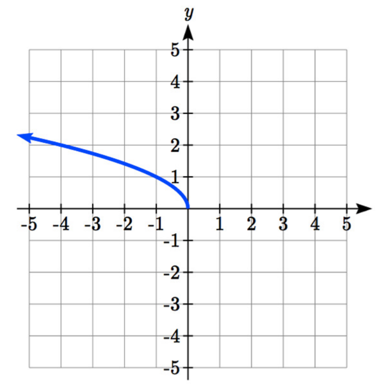
Notice this is an outside change or vertical change that affects the output *s(t)* values so the negative sign belongs outside of the function.

Reflecting horizontally, each input value will be reflected over the vertical axis.

Since each input value is the opposite of the original input value, we can write

$H(t) = s(-t)$

$H(t) = \sqrt{-t}$



Notice this is an inside change or horizontal change that affects the input values so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the *V(t)* function the range $(-\infty, 0]$, and the horizontal reflection gives the *H(t)* function the domain $(-\infty, 0]$.

Reflections

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = -f(x)$,

then *g(x)* is a **vertical reflection** of the function *f(x)*, sometimes called a reflection about the *x*-axis

If we define a new function *g(x)* as

$g(x) = f(-x)$,

then *g(x)* is a **horizontal reflection** of the function *f(x)*, sometimes called a reflection about the *y*-axis

Example 10

A function *f(x)* is given as a table below. Create a table for the function $g(x) = -f(x)$ and $h(x) = f(-x)$

x	2	4	6	8
f(x)	1	3	7	11

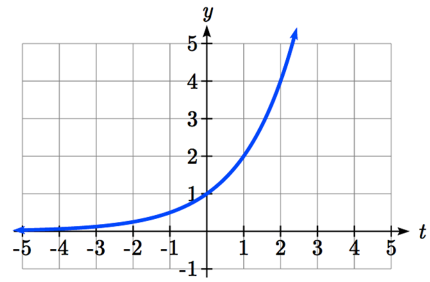
For *g(x)*, this is a vertical reflection, so the *x* values stay the same and each output value will be the opposite of the original output value

For *h(x)*, this is a horizontal reflection, and each input value will be the opposite of the original input value and the *h(x)* values stay the same as the *f(x)* values:

x	-2	-4	-6	-8
h(x)	1	3	7	11

Example 11

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown here. Sketch a graph of $k(t)$.



This equation combines three transformations into one equation.

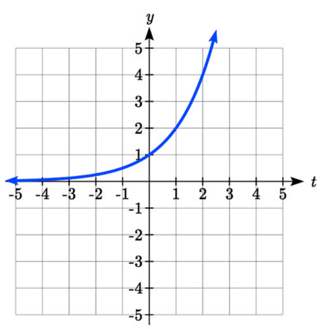
A horizontal reflection: $f(-t) = 2^{-t}$ combined with

A vertical reflection: $-f(-t) = -2^{-t}$ combined with

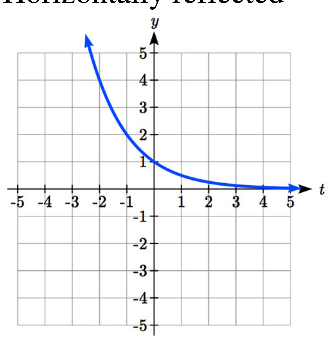
A vertical shift up 1: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function:

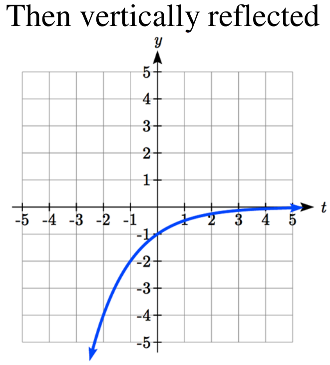
The original graph



Horizontally reflected



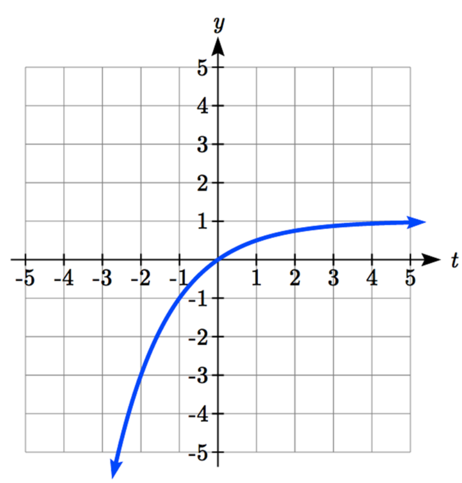
Then vertically reflected



Then, after shifting up 1, we get the final graph.

$k(t) = -f(-t) + 1 = -2^{-t} + 1$.

Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0,1)$.



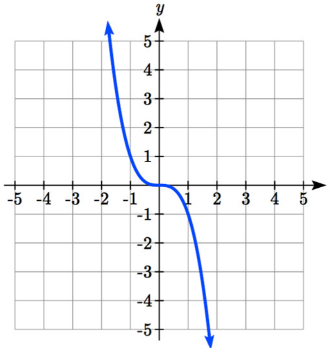
Try it Now

3. Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Do you notice anything surprising?

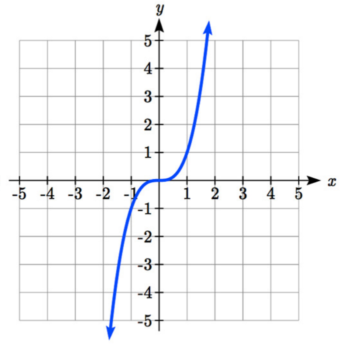
Some functions exhibit symmetry, in which reflections result in the original graph. For example, reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ about the y-axis will result in the original graph. We call these types of graphs symmetric about the y-axis.

Likewise, if the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph:

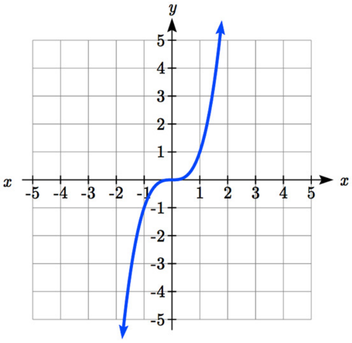
$f(x) = x^3$



$f(-x)$



$-f(-x)$



We call these graphs symmetric about the origin.

Even and Odd Functions

A function is called an **even function** if

$f(x) = f(-x)$

The graph of an even function is symmetric about the vertical axis

A function is called an **odd function** if

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, the $f(x) = 2^x$ function is neither even nor odd.

Example 12

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Without looking at a graph, we can determine this by finding formulas for the reflections, and seeing if they return us to the original function:

$f(-x)=(-x)^3+2(-x)=-x^3-2x$
This does not return us to the original function, so this function is not even.

We can now try also applying a horizontal reflection:
$-f(-x)=-\left(-x^3-2x\right)=x^3+2x$

Since $-f(-x)=f(x)$, this is an odd function.

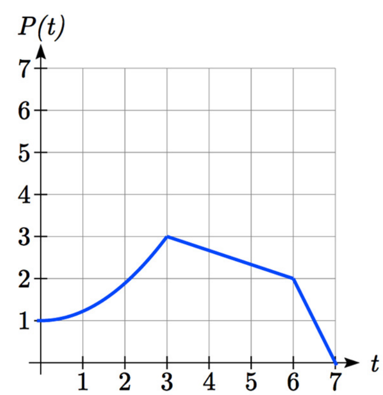
Stretches and Compressions

With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the inputs or outputs.

Remember, we can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

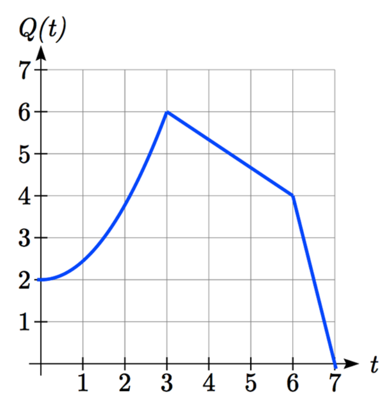
Example 13

A function *P(t)* models the growth of a population of fruit flies. The growth is shown in the graph. A scientist is comparing this to another population, *Q*, that grows the same way, but starts twice as large. Sketch a graph of this population.



Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like the second graph shown.

Symbolically, $Q(t)=2P(t)$



This means that for any input *t*, the value of the *Q* function is twice the value of the *P* function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, *t*, stay the same while the output values are twice as large as before.

Vertical Stretch/Compression

Given a function *f(x)*, if we define a new function *g(x)* as
$g(x) = kf(x)$, where *k* is a constant
then *g(x)* is a **vertical stretch or compression** of the function *f(x)*.

If $k > 1$, then the graph will be stretched
If $0 < k < 1$, then the graph will be compressed
If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

Example 14

A function *f(x)* is given as a table below. Create a table for the function $g(x) = \frac{1}{2}f(x)$

x	2	4	6	8
f(x)	1	3	7	11

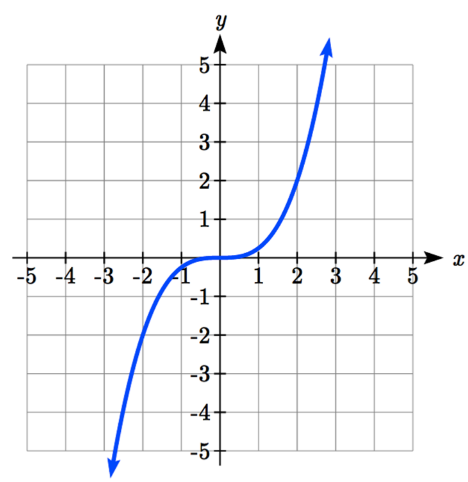
The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of *g* are half of the output values of *f* with the same inputs. For example, we know that $f(4) = 3$. Then $g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$

x	2	4	6	8
g(x)	1/2	3/2	7/2	11/2

The result is that the function *g(x)* has been compressed vertically by ½. Each output value has been cut in half, so the graph would now be half the original height.

Example 15

The graph shown is a transformation of the toolkit function $f(x) = x^3$. Relate this new function *g(x)* to *f(x)*, then find a formula for *g(x)*.



When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$.

Based on that, it appears that the outputs of g are ¼ the outputs of the function f, since $g(2)=\frac{1}{4}f(2)$.

From this we can fairly safely conclude that:
$$g(x)=\frac{1}{4}f(x)$$

We can write a formula for g by using the definition of the function f
$$g(x)=\frac{1}{4}f(x)=\frac{1}{4}x^3$$

Now we consider changes to the inside of a function.

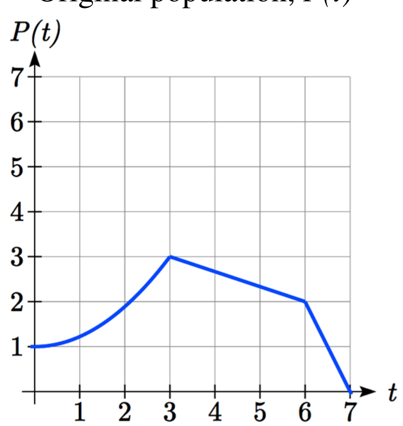
Example 16

Returning to the fruit fly population we looked at earlier, suppose the scientist is now comparing it to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, *R*, will progress in 1 hour the same amount the original population did in 2 hours, and in 2 hours, will progress as much as the original population did in 4 hours. Sketch a graph of this population.

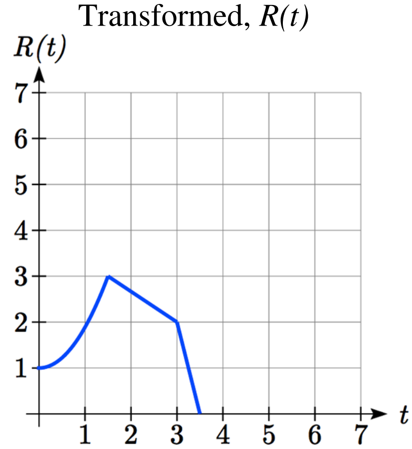
Symbolically, we could write
$R(1)=P(2)$
$R(2)=P(4)$, and in general,
$R(t)=P(2t)$

Graphing this,

Original population, *P(t)*



Transformed, *R(t)*



Note the effect on the graph is a horizontal compression, where all input values are half their original distance from the vertical axis.

Horizontal Stretch/Compression

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = f(kx)$, where k is a constant

then *g(x)* is a **horizontal stretch or compression** of the function *f(x)*.

If $k > 1$, then the graph will be compressed by $1/k$

If $0 < k < 1$, then the graph will be stretched by $1/k$

If $k < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example 17

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$

x	2	4	6	8
$f(x)$	1	3	7	11

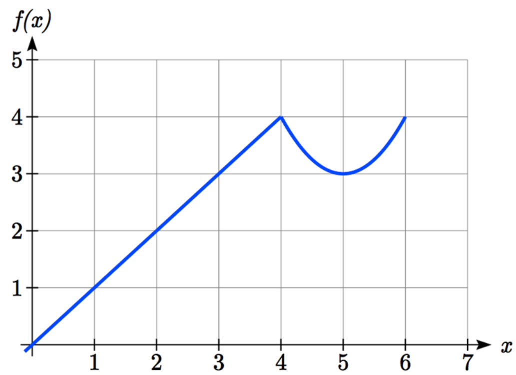
The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we don't have enough information to determine $g(2)$ since $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$ since $g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$.

x	4	8	12	16
$g(x)$	1	3	7	11

Since each input value has been doubled, the result is that the function *g(x)* has been stretched horizontally by 2.

Example 18

Two graphs are shown below. Relate the function *g(x)* to *f(x)*.



The graph of *g(x)* looks like the graph of *f(x)* horizontally compressed. Since *f(x)* ends at (6,4) and *g(x)* ends at (2,4) we can see that the *x* values have been compressed by 1/3, because 6(1/3) = 2. We might also notice that $g(2) = f(6)$, and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by 1/3.

Notice that the coefficient needed for a horizontal stretch or compression is the *reciprocal* of the stretch or compression. To stretch the graph horizontally by 4, we need a coefficient of 1/4 in our function: $f\left(\frac{1}{4}x\right)$. This means the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Try it Now

4. Write a formula for the toolkit square root function horizontally stretched by three.

It is useful to note that for most toolkit functions, a horizontal stretch or vertical stretch can be represented in other ways. For example, a horizontal compression of the function $f(x) = x^2$ by ½ would result in a new function $g(x) = (2x)^2$, but this can also be written as $g(x) = 4x^2$, a vertical stretch of *f(x)* by 4. When writing a formula for a transformed toolkit, we only need to find one transformation that would produce the graph.

Combining Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 then vertically shifting by 3.

When we see an expression like $2f(x)+3$, which transformation should we start with? The answer here follows nicely from order of operations, for outside transformations. Given the output value of *f(x)*, we first multiply by 2, causing the vertical stretch, then add 3, causing the vertical shift. (Multiplication before Addition)

Combining Vertical Transformations
When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by *a*, then vertically shift by *k*.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x+3)$ for example, we have to think about how the inputs to the *g* function relate to the inputs to the *f* function. Suppose we know $f(7)=12$. What input to *g* would produce that output? In other words, what value of *x* will allow $g(x) = f(2x+3) = f(12)$? We would need $2x+3=12$. To solve for *x*, we would first subtract 3, resulting in horizontal shift, then divide by 2, causing a horizontal compression.

Combining Horizontal Transformations
When **combining horizontal transformations** written in the form $f(bx+p)$, first horizontally shift by *p*, then horizontally stretch by 1/*b*.

This format ends up being very difficult to work with, since it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p) = f\left(b\left(x+\frac{p}{b}\right)\right)$$

Factoring in this way allows us to horizontally stretch first, then shift horizontally.

Combining Horizontal Transformations (Factored Form)
When **combining horizontal transformations** written in the form $f(b(x+h))$, first horizontally stretch by 1/*b*, then horizontally shift by *h*.

Independence of Horizontal and Vertical Transformations

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are done first.

Example 19

Given the table of values for the function *f(x)* below, create a table of values for the function $g(x) = 2f(3x) + 1$

x	6	12	18	24
$f(x)$	10	14	15	17

There are 3 steps to this transformation and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by 1/3, which means we multiply each x value by 1/3.

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation.

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values.

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

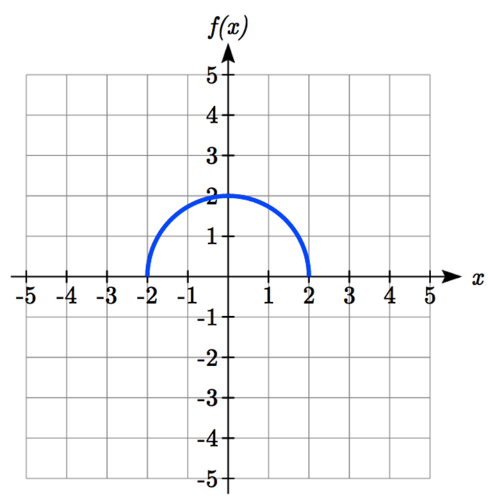
Example 20

Using the graph of *f(x)* below, sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$

To make things simpler, we'll start by factoring out the inside of the function

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3$$

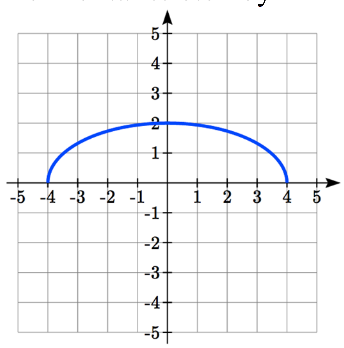
By factoring the inside, we can first horizontally stretch by 2, as indicated by the ½ on the inside of the function. Remember twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0).



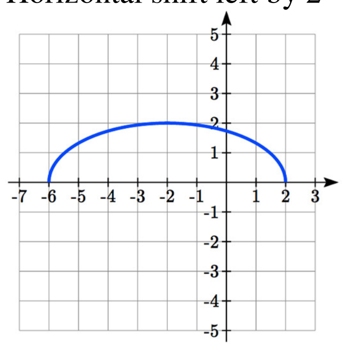
Next, we horizontally shift left by 2 units, as indicated by the $x+2$.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function.

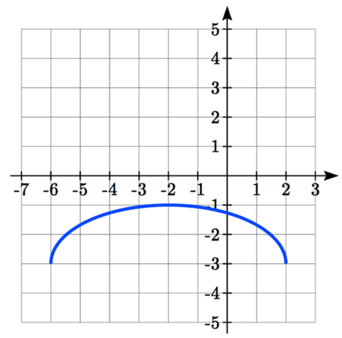
Horizontal stretch by 2



Horizontal shift left by 2

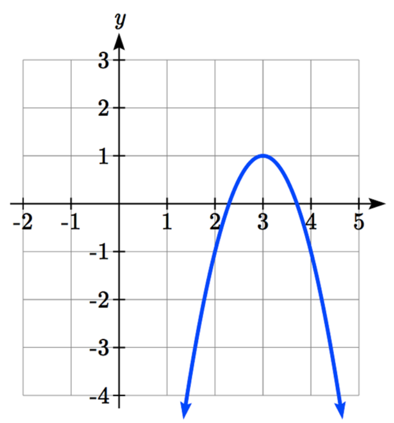


Vertical shift down 3



Example 21

Write an equation for the transformed graph of the quadratic function shown.



Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations:

The original tool kit function has been flipped over the x axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:
Vertical reflection, requiring a negative sign outside the function
Vertical Stretch *or* Horizontal Compression*
Horizontal Shift Right 3 units, which tells us to put x-3 on the inside of the function
Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

* It is unclear from the graph whether it is showing a vertical stretch or a horizontal compression. For the quadratic, it turns out we could represent it either way, so we'll use a vertical stretch. You may be able to determine the vertical stretch by observation.

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

Not everyone can see this by simply looking at the graph. If you can, great, but if not, we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch.

$f(x) = x^2$	The original function
$-f(x) = -x^2$	Vertically reflected
$-af(x) = -ax^2$	Vertically stretched
$-af(x-3) = -a(x-3)^2$	Shifted right 3
$-af(x-3)+1 = -a(x-3)^2+1$	Shifted up 1

We now know our graph is going to have an equation of the form $g(x) = -a(x-3)^2 + 1$. To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2) = -1$. Using our general formula, and substituting 2 for *x*, and -1 for *g(x)*

$$-1 = -a(2-3)^2 + 1$$
$$-1 = -a + 1$$
$$-2 = -a$$
$$2 = a$$

This tells us that to produce the graph we need a vertical stretch by two. The function that produces this graph is therefore $g(x) = -2(x-3)^2 + 1$.

Try it Now

5. Consider the linear function $g(x) = -2x + 1$. Describe its transformation in words using the identity tool kit function *f(x)* = *x* as a reference.

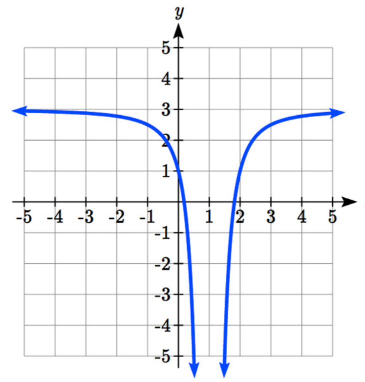
Example 22

On what interval(s) is the function $g(x) = \dfrac{-2}{(x-1)^2} + 3$ increasing and decreasing?

This is a transformation of the toolkit reciprocal squared function, $f(x) = \dfrac{1}{x^2}$:

$-2f(x) = \dfrac{-2}{x^2}$	A vertical flip and vertical stretch by 2
$-2f(x-1) = \dfrac{-2}{(x-1)^2}$	A shift right by 1
$-2f(x-1)+3 = \dfrac{-2}{(x-1)^2} + 3$	A shift up by 3

The basic reciprocal squared function is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. Because of the vertical flip, the *g(x)* function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine *g(x)* will be increasing on $(1,\infty)$ and decreasing on $(-\infty,1)$. We also could graph the transformation to help us determine these intervals.



Try it Now

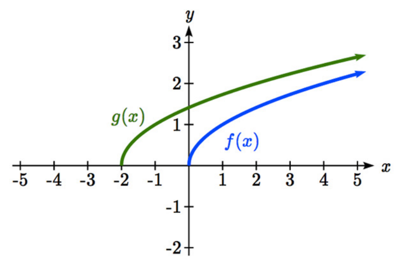
6. On what interval(s) is the function $h(t)=(t-3)^3+2$ concave up and down?

Important Topics of This Section
Transformations
Vertical Shift (up & down)
Horizontal Shifts (left & right)
Reflections over the vertical & horizontal axis
Even & Odd functions
Vertical Stretches & Compressions
Horizontal Stretches & Compressions
Combinations of Transformation

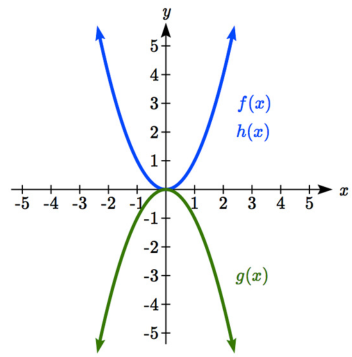
Try it Now Answers

1. $b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

2. a. Horizontal shift
   b. The function is shifted to the LEFT by 2 units.
   c. Shown to the right



3. Shown to the right
   Notice: $g(x) = f(-x)$ looks the same as $f(x)$



4. $g(x) = f\left(\frac{1}{3}x\right)$ so using the square root function we get

   $g(x) = \sqrt{\frac{1}{3}x}$

5. The identity tool kit function $f(x) = x$ has been transformed in 3 steps
   a. Vertically stretched by 2.
   b. Vertically reflected over the x axis.
   c. Vertically shifted up by 1 unit.

6. $h(t)$ is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$

Section 1.5 Exercises

Describe how each function is a transformation of the original function $f(x)$

1. $f(x-49)$
2. $f(x+43)$
3. $f(x+3)$
4. $f(x-4)$
5. $f(x)+5$
6. $f(x)+8$
7. $f(x)-2$
8. $f(x)-7$
9. $f(x-2)+3$
10. $f(x+4)-1$

11. Write a formula for $f(x)=\sqrt{x}$ shifted up 1 unit and left 2 units.
12. Write a formula for $f(x)=|x|$ shifted down 3 units and right 1 unit.
13. Write a formula for $f(x)=\frac{1}{x}$ shifted down 4 units and right 3 units.
14. Write a formula for $f(x)=\frac{1}{x^2}$ shifted up 2 units and left 4 units.

15. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	***f(x)***
-2	-2
-1	-1
0	-3
1	1
2	2

x	***g(x)***
-1	-2
0	-1
1	-3
2	1
3	2

x	***h(x)***
-2	-1
-1	0
0	-2
1	2
2	3

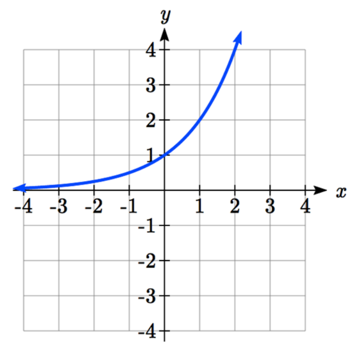
16. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	***f(x)***
-2	-1
-1	-3
0	4
1	2
2	1

x	***g(x)***
-3	-1
-2	-3
-1	4
0	2
1	1

x	***h(x)***
-2	-2
-1	-4
0	3
1	1
2	0

The graph of $f(x) = 2^x$ is shown. Sketch a graph of each transformation of $f(x)$.

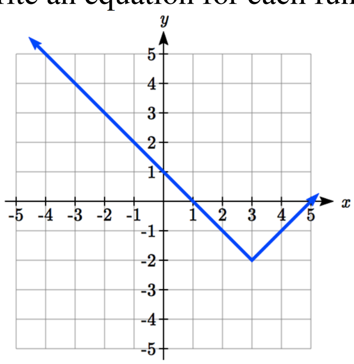


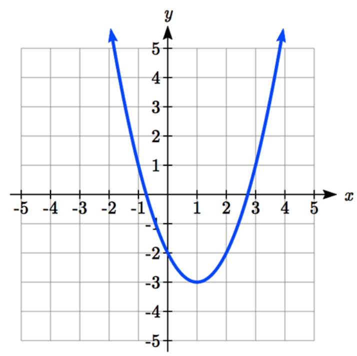
17. $g(x) = 2^x + 1$
18. $h(x) = 2^x - 3$
19. $w(x) = 2^{x-1}$
20. $q(x) = 2^{x+3}$

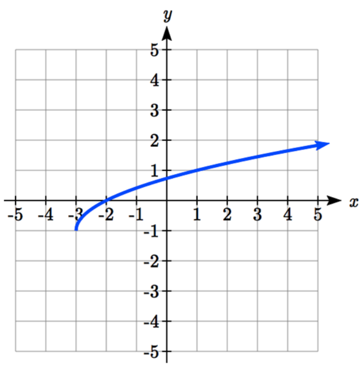
Sketch a graph of each function as a transformation of a toolkit function.

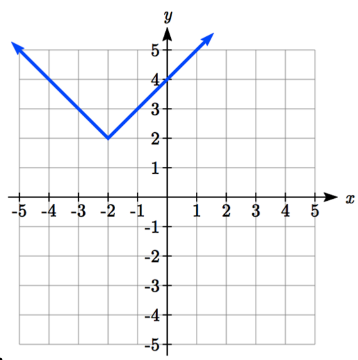
21. $f(t) = (t+1)^2 - 3$
22. $h(x) = |x-1| + 4$
23. $k(x) = (x-2)^3 - 1$
24. $m(t) = 3 + \sqrt{t+2}$

Write an equation for each function graphed below.

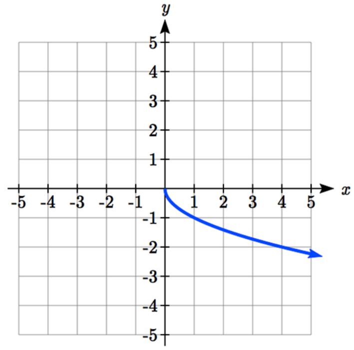
25. 

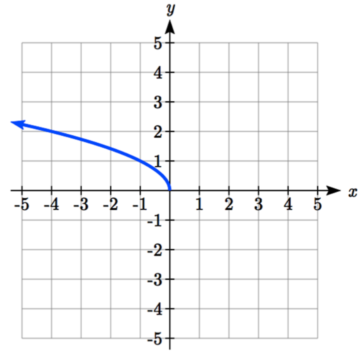
26. 

27. 

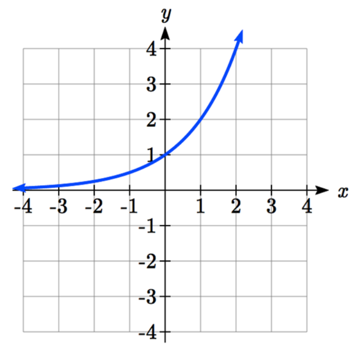
28. 

Find a formula for each of the transformations of the square root whose graphs are given below.

29. 

30. 

The graph of $f(x) = 2^x$ is shown. Sketch a graph of each transformation of $f(x)$



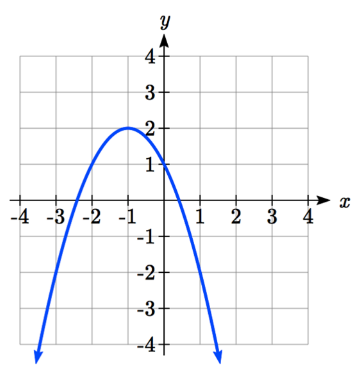
31. $g(x) = -2^x + 1$

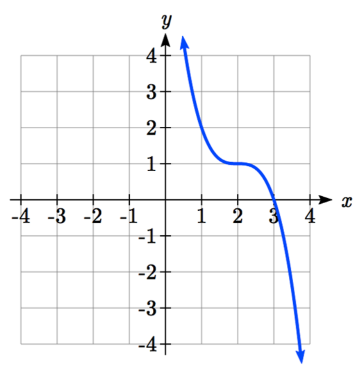
32. $h(x) = 2^{-x}$

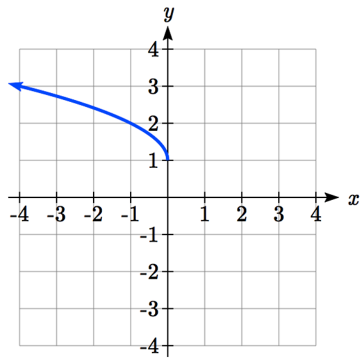
33. Starting with the graph of $f(x) = 6^x$ write the equation of the graph that results from
   a. reflecting $f(x)$ about the x-axis and the y-axis
   b. reflecting $f(x)$ about the x-axis, shifting left 2 units, and down 3 units

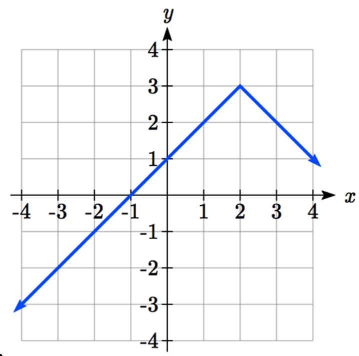
34. Starting with the graph of $f(x) = 4^x$ write the equation of the graph that results from
   a. reflecting $f(x)$ about the x-axis
   b. reflecting $f(x)$ about the y-axis, shifting right 4 units, and up 2 units

Write an equation for each function graphed below.

35. 

36. 

37. 

38. 

39. For each equation below, determine if the function is Odd, Even, or Neither.

a. $f(x) = 3x^4$

b. $g(x) = \sqrt{x}$

c. $h(x) = \dfrac{1}{x} + 3x$

40. For each equation below, determine if the function is Odd, Even, or Neither.

a. $f(x) = (x-2)^2$

b. $g(x) = 2x^4$

c. $h(x) = 2x - x^3$

Describe how each function is a transformation of the original function $f(x)$.

41. $-f(x)$

42. $f(-x)$

43. $4f(x)$

44. $6f(x)$

45. $f(5x)$

46. $f(2x)$

47. $f\left(\dfrac{1}{3}x\right)$

48. $f\left(\dfrac{1}{5}x\right)$

49. $3f(-x)$

50. $-f(3x)$

Write a formula for the function that results when the given toolkit function is transformed as described.

51. $f(x) = |x|$ reflected over the y axis and horizontally compressed by a factor of $\dfrac{1}{4}$.

52. $f(x) = \sqrt{x}$ reflected over the x axis and horizontally stretched by a factor of 2.

53. $f(x) = \dfrac{1}{x^2}$ vertically compressed by a factor of $\dfrac{1}{3}$, then shifted to the left 2 units and down 3 units.

54. $f(x)=\frac{1}{x}$ vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

55. $f(x)=x^2$ horizontally compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

56. $f(x)=x^2$ horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

Describe how each formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

57. $f(x)=4(x+1)^2-5$

58. $g(x)=5(x+3)^2-2$

59. $h(x)=-2|x-4|+3$

60. $k(x)=-3\sqrt{x}-1$

61. $m(x)=\frac{1}{2}x^3$

62. $n(x)=\frac{1}{3}|x-2|$

63. $p(x)=\left(\frac{1}{3}x\right)^2-3$

64. $q(x)=\left(\frac{1}{4}x\right)^3+1$

65. $a(x)=\sqrt{-x+4}$

66. $b(x)=\sqrt[3]{-x-6}$

Determine the interval(s) on which the function is increasing and decreasing.

67. $f(x)=4(x+1)^2-5$

68. $g(x)=5(x+3)^2-2$

69. $a(x)=\sqrt{-x+4}$

70. $k(x)=-3\sqrt{x}-1$

Determine the interval(s) on which the function is concave up and concave down.

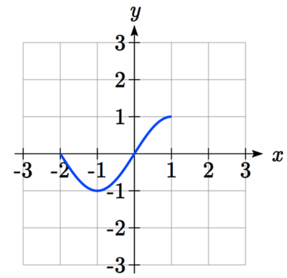
71. $m(x)=-2(x+3)^3+1$

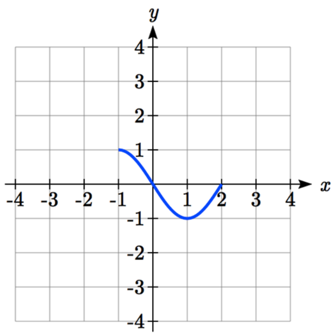
72. $b(x)=\sqrt[3]{-x-6}$

73. $p(x)=\left(\frac{1}{3}x\right)^2-3$

74. $k(x)=-3\sqrt{x}-1$

The function $f(x)$ is graphed here. Write an equation for each graph below as a transformation of $f(x)$.





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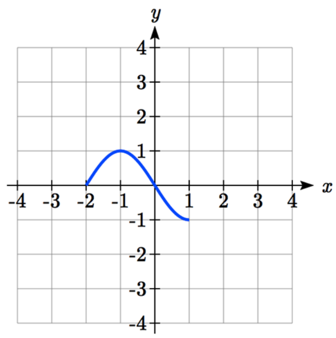

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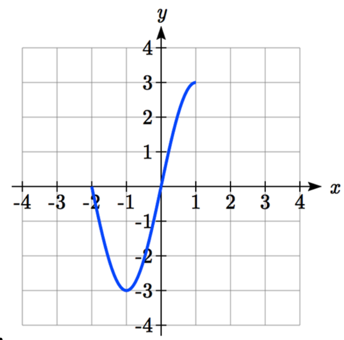

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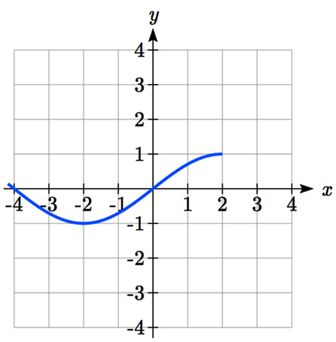

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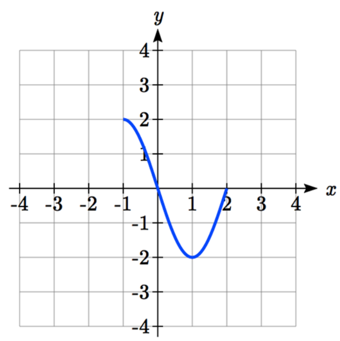

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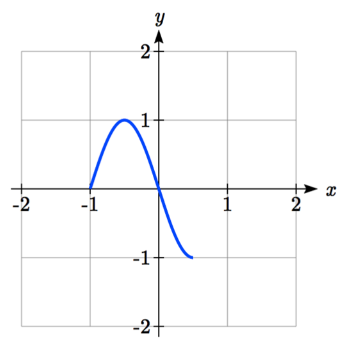

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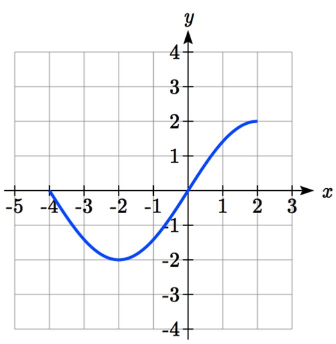

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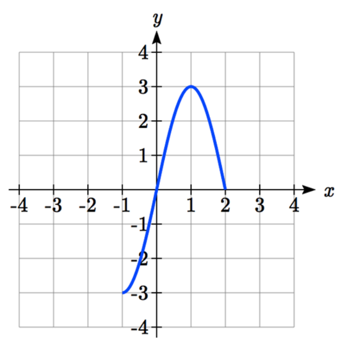

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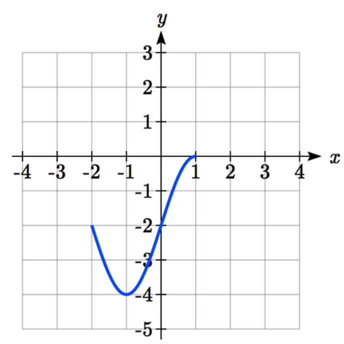

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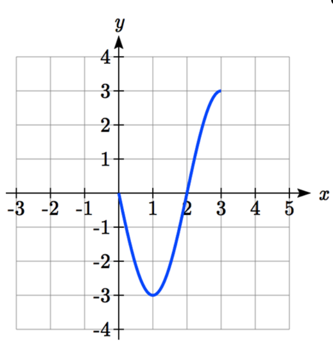

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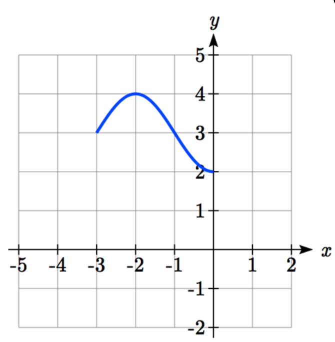

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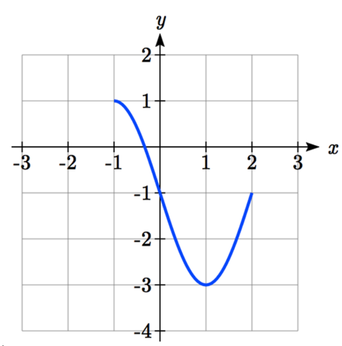
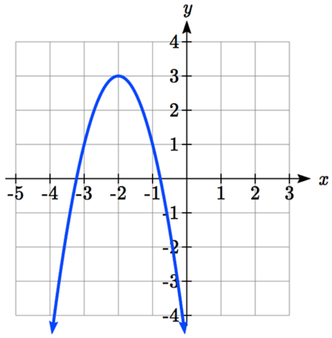


86.

Write an equation for each transformed toolkit function graphed below.



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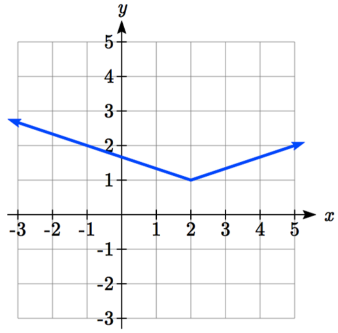

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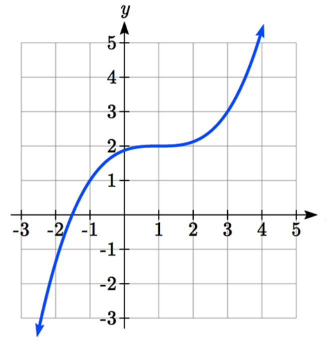

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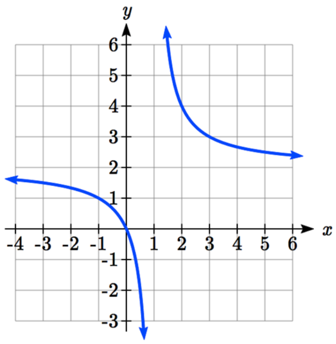

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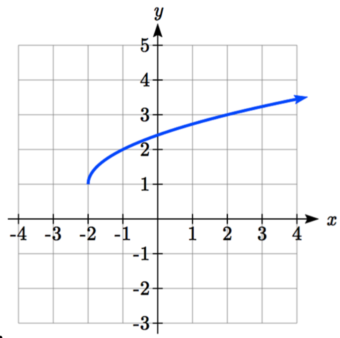

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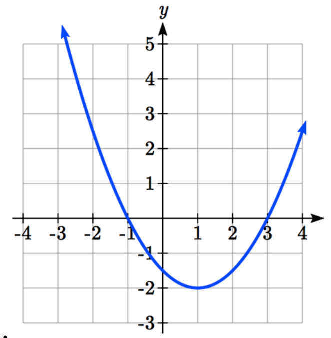

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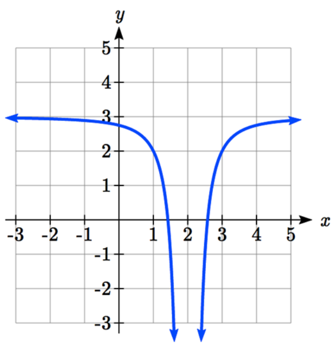

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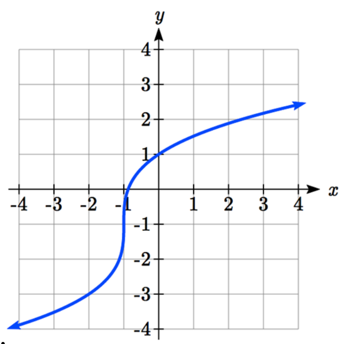

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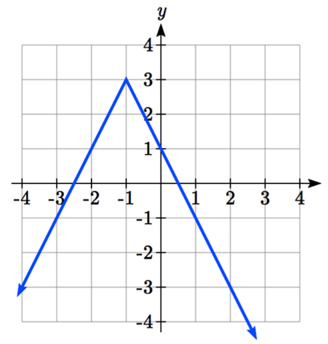

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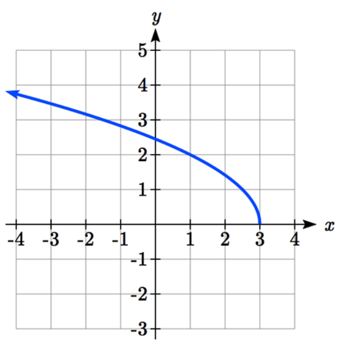

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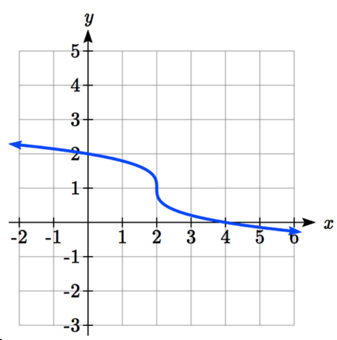

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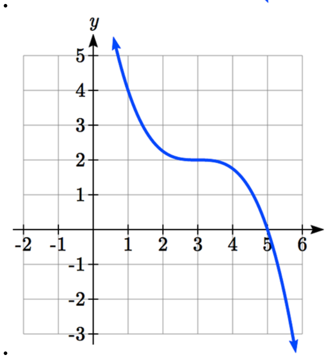
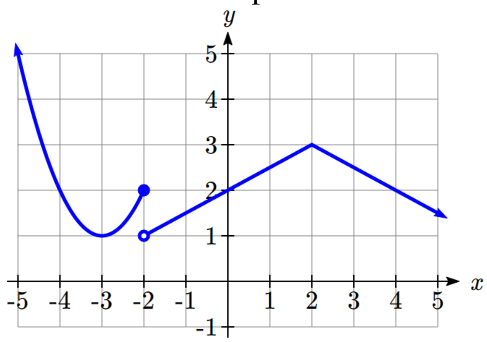
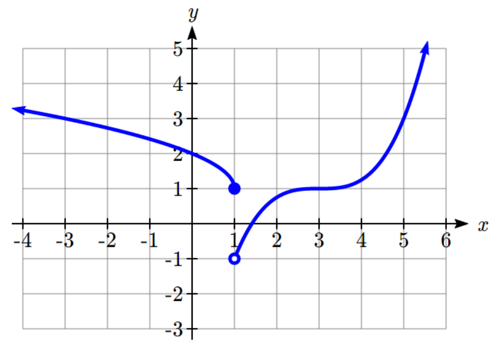
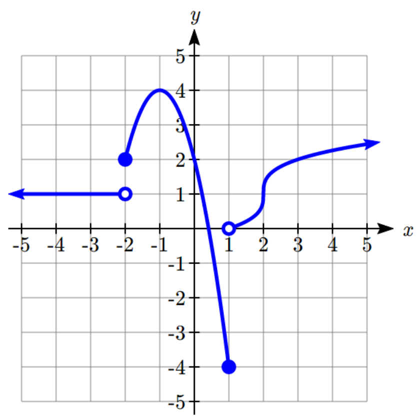


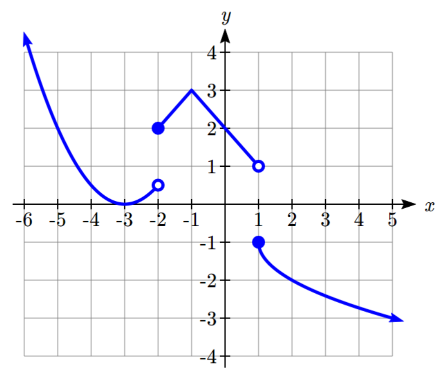
98.

Write a formula for the piecewise function graphed below.

99. 

100. 

101. 

102. 

103. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \le x \le 6$ and the range of $f(x)$ is $-3 \le y \le 5$. [UW]

a. What is the domain of $f(2(x-3))$?

b. What is the range of $f(2(x-3))$?

c. What is the domain of $2f(x)-3$?

d. What is the range of $2f(x)-3$?

e. Can you find constants B and C so that the domain of $f(B(x-C))$ is $8 \le x \le 9$?

f. Can you find constants A and D so that the range of $Af(x)+D$ is $0 \le y \le 1$?

Chapter 3: Polynomial and Rational Functions

Section 3.1 Power Functions & Polynomial Functions

A square is cut out of cardboard, with each side having length L. If we wanted to write a function for the area of the square, with L as the input and the area as output, you may recall that the area of a rectangle can be found by multiplying the length times the width. Since our shape is a square, the length & the width are the same, giving the formula:
$A(L) = L \cdot L = L^2$

Likewise, if we wanted a function for the volume of a cube with each side having some length L, you may recall volume of a rectangular box can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:
$V(L) = L \cdot L \cdot L = L^3$

These two functions are examples of **power functions**, functions that are some power of the variable.

Power Function

A **power function** is a function that can be represented in the form
$f(x) = x^p$
Where the base is a variable and the exponent, p, is a number.

Example 1

Which of our toolkit functions are power functions?

The constant and identity functions are power functions, since they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are both power functions with whole number powers: $f(x) = x^2$ and $f(x) = x^3$.

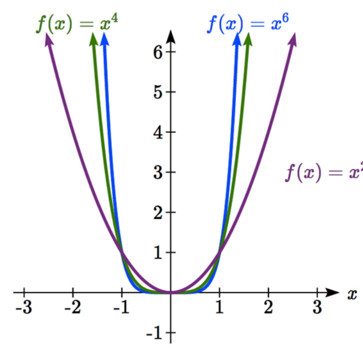
The reciprocal and reciprocal squared functions are both power functions with negative whole number powers since they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are both power functions with fractional powers since they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Try it Now

1. What point(s) do the toolkit power functions have in common?

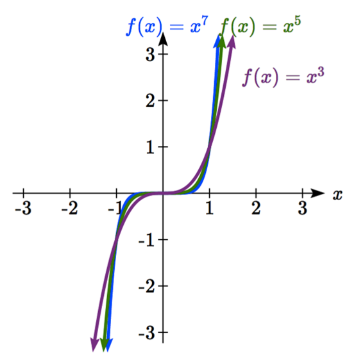
Characteristics of Power Functions



Shown to the right are the graphs of $f(x) = x^2$, $f(x) = x^4$, and $f(x) = x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to the quadratic toolkit, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is ∞, and $-\infty$ for negative infinity. When we say that "*x* approaches infinity", which can be symbolically written as $x \to \infty$, we are describing a behavior – we are saying that *x* is getting large in the positive direction.

With the even power functions, as the *x* becomes large in either the positive or negative direction, the output values become very large positive numbers. Equivalently, we could describe this by saying that as *x* approaches positive or negative infinity, the *f(x)* values approach positive infinity. In symbolic form, we could write: as $x \to \pm\infty$, $f(x) \to \infty$.



Shown here are the graphs of $f(x) = x^3$, $f(x) = x^5$, and $f(x) = x^7$, all odd whole number powers. Notice all these graphs look similar to the cubic toolkit, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.

For these odd power functions, as *x* approaches negative infinity, *f(x)* approaches negative infinity. As *x* approaches positive infinity, *f(x)* approaches positive infinity. In symbolic form we write: as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Long Run Behavior

The behavior of the graph of a function as the input takes on large negative values, $x \to -\infty$, and large positive values, $x \to \infty$, is referred to as the **long run behavior** of the function.

Example 2

Describe the long run behavior of the graph of $f(x) = x^8$.

Since $f(x) = x^8$ has a whole, even power, we would expect this function to behave somewhat like the quadratic function. As the input gets large positive or negative, we would expect the output to grow without bound in the positive direction. In symbolic form, as $x \to \pm\infty$, $f(x) \to \infty$.

Example 3

Describe the long run behavior of the graph of $f(x) = -x^9$

Since this function has a whole odd power, we would expect it to behave somewhat like the cubic function. The negative in front of the x^9 will cause a vertical reflection, so as the inputs grow large positive, the outputs will grow large in the negative direction, and as the inputs grow large negative, the outputs will grow large in the positive direction. In symbolic form, for the long run behavior we would write: as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$.

You may use words or symbols to describe the long run behavior of these functions.

Try it Now

2. Describe in words and symbols the long run behavior of $f(x) = -x^4$

Treatment of the rational and radical forms of power functions will be saved for later.

Polynomials

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. If we wanted to write a formula for the area covered by the oil slick, we could do so by composing two functions together. The first is a formula for the radius, *r*, of the spill, which depends on the number of weeks, *w*, that have passed.

Hopefully you recognized that this relationship is linear:
$r(w) = 24 + 8w$

We can combine this with the formula for the area, A, of a circle:
$A(r) = \pi r^2$

Composing these functions gives a formula for the area in terms of weeks:
$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2$

Multiplying this out gives the formula
$A(w) = 576\pi + 384\pi w + 64\pi w^2$

This formula is an example of a **polynomial**. A polynomial is simply the sum of terms each consisting of a transformed power function with positive whole number power.

Terminology of Polynomial Functions

A **polynomial** is function that can be written as $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Each of the a_i constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any $a_i x^i$. Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending.
$f(x) = a_n x^n + \ldots.. + a_2 x^2 + a_1 x + a_0$

Example 4

Identify the degree, leading term, and leading coefficient of these polynomials:

a) $f(x) = 3 + 2x^2 - 4x^3$ b) $g(t) = 5t^5 - 2t^3 + 7t$ c) $h(p) = 6p - p^3 - 2$

a) For the function *f(x)*, the degree is 3, the highest power on *x*. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

b) For *g(t)*, the degree is 5, the leading term is $5t^5$, and the leading coefficient is 5.

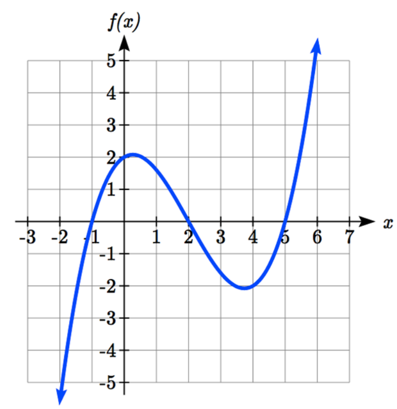
c) For *h(p)*, the degree is 3, the leading term is $-p^3$, so the leading coefficient is -1.

Long Run Behavior of Polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

Example 5

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as $x \to \infty$, $f(x) \to \infty$. Similarly, as $x \to -\infty$, $f(x) \to -\infty$.

In words, we could say that as *x* values approach infinity, the function values approach infinity, and as *x* values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

Try it Now

3. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ use your algebra skills to write the function in standard polynomial form (as a sum of terms) and determine the leading term, degree, and long run behavior of the function.

Short Run Behavior

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

Example 6

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the vertical and horizontal intercepts.

The vertical intercept occurs when the input is zero.
$f(0) = (0-2)(0+1)(0-4) = 8$.

The graph crosses the vertical axis at the point (0, 8).

The horizontal intercepts occur when the output is zero.
$0 = (x-2)(x+1)(x-4)$ when x = 2, -1, or 4.
f(x) has zeros, or roots, at x = 2, -1, and 4.

The graph crosses the horizontal axis at the points (2, 0), (-1, 0), and (4, 0)

Notice that the polynomial in the previous example, which would be degree three if multiplied out, had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it – the reasons will become clear later in this chapter.

Intercepts and Turning Points of Polynomials

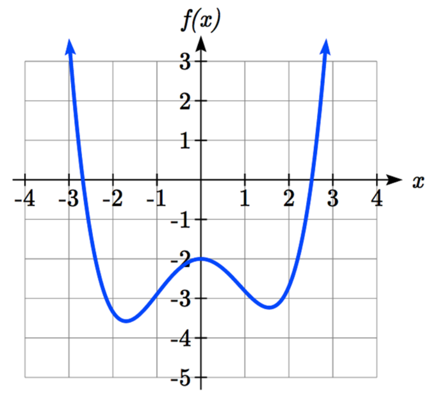
A polynomial of degree n will have:

At most n horizontal intercepts. An odd degree polynomial will always have at least one.

At most $n-1$ turning points

Example 7

What can we conclude about the graph of the polynomial shown here?



Based on the long run behavior, with the graph becoming large positive on both ends of the graph, we can determine that this is the graph of an even degree polynomial. The graph has 2 horizontal intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4, so it is probably a fourth degree polynomial.

Try it Now

4. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the short run behavior.

Important Topics of this Section
Power Functions
Polynomials
Coefficients
Leading coefficient
Term
Leading Term
Degree of a polynomial
Long run behavior
Short run behavior

Try it Now Answers

1. (0, 0) and (1, 1) are common to all power functions.
2. As *x* approaches positive and negative infinity, *f(x)* approaches negative infinity: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the vertical flip.
3. The leading term is $0.2x^3$, so it is a degree 3 polynomial.
   As *x* approaches infinity (or gets very large in the positive direction) *f(x)* approaches infinity; as *x* approaches negative infinity (or gets very large in the negative direction) *f(x)* approaches negative infinity. (Basically the long run behavior is the same as the cubic function).
4. Horizontal intercepts are (2, 0) (-1, 0) and (5, 0), the vertical intercept is (0, 2) and there are 2 turns in the graph.

Section 3.1 Exercises

Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

1. $f(x) = x^4$
2. $f(x) = x^6$
3. $f(x) = x^3$
4. $f(x) = x^5$
5. $f(x) = -x^2$
6. $f(x) = -x^4$
7. $f(x) = -x^7$
8. $f(x) = -x^9$

Find the degree and leading coefficient of each polynomial

9. $4x^7$
10. $5x^6$
11. $5 - x^2$
12. $6 + 3x - 4x^3$
13. $-2x^4 - 3x^2 + x - 1$
14. $6x^5 - 2x^4 + x^2 + 3$
15. $(2x+3)(x-4)(3x+1)$
16. $(3x+1)(x+1)(4x+3)$

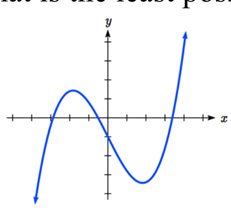
Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

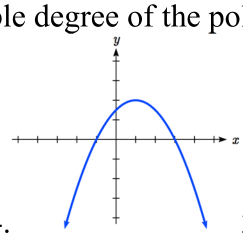
17. $-2x^4 - 3x^2 + x - 1$
18. $6x^5 - 2x^4 + x^2 + 3$
19. $3x^2 + x - 2$
20. $-2x^3 + x^2 - x + 3$

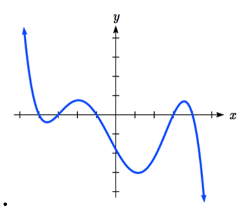
21. What is the maximum number of x-intercepts and turning points for a polynomial of degree 5?

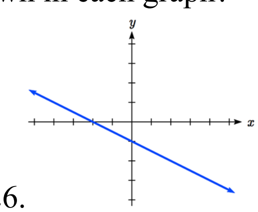
22. What is the maximum number of x-intercepts and turning points for a polynomial of degree 8?

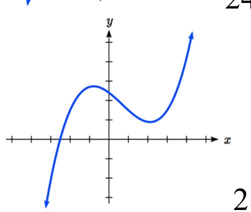
What is the least possible degree of the polynomial function shown in each graph?

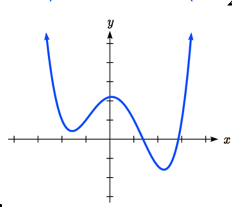
23. 

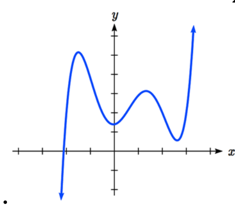
24. 

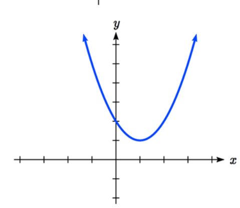
25. 

26. 

27. 

28. 

29. 

30. 

Find the vertical and horizontal intercepts of each function.

31. $f(t) = 2(t-1)(t+2)(t-3)$
32. $f(x) = 3(x+1)(x-4)(x+5)$
33. $g(n) = -2(3n-1)(2n+1)$
34. $k(u) = -3(4-n)(4n+3)$

Section 3.3 Graphs of Polynomial Functions

In the previous section, we explored the short run behavior of quadratics, a special case of polynomials. In this section, we will explore the short run behavior of polynomials in general.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4^{th} degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2) The polynomial is given in factored form.
3) Technology is used to determine the intercepts.

Other techniques for finding the intercepts of general polynomials will be explored in the next section.

Example 1

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$x^6 - 3x^4 + 2x^2 = 0$ Factoring out the greatest common factor

$x^2\left(x^4 - 3x^2 + 2\right) = 0$ Factoring the inside as a quadratic in x^2

$x^2\left(x^2 - 1\right)\left(x^2 - 2\right) = 0$ Then break apart to find solutions

		$\left(x^2 - 1\right) = 0$		$\left(x^2 - 2\right) = 0$
$x^2 = 0$	or	$x^2 = 1$	or	$x^2 = 2$
$x = 0$		$x = \pm 1$		$x = \pm\sqrt{2}$

This gives us 5 horizontal intercepts.

Example 2

Find the vertical and horizontal intercepts of $g(t) = (t-2)^2(2t+3)$

The vertical intercept can be found by evaluating *g(0)*.

$g(0) = (0-2)^2(2(0)+3) = 12$

The horizontal intercepts can be found by solving *g(t)* = 0

$(t-2)^2(2t+3) = 0$ Since this is already factored, we can break it apart:

$(t-2)^2 = 0$ or $(2t+3) = 0$

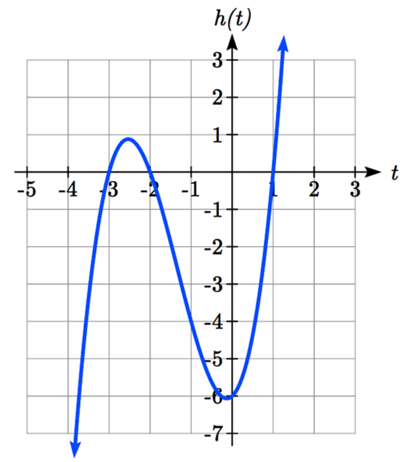
$t - 2 = 0$ $t = \frac{-3}{2}$

$t = 2$

We can always check our answers are reasonable by graphing the polynomial.

Example 3

Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$



Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

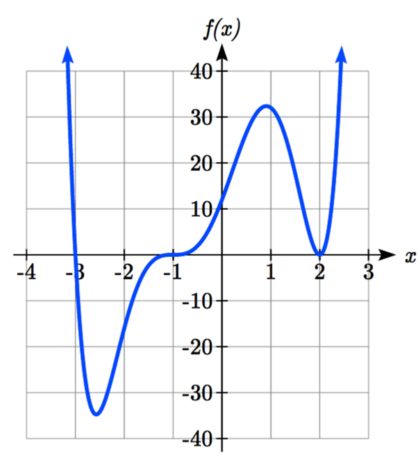
Graphing this function, it appears there are horizontal intercepts at *t* = -3, -2, and 1.

We could check these are correct by plugging in these values for *t* and verifying that $h(-3) = h(-2) = h(1) = 0$.

Try it Now

1. Find the vertical and horizontal intercepts of the function $f(t) = t^4 - 4t^2$.

Graphical Behavior at Intercepts



If we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept *x* = -3, coming from the $(x+3)$ factor of the polynomial, the graph passes directly through the horizontal intercept.

The factor $(x+3)$ is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the horizontal intercept $x = 2$, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity** 2.

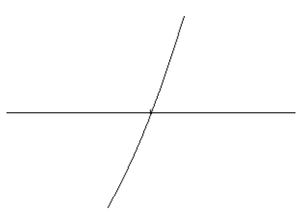
At the horizontal intercept $x = -1$, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit x^3 has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Graphical Behavior of Polynomials at Horizontal Intercepts

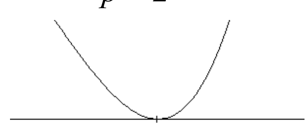
If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

$p = 1$



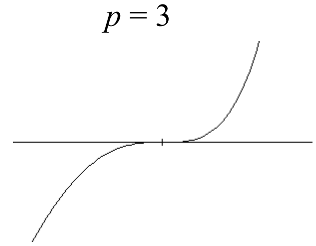
Single zero
Multiplicity 1

$p = 2$



Double zero
Multiplicity 2

$p = 3$



Triple zero
Multiplicity 3

For higher even powers 4,6,8 etc.… the graph will still bounce off the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc… the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 4

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

This graph has two horizontal intercepts. At x = -3, the factor is squared, indicating the graph will bounce at this horizontal intercept. At x = 5, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:
As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

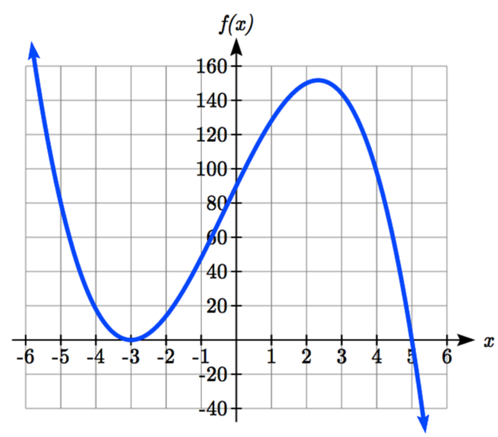
At (-3, 0) the graph bounces off the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0).

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we can verify the shape of the graph.



Try it Now

2. Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 5

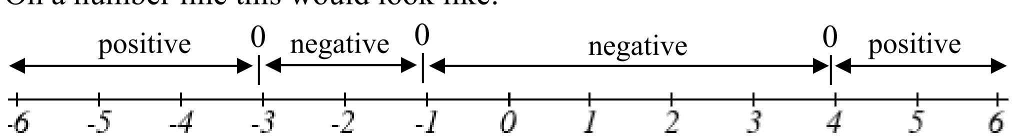
Solve $(x+3)(x+1)^2(x-4) > 0$

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4) = 0$, which has solutions at x = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	*f(* test value*)*	>0 or <0?
x < -3	-4	72	> 0
-3 < x < -1	-2	-6	< 0
-1 < x < 4	0	-12	< 0
x > 4	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when x < -3 or x > 4, or in interval notation, $(-\infty, -3) \cup (4, \infty)$

We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example

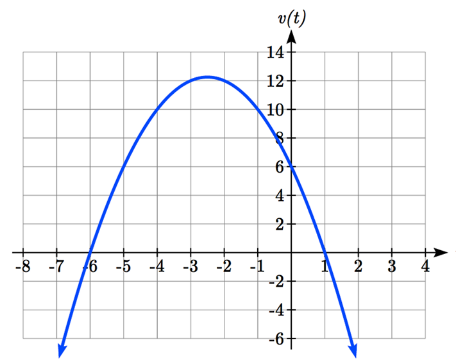
Example 6

Find the domain of the function $v(t) = \sqrt{6 - 5t - t^2}$.

A square root is only defined when the quantity we are taking the square root of, the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when $6 - 5t - t^2 \geq 0$.

We start by solving the equality $6 - 5t - t^2 = 0$. While we could use the quadratic formula, this equation factors nicely to $(6 + t)(1 - t) = 0$, giving horizontal intercepts t = 1 and t = -6.

Sketching a graph of this quadratic will allow us to determine when it is positive.



From the graph we can see this function is positive for inputs between the intercepts. So $6 - 5t - t^2 \geq 0$ for $-6 \leq t \leq 1$, and this will be the domain of the *v(t)* function.

Writing Equations using Intercepts

Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form a function that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

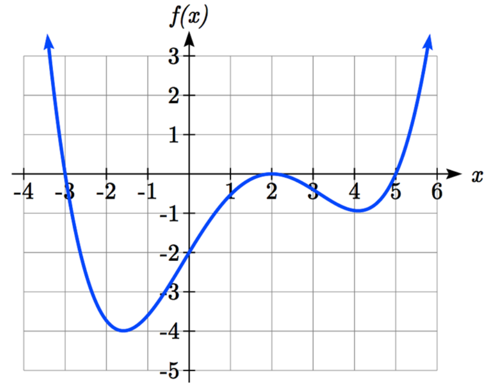
If a polynomial has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form

$$f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$$

where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the horizontal intercept.

Example 7

Write a formula for the polynomial function graphed here.



This graph has three horizontal intercepts: $x = -3$, 2, and 5. At $x = -3$ and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$ the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree (quadratic).

Together, this gives us:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for a:

$$-2 = a(0+3)(0-2)^2(0-5)$$

$$-2 = -60a$$

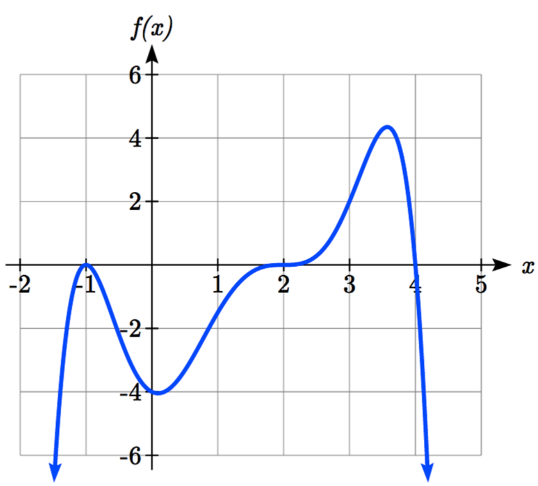
$$a = \frac{1}{30}$$

The graphed polynomial appears to represent the function

$$f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5).$$

Try it Now

3. Given the graph, write a formula for the function shown.

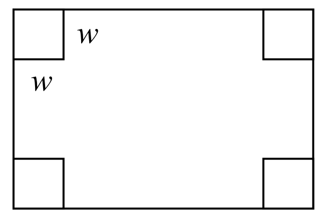


Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 8

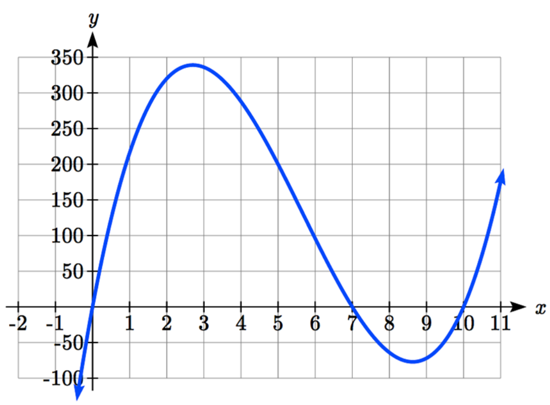
An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.



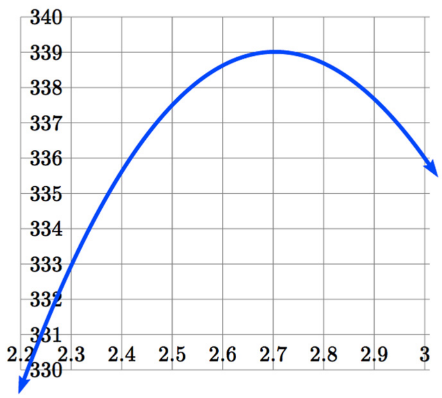
We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, *w*.

Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(120-2w)$ cm rectangle for the base of the box, and the box will be *w* cm tall. This gives the volume:
$V(w) = (14-2w)(20-2w)w = 280w - 68w^2 + 4w^3$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for *w*: values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

Try it Now

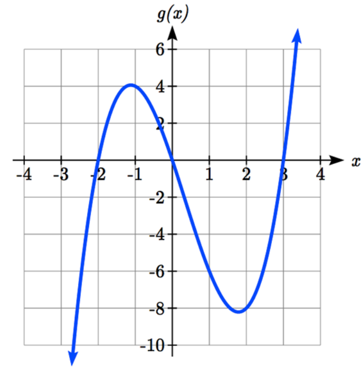
4. Use technology to find the maximum and minimum values on the interval [-1, 4] of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Important Topics of this Section
Short Run Behavior
Intercepts (Horizontal & Vertical)
Methods to find Horizontal intercepts
Factoring Methods
Factored Forms
Technology
Graphical Behavior at intercepts
Single, Double and Triple zeros (or multiplicity 1, 2, and 3 behaviors)
Solving polynomial inequalities using test values & graphing techniques
Writing equations using intercepts
Estimating extrema

Try it Now Answers

1. Vertical intercept (0, 0). $0 = t^4 - 4t^2$ factors as $0 = t^2\left(t^2 - 4\right) = t^2\left(t-2\right)\left(t+2\right)$
   Horizontal intercepts (0, 0), (-2, 0), (2, 0)

2. Vertical intercept (0, 0),
   Horizontal intercepts (-2, 0), (0, 0), (3, 0)
   The function is negative on $(-\infty$, -2) and (0, 3)
   The function is positive on (-2, 0) and $(3, \infty)$
   The leading term is x^3 so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$



3. Double zero at x=-1, triple zero at x=2. Single zero at x=4.
   $f(x) = a(x-2)^3(x+1)^2(x-4)$. Substituting (0,-4) and solving for a,
   $f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

4. The minimum occurs at approximately the point (0, -6.5), and the maximum occurs at approximately the point (3.5, 7).

Section 3.3 Exercises

Find the C and t intercepts of each function.

1. $C(t) = 2(t-4)(t+1)(t-6)$
2. $C(t) = 3(t+2)(t-3)(t+5)$
3. $C(t) = 4t(t-2)^2(t+1)$
4. $C(t) = 2t(t-3)(t+1)^2$
5. $C(t) = 2t^4 - 8t^3 + 6t^2$
6. $C(t) = 4t^4 + 12t^3 - 40t^2$

Use your calculator or other graphing technology to solve graphically for the zeros of the function.

7. $f(x) = x^3 - 7x^2 + 4x + 30$
8. $g(x) = x^3 - 6x^2 + x + 28$

Find the long run behavior of each function as $t \to \infty$ and $t \to -\infty$

9. $h(t) = 3(t-5)^3(t-3)^3(t-2)$
10. $k(t) = 2(t-3)^2(t+1)^3(t+2)$
11. $p(t) = -2t(t-1)(3-t)^2$
12. $q(t) = -4t(2-t)(t+1)^3$

Sketch a graph of each equation.

13. $f(x) = (x+3)^2(x-2)$
14. $g(x) = (x+4)(x-1)^2$
15. $h(x) = (x-1)^3(x+3)^2$
16. $k(x) = (x-3)^3(x-2)^2$
17. $m(x) = -2x(x-1)(x+3)$
18. $n(x) = -3x(x+2)(x-4)$

Solve each inequality.

19. $(x-3)(x-2)^2 > 0$
20. $(x-5)(x+1)^2 > 0$
21. $(x-1)(x+2)(x-3) < 0$
22. $(x-4)(x+3)(x+6) < 0$

Find the domain of each function.

23. $f(x) = \sqrt{-42 + 19x - 2x^2}$
24. $g(x) = \sqrt{28 - 17x - 3x^2}$
25. $h(x) = \sqrt{4 - 5x + x^2}$
26. $k(x) = \sqrt{2 + 7x + 3x^2}$
27. $n(x) = \sqrt{(x-3)(x+2)^2}$
28. $m(x) = \sqrt{(x-1)^2(x+3)}$
29. $p(t) = \dfrac{1}{t^2 + 2t - 8}$
30. $q(t) = \dfrac{4}{x^2 - 4x - 5}$

Write an equation for a polynomial the given features.

31. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. Vertical intercept at (0, -4)

32. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. Vertical intercept at (0, 6)

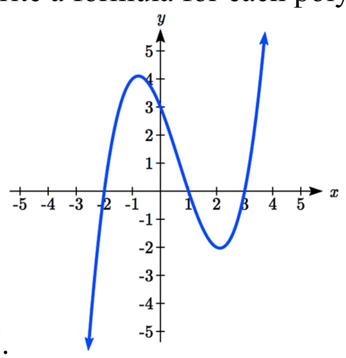
33. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. Vertical intercept at (0, 9)

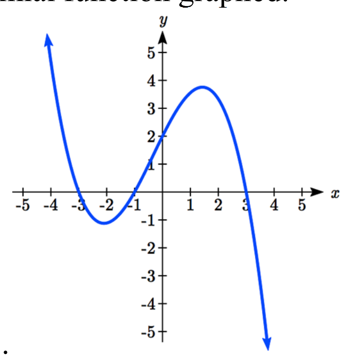
34. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. Vertical intercept at (0, -3)

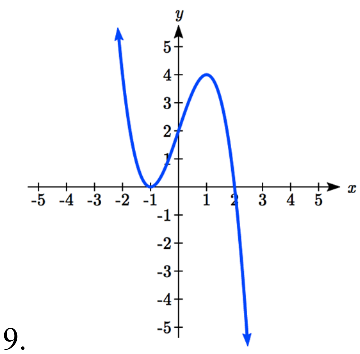
35. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point (2, 15)

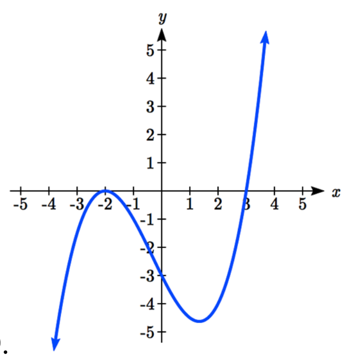
36. Degree 5. Single zero at $x = -2$ and $x = 3$, and triple zero at $x = 1$. Passes through the point (2, 4)

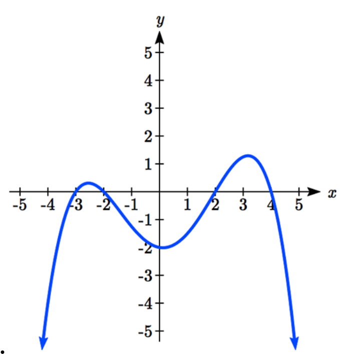
Write a formula for each polynomial function graphed.

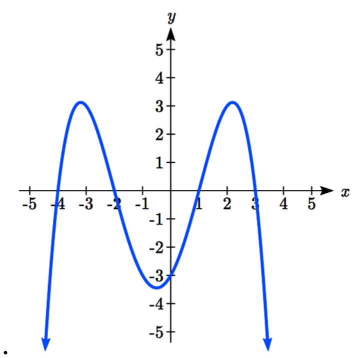
37. 

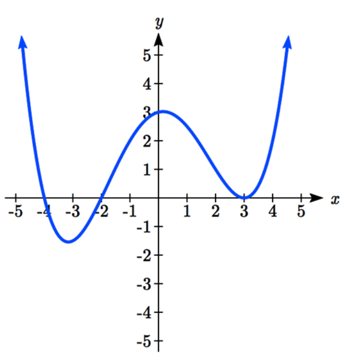
38. 

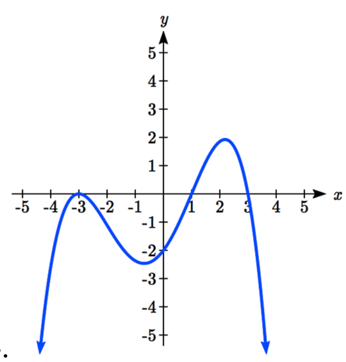
39. 

40. 

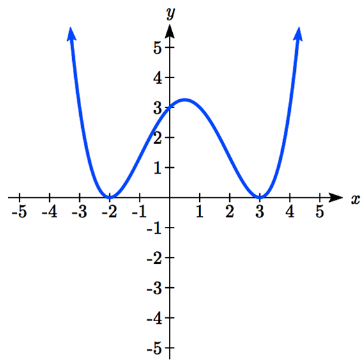
41. 

42. 

43. 

44. 

Write a formula for each polynomial function graphed.



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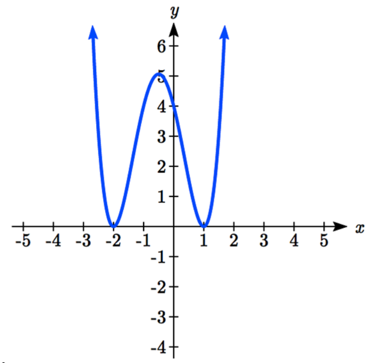

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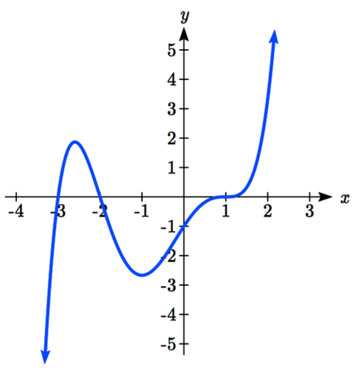

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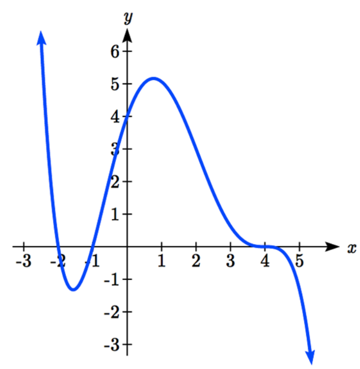

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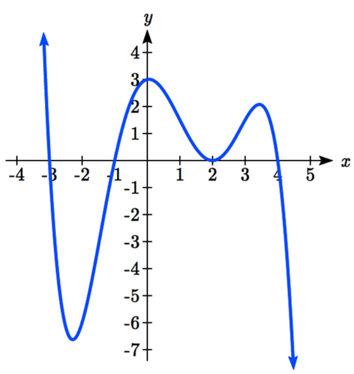

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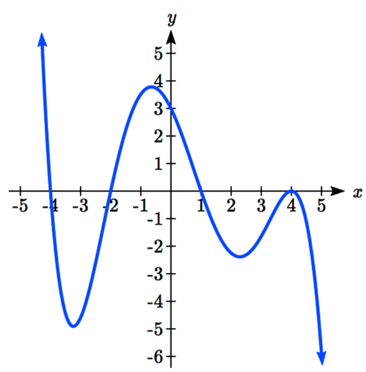


50.

51. A rectangle is inscribed with its base on the x axis and its upper corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle that has the greatest possible area?

52. A rectangle is inscribed with its base on the x axis and its upper corners on the curve $y = 16 - x^4$. What are the dimensions of such a rectangle that has the greatest possible area?

Section 3.4 Factor Theorem and Remainder Theorem

In the last section, we limited ourselves to finding the intercepts, or zeros, of polynomials that factored simply, or we turned to technology. In this section, we will look at algebraic techniques for finding the zeros of polynomials like $h(t) = t^3 + 4t^2 + t - 6$.

Long Division

In the last section we saw that we could write a polynomial as a product of factors, each corresponding to a horizontal intercept. If we knew that $x = 2$ was an intercept of the polynomial $x^3 + 4x^2 - 5x - 14$, we might guess that the polynomial could be factored as $x^3 + 4x^2 - 5x - 14 = (x - 2)(\text{something})$. To find that "something," we can use polynomial division.

Example 1

Divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$

Start by writing the problem out in long division form

$$x - 2\overline{)x^3 + 4x^2 - 5x - 14}$$

Now we divide the leading terms: $x^3 \div x = x^2$. It is best to align it above the same-powered term in the dividend. Now, multiply that x^2 by $x - 2$ and write the result below the dividend.

$$\begin{array}{r} x^2 \\ x - 2\overline{)x^3 + 4x^2 - 5x - 14} \\ x^3 - 2x^2 \end{array}$$

Now subtract that expression from the dividend.

$$\begin{array}{r} x^2 \\ x - 2\overline{)\;x^3 + 4x^2 - 5x - 14} \\ \underline{-(x^3 - 2x^2)} \\ 6x^2 - 5x - 14 \end{array}$$

Again, divide the leading term of the remainder by the leading term of the divisor. $6x^2 \div x = 6x$. We add this to the result, multiply $6x$ by $x - 2$, and subtract.

$$\begin{array}{r|l} & x^2+6x \\ x-2) & x^3+4x^2-5x-14 \\ & -\left(x^3-2x^2\right) \\ & 6x^2-5x-14 \\ & -\left(6x^2-12x\right) \\ & 7x-14 \end{array}$$

Repeat the process one last time.

$$\begin{array}{r|l} & x^2+6x+7 \\ x-2) & x^3+4x^2-5x-14 \\ & -\left(x^3-2x^2\right) \\ & 6x^2-5x-14 \\ & -\left(6x^2-12x\right) \\ & 7x-14 \\ & -(7x-14) \\ & 0 \end{array}$$

This tells us $x^3+4x^2-5x-14$ divided by $x-2$ is x^2+6x+7, with a remainder of zero. This also means that we can factor $x^3+4x^2-5x-14$ as $(x-2)\left(x^2+6x+7\right)$.

This gives us a way to find the intercepts of this polynomial.

Example 2

Find the horizontal intercepts of $h(x)=x^3+4x^2-5x-14$.

To find the horizontal intercepts, we need to solve *h(x)* = 0. From the previous example, we know the function can be factored as $h(x)=(x-2)\left(x^2+6x+7\right)$.

$h(x)=(x-2)\left(x^2+6x+7\right)=0$ when $x = 2$ or when $x^2+6x+7=0$. This doesn't factor nicely, but we could use the quadratic formula to find the remaining two zeros.

$$x=\frac{-6\pm\sqrt{6^2-4(1)(7)}}{2(1)}=-3\pm\sqrt{2}.$$

The horizontal intercepts will be at $(2,0)$, $\left(-3-\sqrt{2},0\right)$, and $\left(-3+\sqrt{2},0\right)$.

Try it Now

1. Divide $2x^3 - 7x + 3$ by $x + 3$ using long division.

The Factor and Remainder Theorems

When we divide a polynomial, *p(x)* by some divisor polynomial *d(x)*, we will get a quotient polynomial *q(x)* and possibly a remainder *r(x)*. In other words, $p(x) = d(x)q(x) + r(x)$.

Because of the division, the remainder will either be zero, or a polynomial of lower degree than *d(x)*. Because of this, if we divide a polynomial by a term of the form $x - c$, then the remainder will be zero or a constant.

If $p(x) = (x - c)q(x) + r$, then $p(c) = (c - c)q(c) + r = 0 + r = r$, which establishes the Remainder Theorem.

The Remainder Theorem

If $p(x)$ is a polynomial of degree 1 or greater and *c* is a real number, then when *p(x)* is divided by $x - c$, the remainder is $p(c)$.

If $x - c$ is a factor of the polynomial *p*, then $p(x) = (x - c)q(x)$ for some polynomial *q*. Then $p(c) = (c - c)q(c) = 0$, showing *c* is a zero of the polynomial. This shouldn't surprise us - we already knew that if the polynomial factors it reveals the roots.

If $p(c) = 0$, then the remainder theorem tells us that if *p* is divided by $x - c$, then the remainder will be zero, which means $x - c$ is a factor of *p*.

The Factor Theorem

If $p(x)$ is a nonzero polynomial, then the real number *c* is a zero of $p(x)$ if and only if $x - c$ is a factor of $p(x)$.

Synthetic Division

Since dividing by $x - c$ is a way to check if a number is a zero of the polynomial, it would be nice to have a faster way to divide by $x - c$ than having to use long division every time. Happily, quicker ways have been discovered.

Let's look back at the long division we did in Example 1 and try to streamline it. First, let's change all the subtractions into additions by distributing through the negatives.

$$\begin{array}{r}
x^2 + 6x + 7 \\
x-2 \,\big)\overline{\, x^3 + 4x^2 - 5x - 14} \\
\underline{-x^3 + 2x^2 } \\
6x^2 - 5x - 14 \\
\underline{-6x^2 + 12x } \\
7x - 14 \\
\underline{-7x + 14} \\
0
\end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$\begin{array}{r}
x^2 + 6x + 7 \\
x-2 \,\big)\overline{\, x^3 + 4x^2 - 5x - 14} \\
\underline{2x^2 } \\
6x^2 \\
\underline{12x } \\
7x \\
\underline{14} \\
0
\end{array}$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$\begin{array}{rrrrr}
 & & x^2 & + 6x & + 7 \\
x-2 \,\big) & x^3 & + 4x^2 & - 5x & - 14 \\
 & & 2x^2 & 12x & 14 \\ \hline
 & x^3 & 6x^2 & 7x & 0
\end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial.

This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} x-2 & x^3 & +4x^2 & -5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the −2 in $x - 2$, then by −1 when we changed the subtraction to addition. Multiplying by −2 then by −1 is the same as multiplying by 2, so we replace the −2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division** tableau for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we write 2 in the place of the divisor and the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. Then "bring down" the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the −5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the −14 to get 0.

2	1	4	-5	-14
	↓	2	12	14
	1	6	7	

2	1	4	-5	-14
	↓	2	12	14
	1	6	7	[0]

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works only for these kinds of divisors. Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division.

Example 3

Use synthetic division to divide $5x^3 - 2x^2 + 1$ by $x - 3$.

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

3	5	-2	0	1
	↓	15	39	117
	5	13	39	[118]

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. This means
$5x^3 - 2x^2 + 1 = (x - 3)(5x^2 + 13x + 39) + 118$.

It also means that $x - 3$ is *not* a factor of $5x^3 - 2x^2 + 1$.

Example 4

Divide $x^3 + 8$ by $x + 2$

For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before.

-2	1	0	0	8
	↓	-2	4	-8
	1	-2	4	[0]

The quotient is $x^2 - 2x + 4$ and the remainder is zero. Since the remainder is zero, $x + 2$ is a factor of $x^3 + 8$.

$$x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$$

Try it Now

2. Divide $4x^4 - 8x^2 - 5x$ by $x - 3$ using synthetic division.

Using this process allows us to find the real zeros of polynomials, presuming we can figure out at least one root. We'll explore how to do that in the next section.

Example 5

The polynomial $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$ has a horizontal intercept at $x = \frac{1}{2}$ with multiplicity 2. Find the other intercepts of *p(x)*.

Since $x = \frac{1}{2}$ is an intercept with multiplicity 2, then $x - \frac{1}{2}$ is a factor twice. Use synthetic division to divide by $x - \frac{1}{2}$ twice.

1/2	4	-4	-11	12	-3
	↓	2	-1	-6	3
	4	-2	-1	-6	0

1/2	4	-2	-1	-6
	↓	2	0	-6
	4	0	-12	0

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - x - 6\right)$

The second division tells us

$$4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(4x^2 - 12\right).$$

To find the remaining intercepts, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$.

Note this also means $4x^4 - 4x^3 - 11x^2 + 12x - 3 = 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$.

Important Topics of this Section
Long division of polynomials
Remainder Theorem
Factor Theorem
Synthetic division of polynomials

Try it Now Answers

1.

$$\begin{array}{r} 2x^2 - 6x + 11 \\ x+3\overline{\big)\ 2x^3 + 0x^2 - 7x + 3} \\ \underline{-\left(2x^3 + 6x^2\right)} \\ -6x^2 - 7x + 3 \\ \underline{-\left(-6x^2 - 18x\right)} \\ 11x + 3 \\ \underline{-\left(11x + 33\right)} \\ -30 \end{array}$$

The quotient is $2x^2 - 6x + 11$ with remainder -30.

2.

3	4	0	-8	-5	0
	↓	12	36	84	237
	4	12	28	79	237

$4x^4 - 8x^2 - 5x$ divided by $x-3$ is $4x^3 + 12x^2 + 28x + 79$ with remainder 237

Section 3.4 Exercises

Use polynomial long division to perform the indicated division.

1. $\left(4x^2+3x-1\right)\div(x-3)$
2. $\left(2x^3-x+1\right)\div\left(x^2+x+1\right)$
3. $\left(5x^4-3x^3+2x^2-1\right)\div\left(x^2+4\right)$
4. $\left(-x^5+7x^3-x\right)\div\left(x^3-x^2+1\right)$
5. $\left(9x^3+5\right)\div(2x-3)$
6. $\left(4x^2-x-23\right)\div\left(x^2-1\right)$

Use synthetic division to perform the indicated division.

7. $\left(3x^2-2x+1\right)\div(x-1)$
8. $\left(x^2-5\right)\div(x-5)$
9. $\left(3-4x-2x^2\right)\div(x+1)$
10. $\left(4x^2-5x+3\right)\div(x+3)$
11. $\left(x^3+8\right)\div(x+2)$
12. $\left(4x^3+2x-3\right)\div(x-3)$
13. $\left(18x^2-15x-25\right)\div\left(x-\frac{5}{3}\right)$
14. $\left(4x^2-1\right)\div\left(x-\frac{1}{2}\right)$
15. $\left(2x^3+x^2+2x+1\right)\div\left(x+\frac{1}{2}\right)$
16. $\left(3x^3-x+4\right)\div\left(x-\frac{2}{3}\right)$
17. $\left(2x^3-3x+1\right)\div\left(x-\frac{1}{2}\right)$
18. $\left(4x^4-12x^3+13x^2-12x+9\right)\div\left(x-\frac{3}{2}\right)$
19. $\left(x^4-6x^2+9\right)\div\left(x-\sqrt{3}\right)$
20. $\left(x^6-6x^4+12x^2-8\right)\div\left(x+\sqrt{2}\right)$

Below you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

21. $x^3-6x^2+11x-6,\ c=1$
22. $x^3-24x^2+192x-512,\ c=8$
23. $3x^3+4x^2-x-2,\ c=\frac{2}{3}$
24. $2x^3-3x^2-11x+6,\ c=\frac{1}{2}$
25. $x^3+2x^2-3x-6,\ c=-2$
26. $2x^3-x^2-10x+5,\ c=\frac{1}{2}$
27. $4x^4-28x^3+61x^2-42x+9$, $c=\frac{1}{2}$ is a zero of multiplicity 2
28. $x^5+2x^4-12x^3-38x^2-37x-12$, $c=-1$ is a zero of multiplicity 3

Chapter 4: Exponential and Logarithmic Functions

Section 4.1 Exponential Functions

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year[1]. We might ask if we can find a formula to model the population, P, as a function of time, t, in years after 2008, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by three each time the input increases by one. This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change.

To see the significance of this difference consider these two companies:

Company A has 100 stores, and expands by opening 50 new stores a year

Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

---

[1] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 20, 2010

Year	Stores, company A		Stores, company B
0	100	Starting with 100 each	100
1	100 + 50 = 150	They both grow by 50 stores in the first year.	100 + 50% of 100 100 + 0.50(100) = 150
2	150 + 50 = 200	Store A grows by 50, Store B grows by 75	150 + 50% of 150 150 + 0.50(150) = 225
3	200 + 50 = 250	Store A grows by 50, Store B grows by 112.5	225 + 50% of 225 225 + 0.50(225) = 337.5

Notice that with the percent growth, each year the company is grows by 50% of the current year's total, so as the company grows larger, the number of stores added in a year grows as well.

To try to simplify the calculations, notice that after 1 year the number of stores for company B was:

$100 + 0.50(100)$ or equivalently by factoring

$100(1 + 0.50) = 150$

We can think of this as "the new number of stores is the original 100% plus another 50%".

After 2 years, the number of stores was:

$150 + 0.50(150)$ or equivalently by factoring

$150(1 + 0.50)$ now recall the 150 came from 100(1+0.50). Substituting that,

$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$

After 3 years, the number of stores was:

$225 + 0.50(225)$ or equivalently by factoring

$225(1 + 0.50)$ now recall the 225 came from $100(1 + 0.50)^2$. Substituting that,

$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.5$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of (1+0.50), so after n years, our equation would be

$B(n) = 100(1 + 0.50)^n$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential functions.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$f(x) = a(1+r)^x$ or $f(x) = ab^x$ where $b = 1+r$

Where

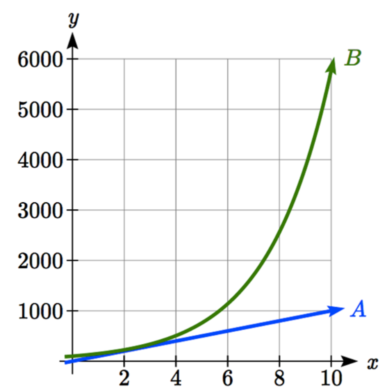
a is the initial or starting value of the function

r is the percent growth or decay rate, written as a decimal

b is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit b to positive values.

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company *A* and *B* described above over a longer time frame if the growth patterns were to continue.

years	Company *A*	Company *B*
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



Example 1

Write an exponential function for India's population, and use it to predict the population in 2020.

At the beginning of the chapter we were given India's population of 1.14 billion in the year 2008 and a percent growth rate of 1.34%. Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134.
Using the basic formula for exponential growth $f(x) = a(1+r)^x$ we can write the formula, $f(t) = 1.14(1+0.0134)^t$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008.
$f(12) = 1.14(1+0.0134)^{12} \approx 1.337$ billion people in 2020

Try it Now

1. Given the three statements below, identify which represent exponential functions.

A. The cost of living allowance for state employees increases salaries by 3.1% each year.
B. State employees can expect a $300 raise each year they work for the state.
C. Tuition costs have increased by 2.8% each year for the last 3 years.

Example 2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment grow to over those 24 months?

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly. To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: 1.2%/12 = 0.1%. Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of r = 0.001, and our input m measured in months.

$$f(m) = 1000\left(1 + \frac{.012}{12}\right)^m$$

$$f(m) = 1000(1 + 0.001)^m$$

After 24 months, the account will have grown to $f(24) = 1000(1 + 0.001)^{24} = \$1024.28$

Try it Now

2. Looking at these two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

$A(t) = 1000(1.05)^t \qquad B(t) = 900(1.075)^t$

In all the preceding examples, we saw exponential growth. Exponential functions can also be used to model quantities that are decreasing at a constant percent rate. An example of this is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

Example 3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is a = 100 mg, and our growth rate will be negative 13%, since we are decreasing: r = -0.13. This gives the equation:
$Q(d) = 100(1-0.13)^d = 100(0.87)^d$
This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be
$Q(7) = 100(0.87)^7 = 37.73$ mg of Bismuth-210 remains.

Try it Now

3. A population of 1000 is decreasing 3% each year. Find the population in 30 years.

Example 4

T(q) represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2016,
Interpret all the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2016 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,306 Android smart phone contracts.

Finding Equations of Exponential Functions

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function.

Example 5

In 2009, 80 deer were reintroduced into a wildlife refuge area from which the population had previously been hunted to elimination. By 2015, the population had grown to 180 deer. If this population grows exponentially, find a formula for the function.

By defining our input variable to be t, years after 2009, the information listed can be written as two input-output pairs: (0,80) and (6,180). Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: $a = 80$. This gives us an equation of the form

$f(t) = 80b^t$.

Substituting in our second input-output pair allows us to solve for b:

$180 = 80b^6$ Divide by 80

$b^6 = \frac{180}{80} = \frac{9}{4}$ Take the 6th root of both sides.

$b = \sqrt[6]{\frac{9}{4}} = 1.1447$

This gives us our equation for the population:

$f(t) = 80(1.1447)^t$

Recall that since $b = 1+r$, we can interpret this to mean that the population growth rate is $r = 0.1447$, and so the population is growing by about 14.47% each year.

In this example, you could also have used (9/4)^(1/6) to evaluate the 6th root if your calculator doesn't have an nth root button.

In the previous example, we chose to use the $f(x) = ab^x$ form of the exponential function rather than the $f(x) = a(1+r)^x$ form. This choice was entirely arbitrary – either form would be fine to use.

When finding equations, the value for b or r will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 significant digits in the growth rate. For example, if your value for b was 1.00317643, you would want to round this no further than to 1.00318.

In the previous example, we were able to "give" ourselves the initial value by clever definition of our input variable. Next, we consider a situation where we can't do this.

Example 6

Find a formula for an exponential function passing through the points (-2,6) and (2,1).

Since we don't have the initial value, we will take a general approach that will work for any function form with unknown parameters: we will substitute in both given input-output pairs in the function form $f(x) = ab^x$ and solve for the unknown values, *a* and *b*.
Substituting in (-2, 6) gives $6 = ab^{-2}$
Substituting in (2, 1) gives $1 = ab^2$

We now solve these as a system of equations. To do so, we could try a substitution approach, solving one equation for a variable, then substituting that expression into the second equation.
Solving $6 = ab^{-2}$ for *a*:

$$a = \frac{6}{b^{-2}} = 6b^2$$

In the second equation, $1 = ab^2$, we substitute the expression above for *a*:

$$1 = (6b^2)b^2$$

$$1 = 6b^4$$

$$\frac{1}{6} = b^4$$

$$b = \sqrt[4]{\frac{1}{6}} \approx 0.6389$$

Going back to the equation $a = 6b^2$ lets us find *a*:

$$a = 6b^2 = 6(0.6389)^2 = 2.4492$$

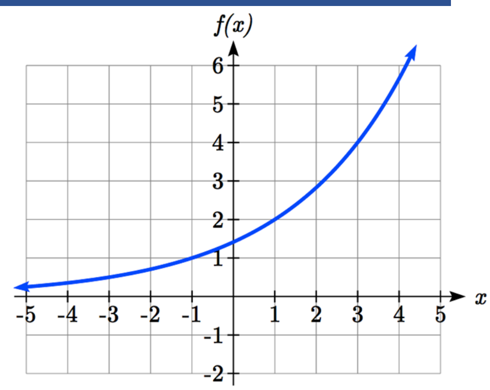
Putting this together gives the equation $f(x) = 2.4492(0.6389)^x$

Try it Now

4. Given the two points (1, 3) and (2, 4.5) find the equation of an exponential function that passes through these two points.

Example 7

Find an equation for the exponential function graphed.



The initial value for the function is not clear in this graph, so we will instead work using two clearer points. There are three clear points: (-1, 1), (1, 2), and (3, 4). As we saw in the last example, two points are sufficient to find the equation for a standard exponential, so we will use the latter two points.

Substituting in (1,2) gives $2 = ab^1$
Substituting in (3,4) gives $4 = ab^3$

Solving the first equation for *a* gives $a = \frac{2}{b}$.

Substituting this expression for *a* into the second equation:

$4 = ab^3$

$4 = \frac{2}{b}b^3 = \frac{2b^3}{b}$ Simplify the right-hand side

$4 = 2b^2$

$2 = b^2$

$b = \pm\sqrt{2}$

Since we restrict ourselves to positive values of *b*, we will use $b = \sqrt{2}$. We can then go back and find *a*:

$a = \frac{2}{b} = \frac{2}{\sqrt{2}} = \sqrt{2}$

This gives us a final equation of $f(x) = \sqrt{2}(\sqrt{2})^x$.

Compound Interest

In the bank certificate of deposit (CD) example earlier in the section, we encountered compound interest. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account.

In the example from earlier, the interest was compounded monthly, so we took the annual interest rate, usually called the **nominal rate** or **annual percentage rate (APR)** and divided by 12, the number of compounds in a year, to find the monthly interest. The exponent was then measured in months.

Generalizing this, we can form a general formula for compound interest. If the APR is written in decimal form as *r*, and there are *k* compounding periods per year, then the interest per compounding period will be *r/k*. Likewise, if we are interested in the value after *t* years, then there will be *kt* compounding periods in that time.

Compound Interest Formula

Compound Interest can be calculated using the formula

$$A(t) = a\left(1+\frac{r}{k}\right)^{kt}$$

Where

A(t) is the account value

t is measured in years

a is the starting amount of the account, often called the principal

r is the annual percentage rate (APR), also called the nominal rate

k is the number of compounding periods in one year

Example 8

If you invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Since we are starting with \$3000, $a = 3000$
Our interest rate is 3%, so $r = 0.03$
Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$
We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1+\frac{0.03}{4}\right)^{4(10)} = \$4045.05$$

The account will be worth \$4045.05 in 10 years.

Example 9

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to $40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Since the account is earning 6%, $r = 0.06$
Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be $40,000, so we will be looking for the value of a so that $A(18) = 40{,}000$.

$$40{,}000 = A(18) = a\left(1 + \frac{0.06}{2}\right)^{2(18)}$$

$$40{,}000 = a(2.8983)$$

$$a = \frac{40{,}000}{2.8983} \approx \$13{,}801$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Try it now

5. Recalculate example 2 from above with quarterly compounding.

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If $1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

If we were to compute the actual percentage increase for the daily compounding, there was an increase of $105.16 from an original amount of $1,000, for a percentage increase of $\frac{105.16}{1000} = 0.10516 = 10.516\%$ increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year would be

$A(1) = a\left(1+\frac{r}{k}\right)^k$. To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount:

$$\frac{a\left(1+\frac{r}{k}\right)^k - a}{a} = \left(1+\frac{r}{k}\right)^k - 1$$

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APY = \left(1+\frac{r}{k}\right)^k - 1$$

This is equivalent to finding the value of \$1 after 1 year, and subtracting the original dollar.

Example 10

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield – the actual percent increase in a year.

Bank A: $APY = \left(1+\frac{0.012}{4}\right)^4 - 1 = 0.012054 = 1.2054\%$

Bank B: $APY = \left(1+\frac{0.011}{12}\right)^{12} - 1 = 0.011056 = 1.1056\%$

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

A Limit to Compounding

As we saw earlier, the amount we earn increases as we increase the compounding frequency. The table, though, shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	**Value**
Annual	$2
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Once per minute	$2.718279
Once per second	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number.

Euler's Number: *e*

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets big.

$e \approx 2.718282$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function *exp(x)*, where $exp(x) = e^x$.

Because e arises when the time between compounds becomes very small, e allows us to define **continuous growth** and allows us to define a new toolkit function, $f(x) = e^x$.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$f(x) = ae^{rx}$

where

a is the starting amount

r is the continuous growth rate

This type of equation is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay.

Example 11

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: r = -0.173
$f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

Try it Now

6. Interpret the following: $S(t) = 20e^{0.12t}$ if *S(t)* represents the growth of a substance in grams, and time is measured in days.

Continuous growth is also often applied to compound interest, allowing us to talk about continuous compounding.

Example 12

If \$1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Here, the continuous growth rate is 10%, so r = 0.10. We start with \$1000, so a = 1000. To find the value after 1 year,
$f(1) = 1000e^{0.10(1)} \approx \$1105.17$

Notice this is a \$105.17 increase for the year. As a percent increase, this is
$\frac{105.17}{1000} = 0.10517 = 10.517\%$ increase over the original \$1000.

Notice that this value is slightly larger than the amount generated by daily compounding in the table computed earlier.

The continuous growth rate is like the nominal growth rate (or APR) – it reflects the growth rate before compounding takes effect. This is different than the annual growth rate used in the formula $f(x) = a(1+r)^x$, which is like the annual percentage yield – it reflects the *actual* amount the output grows in a year.

While the continuous growth rate in the example above was 10%, the actual annual yield was 10.517%. This means we could write two different looking but equivalent formulas for this account's growth:

$f(t) = 1000e^{0.10t}$ using the 10% continuous growth rate

$f(t) = 1000(1.10517)^t$ using the 10.517% actual annual yield rate.

Important Topics of this Section
Percent growth
Exponential functions
Finding formulas
Interpreting equations
Graphs
Exponential Growth & Decay
Compound interest
Annual Percent Yield
Continuous Growth

Try it Now Answers

1. A & C are exponential functions, they grow by a % not a constant number.

2. B(t) is growing faster ($r = 0.075 > 0.05$), but after 3 years A(t) still has a higher account balance

3. $P(t) = 1000(1-0.03)^t = 1000(0.97)^t$

   $P(30) = 1000(0.97)^{30} = 401.0071$

4. $3 = ab^1$, so $a = \dfrac{3}{b}$,

   $4.5 = ab^2$, so $4.5 = \dfrac{3}{b}b^2$. $4.5 = 3b$

   $b = 1.5$. $a = \dfrac{3}{1.5} = 2$

   $f(x) = 2(1.5)^x$

5. 24 months = 2 years. $1000\left(1 + \dfrac{.012}{4}\right)^{4(2)}$ = \$1024.25

6. An initial substance weighing 20g is growing at a continuous rate of 12% per day.

Section 4.1 Exercises

For each table below, could the table represent a function that is linear, exponential, or neither?

1.

x	1	2	3	4
f(x)	70	40	10	-20

2.

x	1	2	3	4
g(x)	40	32	26	22

3.

x	1	2	3	4
h(x)	70	49	34.3	24.01

4.

x	1	2	3	4
k(x)	90	80	70	60

5.

x	1	2	3	4
m(x)	80	61	42.9	25.61

6.

x	1	2	3	4
n(x)	90	81	72.9	65.61

7. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

8. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

9. The fox population in a certain region has an annual growth rate of 9 percent per year. It is estimated that the population in the year 2010 was 23,900. Estimate the fox population in the year 2018.

10. The amount of area covered by blackberry bushes in a park has been growing by 12% each year. It is estimated that the area covered in 2009 was 4,500 square feet. Estimate the area that will be covered in 2020.

11. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

12. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

Find a formula for an exponential function passing through the two points.

13. $(0,6)$, $(3,750)$

14. $(0,3)$, $(2,75)$

15. $(0,2000)$, $(2,20)$

16. $(0,9000)$, $(3,72)$

17. $\left(-1,\frac{3}{2}\right),(3,24)$

18. $\left(-1,\frac{2}{5}\right),(1,10)$

19. $(-2,6),(3,1)$

20. $(-3,4)$, $(3,2)$

21. $(3,1)$, $(5,4)$

22. $(2,5)$, $(6,9)$

23. A radioactive substance decays exponentially. A scientist begins with 100 milligrams of a radioactive substance. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?

24. A radioactive substance decays exponentially. A scientist begins with 110 milligrams of a radioactive substance. After 31 hours, 55 mg of the substance remains. How many milligrams will remain after 42 hours?

25. A house was valued at $110,000 in the year 1985. The value appreciated to $145,000 by the year 2005. What was the annual growth rate between 1985 and 2005? Assume that the house value continues to grow by the same percentage. What did the value equal in the year 2010?

26. An investment was valued at $11,000 in the year 1995. The value appreciated to $14,000 by the year 2008. What was the annual growth rate between 1995 and 2008? Assume that the value continues to grow by the same percentage. What did the value equal in the year 2012?

27. A car was valued at $38,000 in the year 2003. The value depreciated to $11,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What was the value in the year 2013?

28. A car was valued at $24,000 in the year 2006. The value depreciated to $20,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What was the value in the year 2014?

29. If $4,000 is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.

30. If \$6,000 is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously.

31. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded quarterly.

32. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 5% compounded monthly.

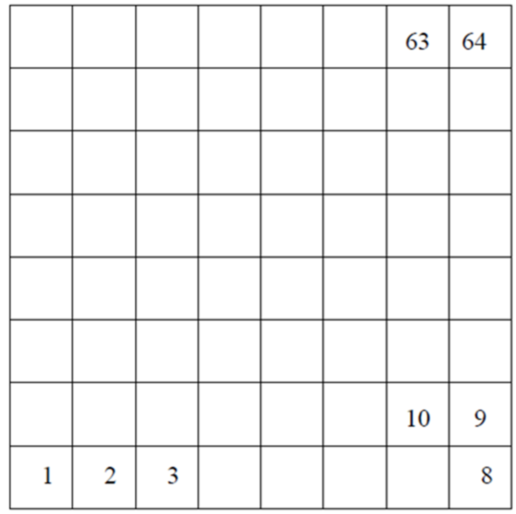
33. A population of bacteria is growing according to the equation $P(t)=1600e^{0.21t}$, with t measured in years. Estimate when the population will exceed 7569.

34. A population of bacteria is growing according to the equation $P(t)=1200e^{0.17t}$, with t measured in years. Estimate when the population will exceed 3443.

35. In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960. [UW]
    a. Find a formula for $w(t)$.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts?

36. In 1989, research scientists published a model for predicting the cumulative number of AIDS cases (in thousands) reported in the United States: $a(t)=155\left(\frac{t-1980}{10}\right)^3$, where t is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a new exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief." [UW]

37. You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always doubling the number from the previous square. [UW]



a. How many dimes will you have stacked on the 10th square?
b. How many dimes will you have stacked on the nth square?
c. How many dimes will you have stacked on the 64th square?
d. Assuming a dime is 1 mm thick, how high will this last pile be?
e. The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

Section 4.2 Graphs of Exponential Functions

Like with linear functions, the graph of an exponential function is determined by the values for the parameters in the function's formula.

To get a sense for the behavior of exponentials, let us begin by looking more closely at the function $f(x) = 2^x$. Listing a table of values for this function:

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice that:

1) This function is positive for all values of x.
2) As x increases, the function grows faster and faster (the rate of change increases).
3) As x decreases, the function values grow smaller, approaching zero.
4) This is an example of exponential growth.

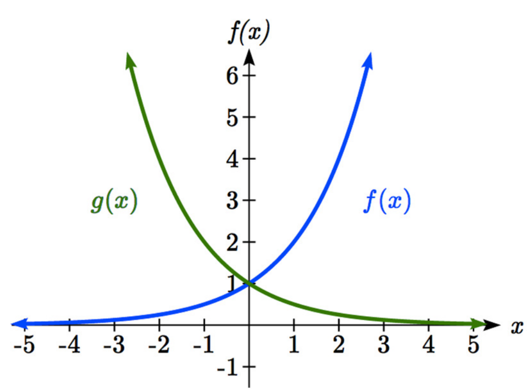
Looking at the function $g(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
$g(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Note this function is also positive for all values of x, but in this case grows as x decreases, and decreases towards zero as x increases. This is an example of exponential decay. You may notice from the table that this function appears to be the horizontal reflection of the $f(x) = 2^x$ table. This is in fact the case:

$$f(-x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x = g(x)$$

Looking at the graphs also confirms this relationship.



Consider a function of the form $f(x) = ab^x$. Since a, which we called the initial value in the last section, is the function value at an input of zero, a will give us the vertical intercept of the graph.

From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the growth factor. This horizontal asymptote will also help us determine the long run behavior and is easy to determine from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor b larger than one. When it's negative, the growth factor will be less than one.

Graphical Features of Exponential Functions

Graphically, in the function $f(x) = ab^x$

a is the vertical intercept of the graph

b determines the rate at which the graph grows. When a is positive,

the function will increase if $b > 1$

the function will decrease if $0 < b < 1$

The graph will have a horizontal asymptote at $y = 0$

The graph will be concave up if $a > 0$; concave down if $a < 0$.

The domain of the function is all real numbers

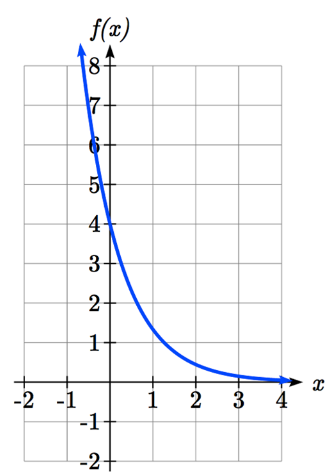
The range of the function is $(0, \infty)$

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$.

The value b will determine the function's long run behavior:
If $b > 1$, as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to 0$.
If $0 < b < 1$, as $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to \infty$.

Example 1

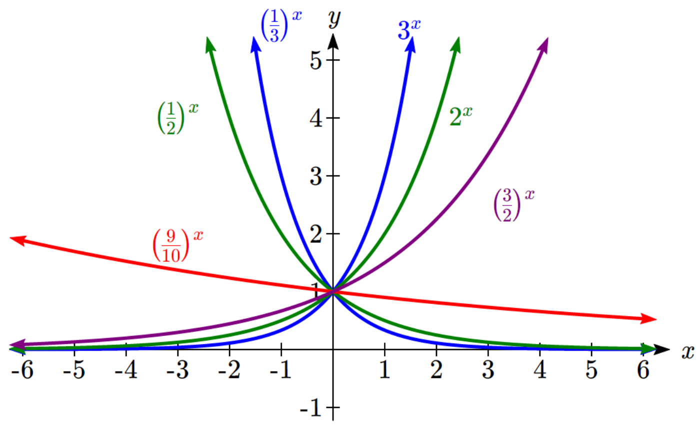
Sketch a graph of $f(x) = 4\left(\frac{1}{3}\right)^x$



This graph will have a vertical intercept at (0,4), and pass through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero. Since $a > 0$, the graph will be concave up.

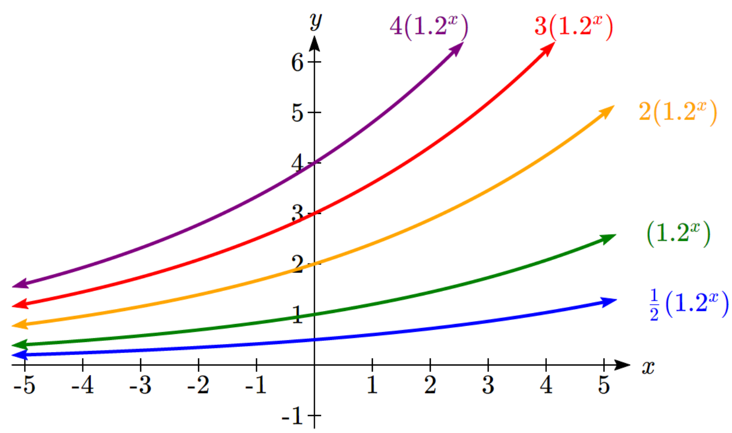
We can also see from the graph the long run behavior: as $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to \infty$.

To get a better feeling for the effect of *a* and *b* on the graph, examine the sets of graphs below. The first set shows various graphs, where *a* remains the same and we only change the value for *b*.



Notice that the closer the value of *b* is to 1, the less steep the graph will be.

In the next set of graphs, *a* is altered and our value for *b* remains the same.



Notice that changing the value for *a* changes the vertical intercept. Since *a* is multiplying the b^x term, *a* acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these *a* values introduced a vertical flip.

Example 2

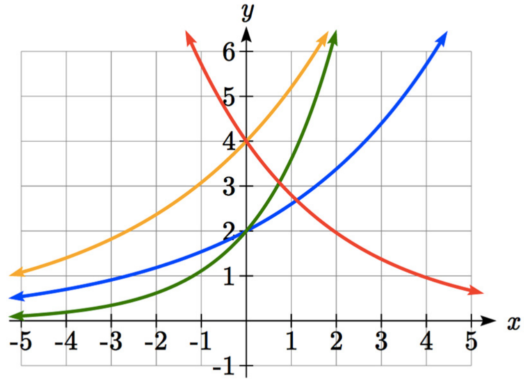
Match each equation with its graph.

$f(x) = 2(1.3)^x$

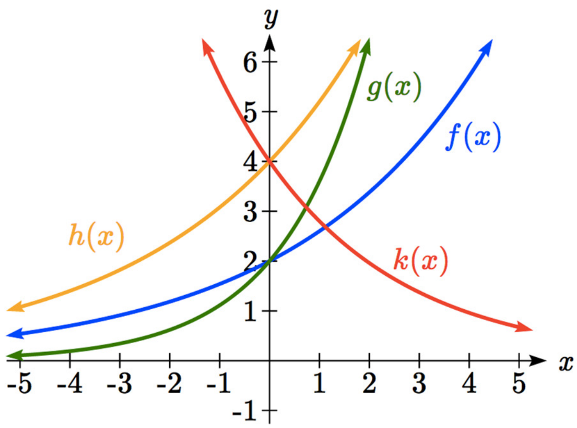
$g(x) = 2(1.8)^x$

$h(x) = 4(1.3)^x$

$k(x) = 4(0.7)^x$



The graph of *k(x)* is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of *h(x)* can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of *f(x)* and *g(x)* both have a vertical intercept at (0,2), but since *g(x)* has a larger growth factor, we can identify it as the graph increasing faster.



Try it Now

1. Graph the following functions on the same axis:

$f(x) = (2)^x$; $g(x) = 2(2)^x$; $h(x) = 2(1/2)^x$.

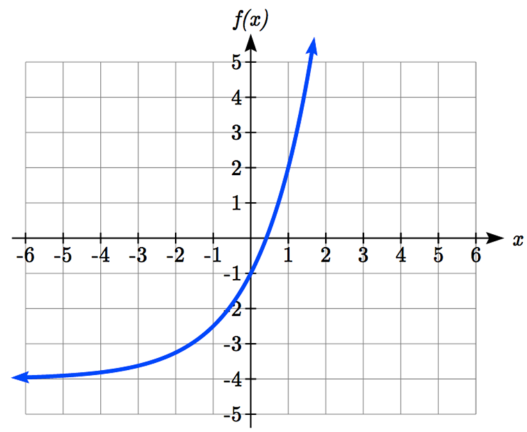
Transformations of Exponential Graphs

While exponential functions can be transformed following the same rules as any function, there are a few interesting features of transformations that can be identified. The first was seen at the beginning of the section – that a horizontal reflection is equivalent to a change in the growth factor. Likewise, since *a* is itself a stretch factor, a vertical stretch of an exponential corresponds with a change in the initial value of the function.

Next consider the effect of a horizontal shift on an exponential function. Shifting the function $f(x) = 3(2)^x$ four units to the left would give $f(x+4) = 3(2)^{x+4}$. Employing exponent rules, we could rewrite this:
$f(x+4) = 3(2)^{x+4} = 3(2)^x(2^4) = 48(2)^x$

Interestingly, it turns out that a horizontal shift of an exponential function corresponds with a change in initial value of the function.

Lastly, consider the effect of a vertical shift on an exponential function. Shifting $f(x) = 3(2)^x$ down 4 units would give the equation $f(x) = 3(2)^x - 4$.



Graphing that, notice it is substantially different than the basic exponential graph. Unlike a basic exponential, this graph does not have a horizontal asymptote at $y = 0$; due to the vertical shift, the horizontal asymptote has also shifted to $y = -4$. We can see that as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -4$.

We have determined that a vertical shift is the only transformation of an exponential function that changes the graph in a way that cannot be achieved by altering the parameters *a* and *b* in the basic exponential function $f(x) = ab^x$.

Transformations of Exponentials
Any transformed exponential can be written in the form $f(x) = ab^x + c$ where $y = c$ is the horizontal asymptote.

Note that, due to the shift, the vertical intercept is shifted to $(0, a+c)$.

Try it Now

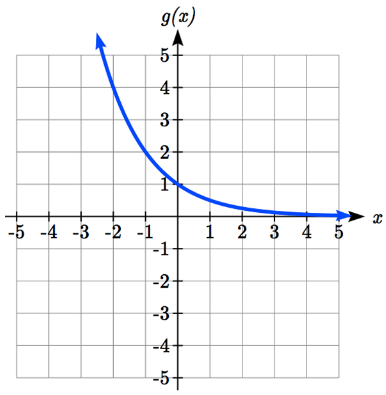
2. Write the equation and graph the exponential function described as follows:
$f(x) = e^x$ is vertically stretched by a factor of 2, flipped across the *y* axis and shifted up 4 units.

Example 3

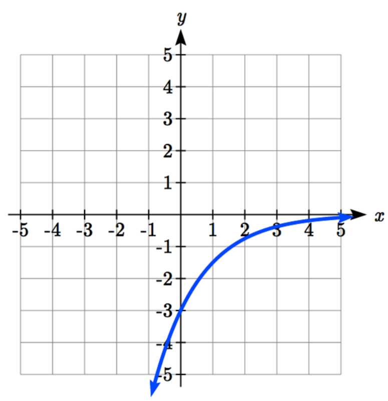
Sketch a graph of $f(x) = -3\left(\frac{1}{2}\right)^x + 4$.

Notice that in this exponential function, the negative in the stretch factor -3 will cause a vertical reflection, and the vertical shift up 4 will move the horizontal asymptote to $y = 4$. Sketching this as a transformation of $g(x) = \left(\frac{1}{2}\right)^x$,

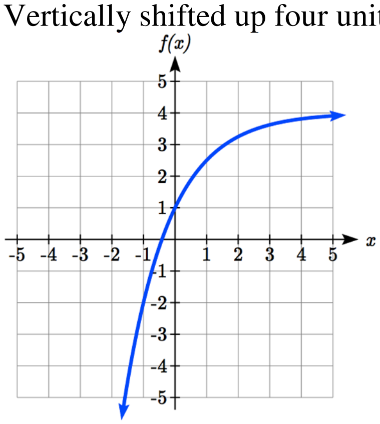
The basic $g(x) = \left(\frac{1}{2}\right)^x$



Vertically reflected and stretched by 3



Vertically shifted up four units



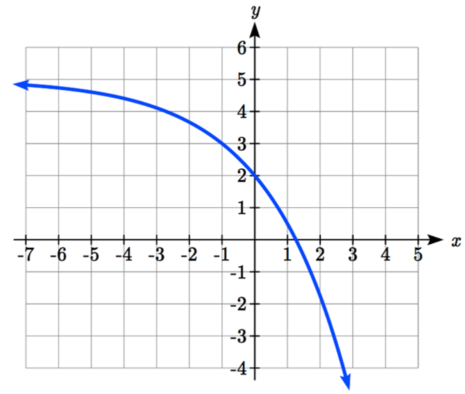
Notice that while the domain of this function is unchanged, due to the reflection and shift, the range of this function is $(-\infty, 4)$.

As $x \to \infty$, $f(x) \to 4$ and as $x \to -\infty$, $f(x) \to -\infty$.

Functions leading to graphs like the one above are common as models for learning and models of growth approaching a limit.

Example 4

Find an equation for the function graphed.



Looking at this graph, it appears to have a horizontal asymptote at $y = 5$, suggesting an equation of the form $f(x) = ab^x + 5$. To find values for a and b, we can identify two other points on the graph. It appears the graph passes through (0,2) and (-1,3), so we can use those points. Substituting in (0,2) allows us to solve for a.

$$2 = ab^0 + 5$$
$$2 = a + 5$$
$$a = -3$$

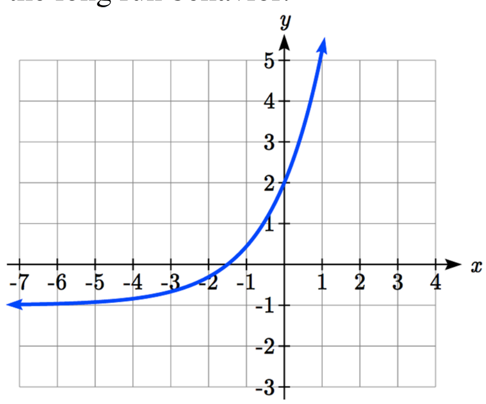
Substituting in (-1,3) allows us to solve for b

$$3 = -3b^{-1} + 5$$
$$-2 = \frac{-3}{b}$$
$$-2b = -3$$
$$b = \frac{3}{2} = 1.5$$

The final formula for our function is $f(x) = -3(1.5)^x + 5$.

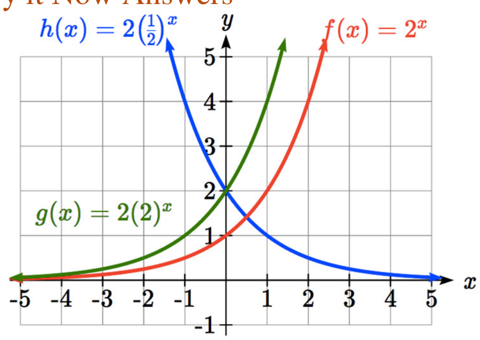
Try it Now

3. Given the graph of the transformed exponential function, find a formula and describe the long run behavior.

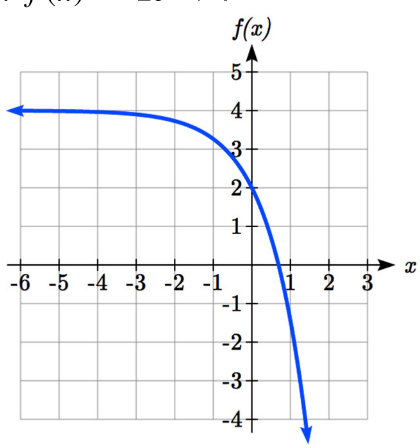


Important Topics of this Section
Graphs of exponential functions
Intercept
Growth factor
Exponential Growth
Exponential Decay
Horizontal intercepts
Long run behavior
Transformations

Try it Now Answers

1. 

2. $f(x) = -2e^x + 4$

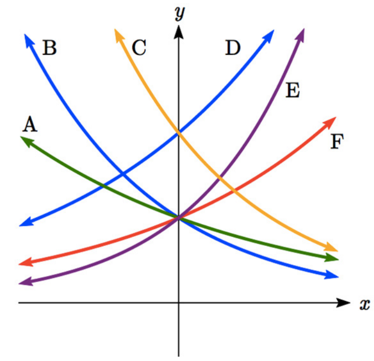


3. Horizontal asymptote at y = -1, so $f(x) = ab^x - 1$. Substitute (0, 2) to find a = 3.
Substitute (1,5) to find $5 = 3b^1 - 1$, b = 2.
$f(x) = 3(2^x) - 1$ or $f(x) = 3(.5)^{-x} - 1$
As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -1$

Section 4.2 Exercises

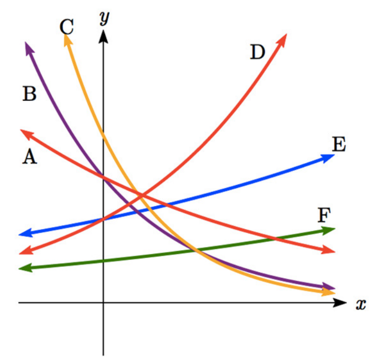
Match each function with one of the graphs below.

1. $f(x) = 2(0.69)^x$
2. $f(x) = 2(1.28)^x$
3. $f(x) = 2(0.81)^x$
4. $f(x) = 4(1.28)^x$
5. $f(x) = 2(1.59)^x$
6. $f(x) = 4(0.69)^x$



If all the graphs to the right have equations with form $f(x) = ab^x$,

7. Which graph has the largest value for b?
8. Which graph has the smallest value for b?
9. Which graph has the largest value for a?
10. Which graph has the smallest value for a?



Sketch a graph of each of the following transformations of $f(x) = 2^x$

11. $f(x) = 2^{-x}$
12. $g(x) = -2^x$
13. $h(x) = 2^x + 3$
14. $f(x) = 2^x - 4$
15. $f(x) = 2^{x-2}$
16. $k(x) = 2^{x-3}$

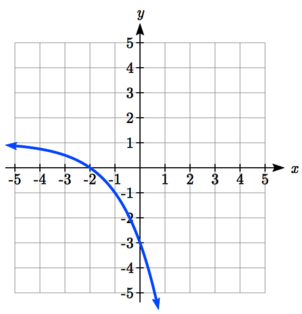
Starting with the graph of $f(x) = 4^x$, find a formula for the function that results from

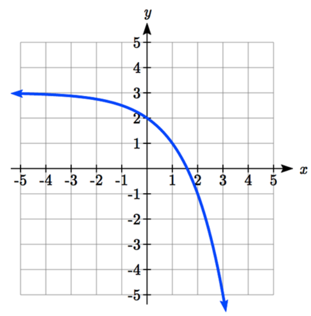
17. Shifting $f(x)$ 4 units upwards
18. Shifting $f(x)$ 3 units downwards
19. Shifting $f(x)$ 2 units left
20. Shifting $f(x)$ 5 units right
21. Reflecting $f(x)$ about the x-axis
22. Reflecting $f(x)$ about the y-axis

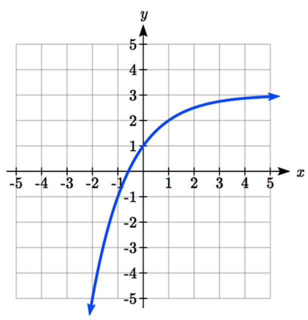
Describe the long run behavior, as $x \to \infty$ and $x \to -\infty$ of each function

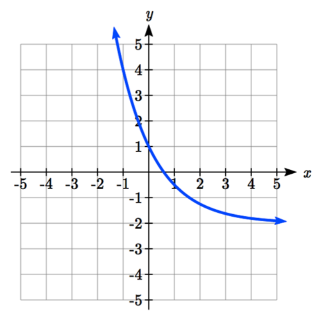
23. $f(x) = -5(4^x) - 1$
24. $f(x) = -2(3^x) + 2$
25. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$
26. $f(x) = 4\left(\frac{1}{4}\right)^x + 1$
27. $f(x) = 3(4)^{-x} + 2$
28. $f(x) = -2(3)^{-x} - 1$

Find a formula for each function graphed as a transformation of $f(x) = 2^x$.

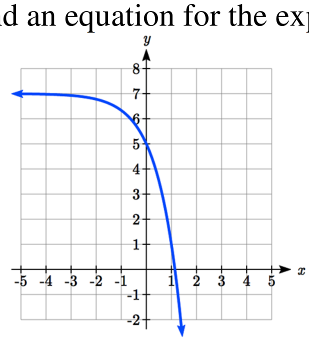
29. 

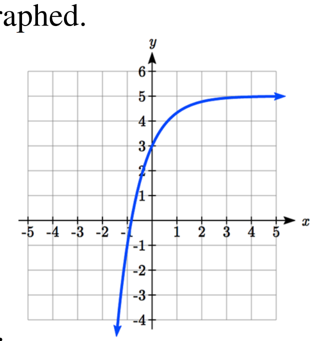
30. 

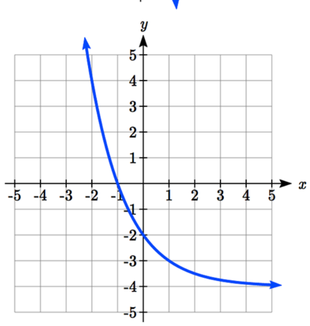
31. 

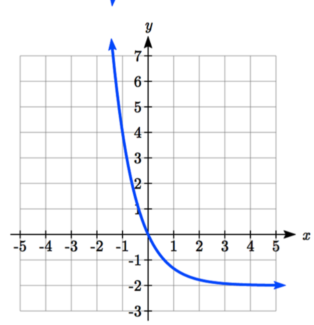
32. 

Find an equation for the exponential function graphed.

33. 

34. 

35. 

36. 

Section 4.3 Logarithmic Functions

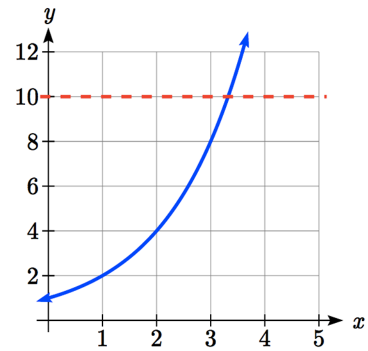
A population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where *x* represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$500 = 50(2)^x$ Dividing both sides by 50 to isolate the exponential

$10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that *x* must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution.

From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, and since the exponential function is one-to-one, it would be great to have an inverse function. None of the functions we have already discussed would serve as an inverse function and so we must introduce a new function, named **log** as the inverse of an exponential function. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.



Logarithm
The logarithm (base *b*) function, written $\log_b(x)$, is the inverse of the exponential function (base *b*), b^x.

Since the logarithm and exponential are inverses, it follows that:

Properties of Logs: Inverse Properties
$\log_b\left(b^x\right) = x$ $b^{\log_b x} = x$

Recall from the definition of an inverse function that if $f(a) = c$, then $f^{-1}(c) = a$. Applying this to the exponential and logarithmic functions, we can convert between a logarithmic equation and its equivalent exponential.

Logarithm Equivalent to an Exponential
The statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Alternatively, we could show this by starting with the exponential function $c = b^a$, then taking the log base *b* of both sides, giving $\log_b(c) = \log_b b^a$. Using the inverse property of logs, we see that $\log_b(c) = a$.

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 1

Write these exponential equations as logarithmic equations:

a) $2^3 = 8$ b) $5^2 = 25$ c) $10^{-4} = \dfrac{1}{10000}$

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$

b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$

c) $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$

Example 2

Write these logarithmic equations as exponential equations:

a) $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ b) $\log_3(9) = 2$

a) $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ is equivalent to $6^{1/2} = \sqrt{6}$

b) $\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try it Now

1. Write the exponential equation $4^2 = 16$ as a logarithmic equation.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 3

Solve $\log_4(x) = 2$ for *x*.

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$.

Example 4

Solve $2^x = 10$ for *x*.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base *e*, and is typically written $\ln(x)$.

Example 5

Evaluate $\log(1000)$ using the definition of the common log.

To evaluate $\log(1000)$, we can let $x = \log(1000)$, then rewrite into exponential form using the common log base of 10: $10^x = 1000$.

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

We also can use the inverse property of logs to write $\log_{10}(10^3) = 3$.

Try it Now

2. Evaluate $\log(1000000)$.

Example 6

Evaluate $\ln\left(\sqrt{e}\right)$.

We can rewrite $\ln\left(\sqrt{e}\right)$ as $\ln\left(e^{1/2}\right)$. Since ln is a log base *e*, we can use the inverse property for logs: $\ln\left(e^{1/2}\right) = \log_e\left(e^{1/2}\right) = \frac{1}{2}$.

Example 7

Evaluate log(500) using your calculator or computer.

Using a computer, we can evaluate $\log(500) \approx 2.69897$

To utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need to establish some additional properties.

Properties of Logs: Exponent Property
$\log_b\left(A^r\right) = r\log_b(A)$

To show why this is true, we offer a proof:

Since the logarithmic and exponential functions are inverses, $b^{\log_b A} = A$.

Raising both sides to the *r* power, we get $A^r = \left(b^{\log_b A}\right)^r$.

Utilizing the exponential rule that states $\left(x^p\right)^q = x^{pq}$, $A^r = \left(b^{\log_b A}\right)^r = b^{r\log_b A}$

Taking the log of both sides, $\log_b\left(A^r\right) = \log_b\left(b^{r\log_b A}\right)$

Utilizing the inverse property on the right side yields the result: $\log_b\left(A^r\right) = r\log_b A$

Example 8

Rewrite $\log_3(25)$ using the exponent property for logs.

Since $25 = 5^2$,

$\log_3(25) = \log_3\left(5^2\right) = 2\log_3(5)$

Example 9

Rewrite $4\ln(x)$ using the exponent property for logs.

Using the property in reverse, $4\ln(x) = \ln\left(x^4\right)$.

Try it Now

3. Rewrite using the exponent property for logs: $\ln\left(\frac{1}{x^2}\right)$.

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$$

Proof:

Let $\log_b(A) = x$.

Rewriting as an exponential gives $b^x = A$.

Taking the log base *c* of both sides of this equation gives $\log_c b^x = \log_c A$,

Now utilizing the exponent property for logs on the left side, $x\log_c b = \log_c A$

Dividing, we obtain $x = \frac{\log_c A}{\log_c b}$. Replacing our original expression for *x*,

$$\log_b A = \frac{\log_c A}{\log_c b}$$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

Example 10

Evaluate $\log_2(10)$ using the change of base formula.

According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base *e*:

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} = \frac{\ln 10}{\ln 2}$$

Using our calculators to evaluate this,

$$\frac{\ln 10}{\ln 2} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

Example 11

Evaluate $\log_5(100)$ using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we can solve the basic exponential equation $2^x = 10$ by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 12

Solve $2^x = 10$ for *x*.

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$\ln(2^x) = \ln(10)$ Utilizing the exponent property for logs,

$x\ln(2) = \ln(10)$ Now dividing by ln(2),

$$x = \frac{\ln(10)}{\ln(2)} \approx 3.3219$$

Notice that this result matches the result we found using the change of base formula.

Example 13

In the first section, we predicted the population (in billions) of India *t* years after 2008 by using the function $f(t) = 1.14(1+0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for time *t* so that *f(t)* = 2.

$2 = 1.14(1.0134)^t$ Divide by 1.14 to isolate the exponential expression

$\frac{2}{1.14} = 1.0134^t$ Take the logarithm of both sides of the equation

$\ln\left(\frac{2}{1.14}\right) = \ln(1.0134^t)$ Apply the exponent property on the right side

$\ln\left(\frac{2}{1.14}\right) = t\ln(1.0134)$ Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Try it Now

4. Solve $5(0.93)^x = 10$.

Example 14

Solve $5(1.07)^{3t} = 2$

To start, we want to isolate the exponential part of the expression, the $(1.07)^{3t}$, so it is alone on one side of the equation. Then we can use the log to solve the equation. We can use any base log; this time we'll use the common log.

$5(1.07)^{3t} = 2$	Divide both sides by 5 to isolate the exponential
$(1.07)^{3t} = \frac{2}{5}$	Take the log of both sides.
$\log\left((1.07)^{3t}\right) = \log\left(\frac{2}{5}\right)$	Use the exponent property for logs
$3t\log(1.07) = \log\left(\frac{2}{5}\right)$	Divide by $3\log(1.07)$ on both sides
$\frac{3t\log(1.07)}{3\log(1.07)} = \frac{\log\left(\frac{2}{5}\right)}{3\log(1.07)}$	Simplify and evaluate
$t = \frac{\log\left(\frac{2}{5}\right)}{3\log(1.07)} \approx -4.5143$	

Note that when entering that expression on your calculator, be sure to put parentheses around the whole denominator to ensure the proper order of operations:

```
log(2/5)/(3*log(1.07))
```

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 15

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation $pH = -\log\left(\left[H^+\right]\right)$.

If a liquid has concentration of 0.0001 moles per liber, determine the pH.
Determine the hydrogen ion concentration of a liquid with pH of 7.

To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$$-\log(0.0001) = -\log\left(10^{-4}\right) = -(-4) = 4$$

To answer the second question, we need to solve the equation $7 = -\log\left(\left[H^+\right]\right)$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1: $-7 = \log\left(\left[H^+\right]\right)$. Rewriting into exponential form yields the answer: $\left[H^+\right] = 10^{-7} = 0.0000001$ moles per liter.

Logarithms also provide us a mechanism for finding continuous growth models for exponential growth given two data points.

Example 15

A population grows from 100 to 130 in 2 weeks. Find the continuous growth rate.

Measuring t in weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0) = 100$ and $P(2) = 130$. Using the first pair of values,
$100 = ae^{r \cdot 0}$, so $a = 100$.

Using the second pair of values,

$130 = 100e^{r \cdot 2}$	Divide by 100
$\frac{130}{100} = e^{r2}$	Take the natural log of both sides
$\ln(1.3) = \ln\left(e^{r2}\right)$	Use the inverse property of logs
$\ln(1.3) = 2r$	
$r = \frac{\ln(1.3)}{2} \approx 0.1312$	

This population is growing at a continuous rate of 13.12% per week.

In general, we can relate the standard form of an exponential with the continuous growth form by noting (using k to represent the continuous growth rate to avoid the confusion of using r in two different ways in the same formula):

$$a(1+r)^x = ae^{kx}$$
$$(1+r)^x = e^{kx}$$
$$1+r = e^k$$

Converting Between Periodic to Continuous Growth Rate

In the equation $f(x) = a(1+r)^x$, r is the **periodic growth rate**, the percent growth each time period (weekly growth, annual growth, etc.).

In the equation $f(x) = ae^{kx}$, k is the **continuous growth rate**.

You can convert between these using: $1+r = e^k$.

Remember that the continuous growth rate k represents the nominal growth rate before accounting for the effects of continuous compounding, while r represents the actual percent increase in one time unit (one week, one year, etc.).

Example 16

A company's sales can be modeled by the function $S(t) = 5000e^{0.12t}$, with t measured in years. Find the annual growth rate.

Noting that $1 + r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form $S(t) = 5000(1 + 0.1275)^t$.

Important Topics of this Section
The Logarithmic function as the inverse of the exponential function
Writing logarithmic & exponential expressions
Properties of logs
Inverse properties
Exponential properties
Change of base
Common log
Natural log
Solving exponential equations
Converting between periodic and continuous growth rate.

Try it Now Answers

1. $\log_4(16) = 2 = \log_4 4^2 = 2\log_4 4$

2. $\log(1000000) = \log(10^6) = 6$

3. $\ln\left(\frac{1}{x^2}\right) = \ln\left(x^{-2}\right) = -2\ln(x)$

4. $5(0.93)^x = 10$

$$(0.93)^x = 2$$
$$\ln\left(0.93^x\right) = \ln(2)$$
$$x\ln(0.93) = \ln(2)$$
$$\frac{\ln(2)}{\ln(0.93)} \approx -9.5513$$

Section 4.3 Exercises

Rewrite each equation in exponential form

1. $\log_4(q) = m$
2. $\log_3(t) = k$
3. $\log_a(b) = c$
4. $\log_p(z) = u$
5. $\log(v) = t$
6. $\log(r) = s$
7. $\ln(w) = n$
8. $\ln(x) = y$

Rewrite each equation in logarithmic form.

9. $4^x = y$
10. $5^y = x$
11. $c^d = k$
12. $n^z = L$
13. $10^a = b$
14. $10^p = v$
15. $e^k = h$
16. $e^y = x$

Solve for *x*.

17. $\log_3(x) = 2$
18. $\log_4(x) = 3$
19. $\log_2(x) = -3$
20. $\log_5(x) = -1$
21. $\log(x) = 3$
22. $\log(x) = 5$
23. $\ln(x) = 2$
24. $\ln(x) = -2$

Simplify each expression using logarithm properties.

25. $\log_5(25)$
26. $\log_2(8)$
27. $\log_3\left(\frac{1}{27}\right)$
28. $\log_6\left(\frac{1}{36}\right)$
29. $\log_6\left(\sqrt{6}\right)$
30. $\log_5\left(\sqrt[3]{5}\right)$
31. $\log(10{,}000)$
32. $\log(100)$
33. $\log(0.001)$
34. $\log(0.00001)$
35. $\ln\left(e^{-2}\right)$
36. $\ln\left(e^{3}\right)$

Evaluate using your calculator.

37. $\log(0.04)$
38. $\log(1045)$
39. $\ln(15)$
40. $\ln(0.02)$

Solve each equation for the variable.

41. $5^x = 14$
42. $3^x = 23$
43. $7^x = \frac{1}{15}$
44. $3^x = \frac{1}{4}$
45. $e^{5x} = 17$
46. $e^{3x} = 12$
47. $3^{4x-5} = 38$
48. $4^{2x-3} = 44$
49. $1000(1.03)^t = 5000$
50. $200(1.06)^t = 550$
51. $3(1.04)^{3t} = 8$
52. $2(1.08)^{4t} = 7$
53. $50e^{-0.12t} = 10$
54. $10e^{-0.03t} = 4$
55. $10 - 8\left(\frac{1}{2}\right)^x = 5$
56. $100 - 100\left(\frac{1}{4}\right)^x = 70$

Convert the equation into continuous growth form, $f(t) = ae^{kt}$.

57. $f(t) = 300(0.91)^t$

58. $f(t) = 120(0.07)^t$

59. $f(t) = 10(1.04)^t$

60. $f(t) = 1400(1.12)^t$

Convert the equation into annual growth form, $f(t) = ab^t$.

61. $f(t) = 150e^{0.06t}$

62. $f(t) = 100e^{0.12t}$

63. $f(t) = 50e^{-0.012t}$

64. $f(t) = 80e^{-0.85t}$

65. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

66. The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million?

67. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. If the population continues to grow exponentially at the same rate, when will the population exceed 1 million people?

68. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. If it continues to grow exponentially at the same rate, when will median income exceed $50,000?

69. A scientist begins with 100 mg of a radioactive substance. After 4 hours, it has decayed to 80 mg. How long after the process began will it take to decay to 15 mg?

70. A scientist begins with 100 mg of a radioactive substance. After 6 days, it has decayed to 60 mg. How long after the process began will it take to decay to 10 mg?

71. If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500?

72. If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300?

Section 4.4 Logarithmic Properties

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

Properties of Logs
Inverse Properties:
$\log_b\left(b^x\right) = x$
$b^{\log_b x} = x$
Exponential Property:
$\log_b\left(A^r\right) = r\log_b\left(A\right)$
Change of Base:
$\log_b\left(A\right) = \dfrac{\log_c(A)}{\log_c(b)}$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems involving exponential and logarithmic equations.

Properties of Logs
Sum of Logs Property:
$\log_b\left(A\right) + \log_b\left(C\right) = \log_b(AC)$
Difference of Logs Property:
$\log_b\left(A\right) - \log_b\left(C\right) = \log_b\left(\dfrac{A}{C}\right)$

It's just as important to know what properties logarithms do *not* satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute: $\log(A + B) \neq \log(A) + \log(B)$.

To help in this process we offer a proof to help solidify our new rules and show how they follow from properties you've already seen.

Let $a = \log_b(A)$ and $c = \log_b(C)$.
By definition of the logarithm, $b^a = A$ and $b^c = C$.
Using these expressions, $AC = b^a b^c$
Using exponent rules on the right, $AC = b^{a+c}$
Taking the log of both sides, and utilizing the inverse property of logs,
$\log_b(AC) = \log_b\left(b^{a+c}\right) = a + c$
Replacing *a* and *c* with their definition establishes the result
$\log_b(AC) = \log_b A + \log_b C$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 1

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Using the sum of logs property on the first two terms,
$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$

This reduces our original expression to $\log_3(40) - \log_3(2)$

Then using the difference of logs property,
$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$

Example 2

Evaluate $2\log(5) + \log(4)$ without a calculator by first rewriting as a single logarithm.

On the first term, we can use the exponent property of logs to write
$2\log(5) = \log\left(5^2\right) = \log(25)$

With the expression reduced to a sum of two logs, $\log(25) + \log(4)$, we can utilize the sum of logs property
$\log(25) + \log(4) = \log(4 \cdot 25) = \log(100)$

Since 100 = 10^2, we can evaluate this log without a calculator:
$\log(100) = \log\left(10^2\right) = 2$

Try it Now

1. Without a calculator evaluate by first rewriting as a single logarithm:
$\log_2(8)+\log_2(4)$

Example 3

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4 y}{7}\right)=\ln\left(x^4 y\right)-\ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln\left(x^4 y\right)-\ln(7)=\ln\left(x^4\right)+\ln(y)-\ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln\left(x^4\right)+\ln(y)-\ln(7)=4\ln(x)+\ln(y)-\ln(7)$$

Interestingly, solving exponential equations was not the reason logarithms were originally developed. Historically, up until the advent of calculators and computers, the power of logarithms was that these log properties reduced multiplication, division, roots, or powers to be evaluated using addition, subtraction, division and multiplication, respectively, which are much easier to compute without a calculator. Large books were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the sum of log property was used. Suppose for example we didn't know the value of 2 times 3. Using the sum property of logs:

value	log(value)
1	0.0000000
2	0.3010300
3	0.4771213
4	0.6020600
5	0.6989700
6	0.7781513
7	0.8450980
8	0.9030900
9	0.9542425
10	1.0000000

$$\log(2\cdot 3)=\log(2)+\log(3)$$

Using the log table,

$$\log(2\cdot 3)=\log(2)+\log(3)=0.3010300+0.4771213=0.7781513$$

We can then use the table again in reverse, looking for 0.7781513 as an output of the logarithm. From that we can determine:

$$\log(2\cdot 3)=0.7781513=\log(6).$$

By using addition and the table of logs, we were able to determine $2\cdot 3=6$.

Likewise, to compute a cube root like $\sqrt[3]{8}$

$$\log(\sqrt[3]{8}) == \log\left(8^{1/3}\right) = \frac{1}{3}\log(8) = \frac{1}{3}(0.9030900) = 0.3010300 = \log(2)$$

So $\sqrt[3]{8} = 2$.

Although these calculations are simple and insignificant, they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

These properties still have other practical applications for interpreting changes in exponential and logarithmic relationships.

Example 4

Recall that in chemistry, $pH = -\log\left(\left[H^+\right]\right)$. If the concentration of hydrogen ions in a liquid is doubled, what is the affect on pH?

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the sum property of logs,

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$pH = P - \log(2) = P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by 0.301.

Log properties in solving equations

The logarithm properties often arise when solving problems involving logarithms. First, we'll look at a simpler log equation.

Example 5

Solve $\log(2x-6) = 3$.

To solve for x, we need to get it out from inside the log function. There are two ways we can approach this.

Method 1: Rewrite as an exponential.

Recall that since the common log is base 10, $\log(A) = B$ can be rewritten as the exponential $10^B = A$. Likewise, $\log(2x-6) = 3$ can be rewritten in exponential form as $10^3 = 2x - 6$

Method 2: Exponentiate both sides.

If $A = B$, then $10^A = 10^B$. Using this idea, since $\log(2x-6) = 3$, then $10^{\log(2x-6)} = 10^3$. Use the inverse property of logs to rewrite the left side and get $2x - 6 = 10^3$.

Using either method, we now need to solve $2x - 6 = 10^3$. Evaluate 10^3 to get

$2x - 6 = 1000$ Add 6 to both sides

$2x = 1006$ Divide both sides by 2

$x = 503$

Occasionally the solving process will result in extraneous solutions – answers that are outside the domain of the original equation. In this case, our answer looks fine.

Example 6

Solve $\log(50x + 25) - \log(x) = 2$.

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x+25}{x}\right) = 2$$

Rewriting in exponential form reduces this to an algebraic equation:

$\frac{50x+25}{x} = 10^2 = 100$ Multiply both sides by x

$50x + 25 = 100x$ Combine like terms

$25 = 50x$ Divide by 50

$x = \frac{25}{50} = \frac{1}{2}$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Try it Now

2. Solve $\log(x^2 - 4) = 1 + \log(x + 2)$.

Example 7

Solve $\ln(x+2)+\ln(x+1)=\ln(4x+14)$.

$\ln(x+2)+\ln(x+1)=\ln(4x+14)$	Use the sum of logs property on the right
$\ln\left((x+2)(x+1)\right)=\ln(4x+14)$	Expand
$\ln\left(x^2+3x+2\right)=\ln(4x+14)$	

We have a log on both side of the equation this time. Rewriting in exponential form would be tricky, so instead we can exponentiate both sides.

$e^{\ln\left(x^2+3x+2\right)}=e^{\ln(4x+13)}$	Use the inverse property of logs
$x^2+3x+2=4x+14$	Move terms to one side
$x^2-x-12=0$	Factor
$(x+4)(x-3)=0$	
$x=-4$ or $x=3$.	

Checking our answers, notice that evaluating the original equation at $x=-4$ would result in us evaluating $\ln(-2)$, which is undefined. That answer is outside the domain of the original equation, so it is an extraneous solution and we discard it. There is one solution: $x=3$.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 8a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year[2]. If these trends continue, when will the population of Kenya match that of Sudan?

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t)=38.8(1+0.0264)^t$$

$$Sudan(t)=41.3(1+0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t=41.3(1.0224)^t$$

[2] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010

For our first approach, we take the log of both sides of the equation.

$$\log\left(38.8(1.0264)^t\right) = \log\left(41.3(1.0224)^t\right)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8) + \log\left(1.0264^t\right) = \log(41.3) + \log\left(1.0224^t\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8) + t\log\left(1.0264\right) = \log(41.3) + t\log\left(1.0224\right)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,

$$t\log\left(1.0264\right) - t\log\left(1.0224\right) = \log(41.3) - \log(38.8)$$

Factoring out the t on the left,

$$t\left(\log\left(1.0264\right) - \log\left(1.0224\right)\right) = \log(41.3) - \log(38.8)$$

Dividing to solve for t

$$t = \frac{\log(41.3) - \log(38.8)}{\log\left(1.0264\right) - \log\left(1.0224\right)} \approx 15.991$$ years until the populations will be equal.

Example 8b

Solve the problem above by rewriting before taking the log.

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t\log\left(\frac{1.0264}{1.0224}\right)=\log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t=\frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)}\approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t=\frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)}=\frac{\log(41.3)-\log(38.8)}{\log(1.0264)-\log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Try it Now

3. Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Important Topics of this Section
Inverse
Exponential
Change of base
Sum of logs property
Difference of logs property
Solving equations using log rules

Try it Now Answers

1. $\log_2(8\cdot 4)=\log_2(32)=\log_2(2^5)=5$

2. $\log(x^2-4)=1+\log(x+2)$ — Move both logs to one side

$\log(x^2-4)-\log(x+2)=1$ — Use the difference property of logs

$\log\left(\dfrac{x^2-4}{x+2}\right)=1$ — Factor

$\log\left(\dfrac{(x+2)(x-2)}{x+2}\right)=1$ — Simplify

$\log(x-2)=1$ — Rewrite as an exponential

$10^1=x-2$ — Add 2 to both sides

$x=12$

3. Tank A: $A(t)=10(1-0.35)^t$. Tank B: $B(t)=30(1-0.50)^t$

Solving *A(t)* = *B(t)*,

$10(0.65)^t=30(0.5)^t$ — Using the method from Example 8b

$\dfrac{(0.65)^t}{(0.5)^t}=\dfrac{30}{10}$ — Regroup

$\left(\dfrac{0.65}{0.5}\right)^t=3$ — Simplify

$(1.3)^t=3$ — Take the log of both sides

$\log\left((1.3)^t\right)=\log(3)$ — Use the exponent property of logs

$t\log(1.3)=\log(3)$ — Divide and evaluate

$t=\dfrac{\log(3)}{\log(1.3)}\approx 4.1874$ weeks

Section 4.4 Exercises

Simplify to a single logarithm, using logarithm properties.

1. $\log_3(28)-\log_3(7)$
2. $\log_3(32)-\log_3(4)$
3. $-\log_3\left(\frac{1}{7}\right)$
4. $-\log_4\left(\frac{1}{5}\right)$
5. $\log_3\left(\frac{1}{10}\right)+\log_3(50)$
6. $\log_4(3)+\log_4(7)$
7. $\frac{1}{3}\log_7(8)$
8. $\frac{1}{2}\log_5(36)$
9. $\log\left(2x^4\right)+\log\left(3x^5\right)$
10. $\ln\left(4x^2\right)+\ln\left(3x^3\right)$
11. $\ln\left(6x^9\right)-\ln\left(3x^2\right)$
12. $\log\left(12x^4\right)-\log(4x)$
13. $2\log(x)+3\log(x+1)$
14. $3\log(x)+2\log\left(x^2\right)$
15. $\log(x)-\frac{1}{2}\log(y)+3\log(z)$
16. $2\log(x)+\frac{1}{3}\log(y)-\log(z)$

Use logarithm properties to expand each expression.

17. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
18. $\log\left(\frac{a^2b^3}{c^5}\right)$
19. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
20. $\ln\left(\frac{a^{-2}b^3}{c^{-5}}\right)$
21. $\log\left(\sqrt{x^3y^{-4}}\right)$
22. $\log\left(\sqrt{x^{-3}y^2}\right)$
23. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
24. $\ln\left(\frac{x}{\sqrt{1-x^2}}\right)$
25. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$
26. $\log\left(x^3y^4\sqrt[7]{x^3y^9}\right)$

Solve each equation for the variable.

27. $4^{4x-7} = 3^{9x-6}$
28. $2^{2x-5} = 7^{3x-7}$
29. $17(1.14)^x = 19(1.16)^x$
30. $20(1.07)^x = 8(1.13)^x$
31. $5e^{0.12t} = 10e^{0.08t}$
32. $3e^{0.09t} = e^{0.14t}$
33. $\log_2(7x+6) = 3$
34. $\log_3(2x+4) = 2$
35. $2\ln(3\text{x}) + 3 = 1$
36. $4\ln(5x) + 5 = 2$
37. $\log(x^3) = 2$
38. $\log(x^5) = 3$
39. $\log(x) + \log(x+3) = 3$
40. $\log(x+4) + \log(x) = 9$
41. $\log(x+4) - \log(x+3) = 1$
42. $\log(x+5) - \log(x+2) = 2$
43. $\log_6(x^2) - \log_6(x+1) = 1$
44. $\log_3(x^2) - \log_3(x+2) = 5$
45. $\log(x+12) = \log(x) + \log(12)$
46. $\log(x+15) = \log(x) + \log(15)$
47. $\ln(x) + \ln(x-3) = \ln(7x)$
48. $\ln(x) + \ln(x-6) = \ln(6x)$

Solutions to Selected Exercises

Chapter 1

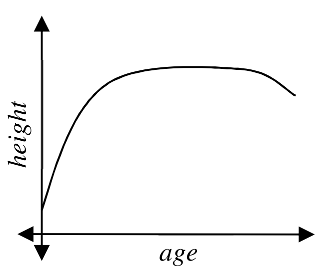
Section 1.1

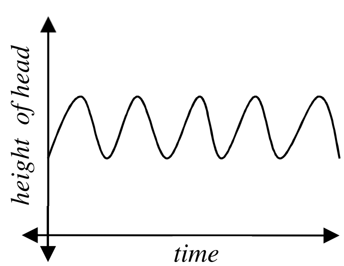
1. a. $f(40)=13$
   b. 2 Tons of garbage per week is produced by a city with a population of 5,000.
3. a. In 1995 there are 30 ducks in the lake
   b. In 2000 there are 40 ducks in the late

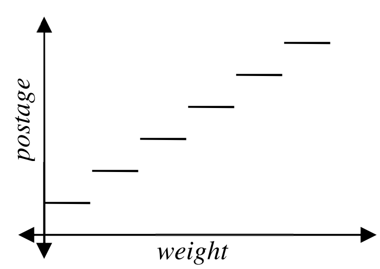
5. a ,b, d, e
7. a, b
9. a, b, d
11. b
13. b, c, e, f
15. $f(1)=1$, $f(3)=1$
17. $g(2)=4$, $g(-3)=2$
19. $f(3)=53$, $f(2)=1$

	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
21.	8	6	4	2	0
23.	49	18	3	4	21
25.	4	-1	0	1	-4
27.	4	4.414	4.732	5	5.236
29.	-4	-6	-6	-4	0
31.	5	DNE	-3	-1	-1/3
33.	1/4	1/2	1	2	4

35. a. -6 b.-16
37. a. 5 b. $-\frac{5}{3}$
39. a. iii b. viii c. I d. ii e. vi f. iv g. v h. vii
41. a. iv b. ii c. v d. I e. vi f. iii
43. $(x-3)^2+(y+9)^2=36$
45. (a) (b) (c)







47a. t b. a c. r d. L: (c, t) and K: (a, p)

Section 1.2

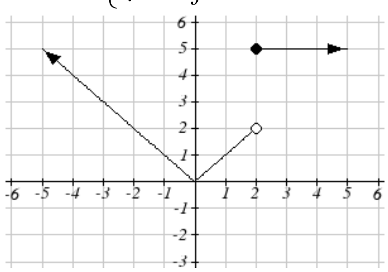
1. D: [-5, 3) R: [0,2]
3. D: $2 < t \le 8$ R: $6 \le g(t) < 8$
5. D: [0,4] R: [-3, 0]
7. $[2,\infty)$
9. $(-\infty,3]$
11. $(-\infty,6)\cup(6,\infty)$
13. $(-\infty,-\frac{1}{2})\cup\left(-\frac{1}{2},\infty\right)$
15. $[-4,4)\cup(4,\infty)$
17. $(-\infty,-11)\cup(-11,2)\cup(2,\infty)$

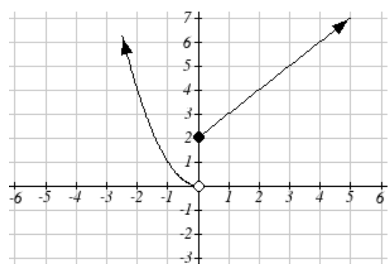
	$f(-1)$	$f(0)$	$f(2)$	$f(4)$
19.	-4	6	20	34
21.	-1	-2	7	5
23.	-5	3	3	16

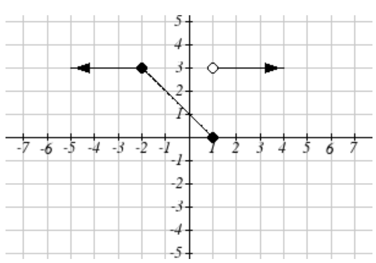
25. $f(x)=\begin{cases} 2 & if \quad -6 \le x \le -1 \\ -2 & if \quad -1 < x \le 2 \\ -4 & if \quad 2 < x \le 4 \end{cases}$

27. $f(x)=\begin{cases} 3 & if \quad x \le 0 \\ x^2 & if \quad x > 0 \end{cases}$

29. $f(x)=\begin{cases} \frac{1}{x} & if \quad x < 0 \\ \sqrt{x} & if \quad x \ge 0 \end{cases}$

31. 

33. 

35. 

Section 1.3

1. a) 6 million dollars per year b) 2 million dollars per year

3. $\frac{4-5}{4-1}=-\frac{1}{3}$

5. 6

7. 27

9. $\frac{352}{27}$

11. 4*b*+4

13. 3

15. $-\frac{1}{13h+169}$

17. $9+9h+3h^2$

19. $4x+2h$

21. Increasing: $(-1.5, 2)$. Decreasing: $(-\infty,-1.5)\cup(2,\infty)$

23. Increasing: $(-\infty,1)\cup(3,4)$. Decreasing: $(1,3)\cup(4,\infty)$

25. Increasing, concave up

27. Decreasing, concave down

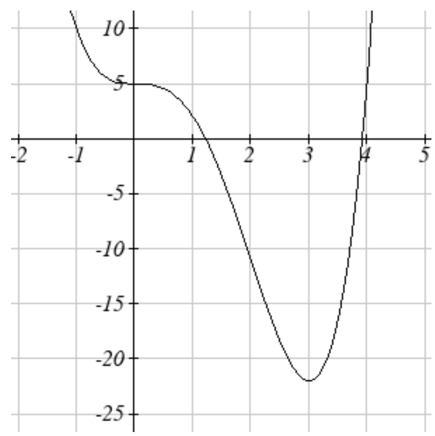
29. Decreasing, concave up

31. Increasing, concave down

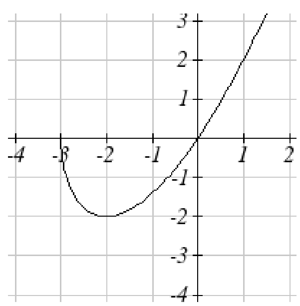
33. Concave up $(-\infty,1)$. Concave down $(1,\infty)$. Inflection point at (1, 2)

35. Concave down $(-\infty,3)\cup(3,\infty)$

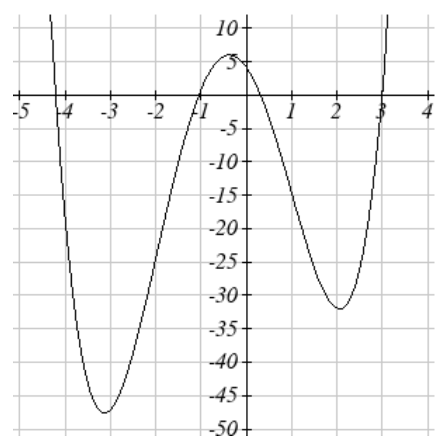
37. Local minimum at (3, -22).
Inflection points at (0,5) and (2, -11).
Increasing on $(3,\infty)$. Decreasing $(-\infty,3)$
Concave up $(-\infty,0)\cup(2,\infty)$. Concave down $(0,2)$



39. Local minimum at (-2, -2)
Decreasing $(-3,-2)$
Increasing $(-2,\infty)$
Concave up $(-3,\infty)$



41. Local minimums at (-3.152, -47.626)
and (2.041, -32.041)
Local maximum at (-0.389, 5.979)
Inflection points at (-2, -24) and (1, -15)
Increasing $(-3.152,-0.389)\cup(2.041,\infty)$
Decreasing $(-\infty,-3.152)\cup(-0.389,2.041)$
Concave up $(-\infty,-2)\cup(1,\infty)$
Concave down $(-2,1)$



Section 1.4

1. $f(g(0)) = 36$. $g(f(0)) = -57$

3. $f(g(0)) = 4$. $g(f(0)) = 4$

5. 4 7. 9 9. 4 11. 7 13. 0 15. 4 17. 3 19. 2

21. $f\left(g\left(x\right)\right) = \frac{x}{7}$ $g\left(f\left(x\right)\right) = 7x - 36$

23. $f\left(g\left(x\right)\right) = x + 3$ $g\left(f\left(x\right)\right) = \sqrt{x^2 + 3}$

25. $f\left(g\left(x\right)\right) = \left|5x + 1\right|$ $g\left(f\left(x\right)\right) = 5\left|x\right| + 1$

27. $f\left(g\left(h\left(x\right)\right)\right) = \left(\sqrt{x} - 6\right)^4 + 6$

29. b 31a. $r\left(V\left(t\right)\right) = \sqrt[3]{\frac{3\left(10 + 20t\right)}{4\pi}}$ b. 4.609in

33. $\left(0, \infty\right)$ 35. $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right) \cup \left(1, \infty\right)$ 37. $\left[2,5\right) \cup \left(5, \infty\right)$

39. $g\left(x\right) = x + 2, f\left(x\right) = x^2$ 41. $f\left(x\right) = \frac{3}{x}, g\left(x\right) = x - 5$

43. $f\left(x\right) = 3 + \sqrt{x}, g\left(x\right) = x - 2$, or $f\left(x\right) = 3 + x, g\left(x\right) = \sqrt{x - 2}$

45a. $f\left(f\left(x\right)\right) = a\left(ax + b\right) + b = \left(a^2\right)x + \left(ab + b\right)$

b. $g\left(x\right) = \sqrt{6}x - \frac{8}{\sqrt{6} + 1}$ or $g\left(x\right) = -\sqrt{6}x - \frac{8}{1 - \sqrt{6}}$

47a. $C\left(f\left(s\right)\right) = \frac{70\left(\frac{s}{60}\right)^2}{10 + \left(\frac{s}{60}\right)^2}$ b. $C\left(g\left(h\right)\right) = \frac{70\left(60h\right)^2}{10 + \left(60h\right)^2}$

c. $v\left(C\left(m\right)\right) = \frac{5280}{3600}\left(\frac{70m^2}{10 + m^2}\right)$

Section 1.5

1. Horizontal shift right 49 units

3. Horizontal shift left 3 units

5. Vertical shift up 5 units

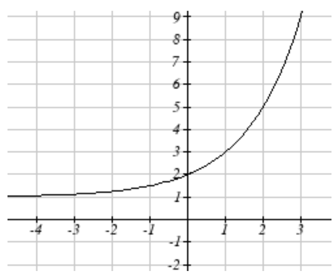
7. Vertical shift down 2 units

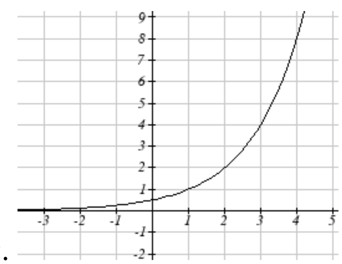
9. Horizontal shift right 2 units, Vertical shift up 3 units

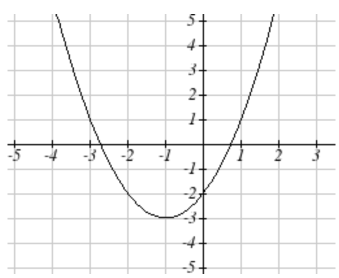
11. $f(x+2)+1=\sqrt{x+2}+1$

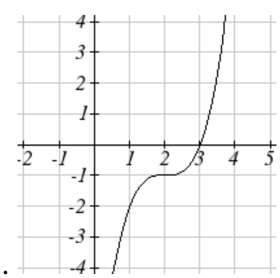
13. $f(x-3)-4=\dfrac{1}{x-3}-4$

15. $g(x)=f(x-1),\quad h(x)=f(x)+1$

17. 

19. 

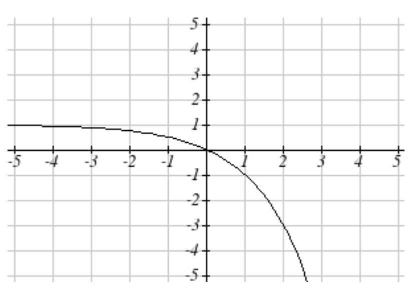
21. 

23. 

25. $y=|x-3|-2$

27. $y=\sqrt{x+3}-1$

29. $y=-\sqrt{x}$

31. 

33a. $-f(-x)=-6^{-x}$

b. $-f(x+2)-3=-6^{x+2}-3$

35. $y=-(x+1)^2+2$

37. $y=\sqrt{-x}+1$

39a. Even b. Neither c. Odd

41. Reflect *f(x)* about the *x*-axis

43. Vertically stretch *y* values by 4

45. Horizontally compress *x* values by 1/5

47. Horizontally stretch *x* values by 3

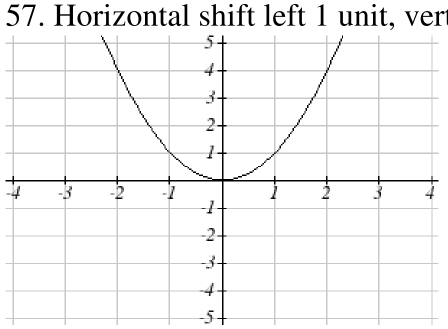
49. Reflect *f(x)* about the *y*-axis and vertically stretch *y* values by 3

51. $f(-4x) = |-4x|$

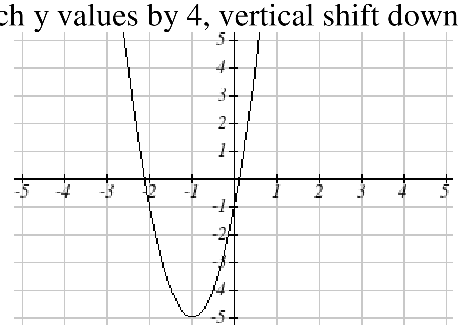
53. $\frac{1}{3} f(x+2) - 3 = \frac{1}{3(x+2)^2} - 3$

55. $f(2(x-5)) + 1 = (2(x-5))^2 + 1$

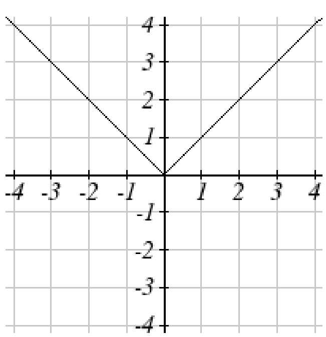
57. Horizontal shift left 1 unit, vertical stretch y values by 4, vertical shift down 5 units



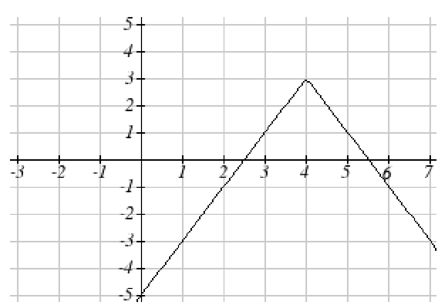
becomes



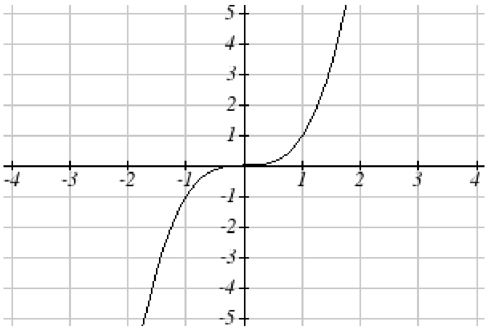
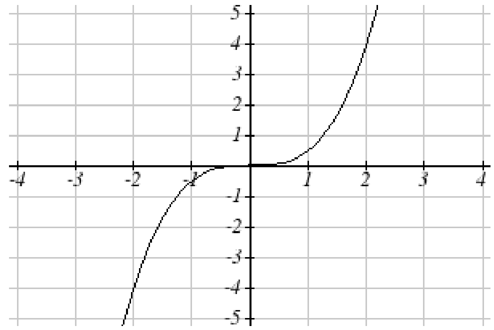
59. Horizontal shift right 4 units, vertical stretch y values by 2, reflect over x axis, vertically shift up 3 units.



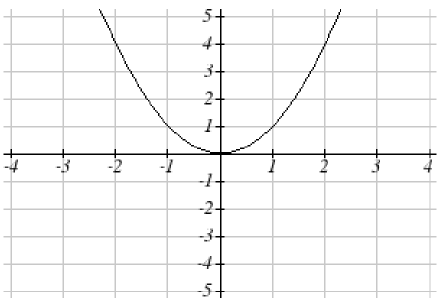
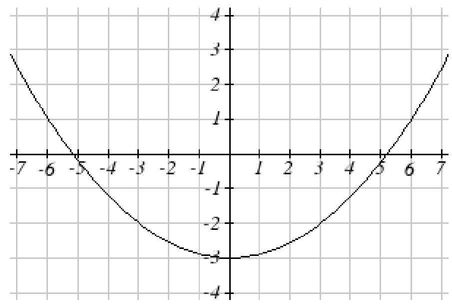
becomes



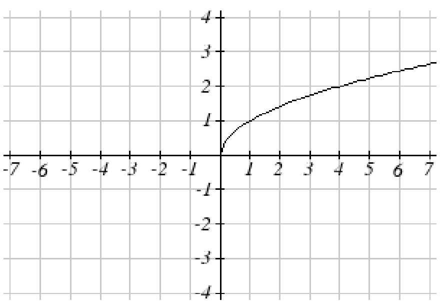
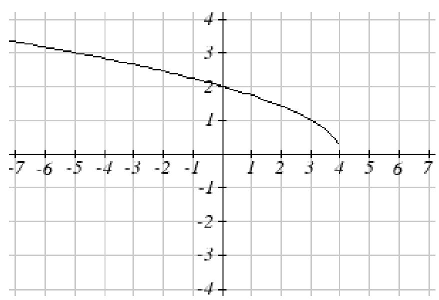
61. Vertically compress y values by ½


becomes


63. Horizontally stretch x values by 3, vertical shift down 3 units


becomes


65. Reflected over the y axis, horizontally shift right 4 units $a(x)=\sqrt{-(x-4)}$


becomes


67. This function is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$

69. This function is decreasing on $(-\infty,4)$

71. This function is concave down on $(-3,\infty)$ and concave up on $(-\infty,-3)$

73. This function is concave up everywhere

75. $f(-x)$

77. $3f(x)$

79. $2f(-x)$

81. $2f\left(\frac{1}{2}x\right)$

83. $2f(x)-2$

85. $-f(x+1)+3$

87. $y=-2(x+2)^2+3$

89. $y=\left(\frac{1}{2}(x-1)\right)^3+2$

91. $y=\sqrt{2(x+2)}+1$

93. $y=\dfrac{-1}{\left(x-2\right)^2}+3$

95. $y=-2\left|x+1\right|+3$

97. $y=\sqrt[3]{-\dfrac{1}{2}\left(x-2\right)}+1$

99. $f\left(x\right)=\begin{cases}(x+3)^2+1 & if\ \ x\le -2\\ \dfrac{1}{2}\left|x-2\right|+3 & if\ \ x>-2\end{cases}$

101. $f\left(x\right)=\begin{cases}1 & if \quad x<-2\\ -2(x+1)^2+4 & if \quad -2\le x\le 1\\ \sqrt[3]{x-2}+1 & if \quad x>1\end{cases}$

103a. $Domain: 3.5\le x\le 6$

d. $Range: -9\le y\le 7$

Chapter 3

Section 3.1

1. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to \infty$
3. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to -\infty$
5. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to -\infty$
7. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to \infty$
9. 7^{th} Degree, Leading coefficient 4
11. 2^{nd} Degree, Leading coefficient -1
13. 4^{th} Degree, Leading coefficient -2
15. 3^{rd} Degree, Leading coefficient 6
17. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to -\infty$
19. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to \infty$
21. intercepts: 5, turning points: 4 23. 3
25. 5 27. 3 29. 5
31. Horizontal Intercepts (1,0), (-2, 0), (3, 0) Vertical Intercept (0, 12)
33. Horizontal Intercepts (1/3, 0) (-1/2, 0) Vertical Intercept (0, 2)

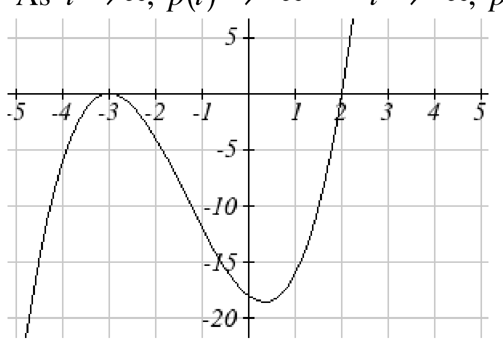
Section 3.3

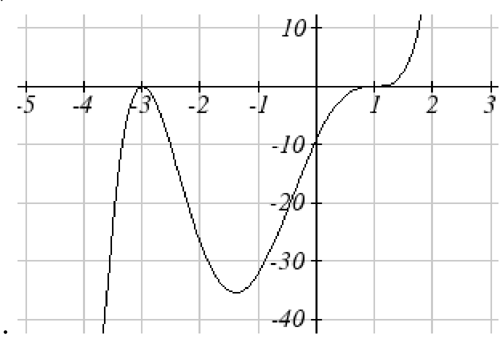
C(t)	*C*, intercepts	*t*, intercepts
1.	(0,48)	(4,0), (-1,0), (6,0)
3.	(0,0)	(0,0), (2,0), (-1,0)
5.	(0,0)	(0,0), (1,0), (3,0)

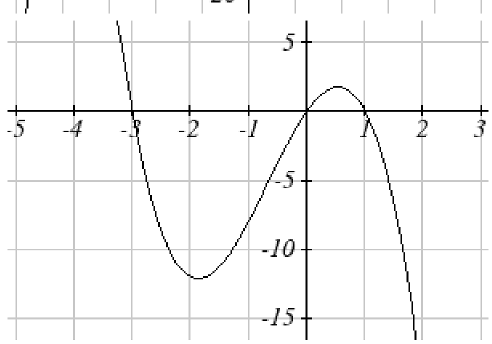
7. (-1.646, 0) (3.646, 0) (5,0)

9. As $t \to \infty,\ h(t) \to \infty \qquad t \to -\infty,\ h(t) \to -\infty$

11. As $t \to \infty,\ p(t) \to -\infty \qquad t \to -\infty,\ p(t) \to -\infty$

13. 

15. 

17. 

19. $(3,\infty)$

21. $(-\infty,-2)\cup(1,3)$

23. $[3.5, 6]$

25. $(-\infty, 1] \cup [4, \infty)$

27. $[-2, -2] \cup [3, \infty)$

29. $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$

31. $y = -\frac{2}{3}(x+2)(x-1)(x-3)$

33. $y = \frac{1}{3}(x-1)^2(x-3)^2(x+3)$

35. $y = -15(x-1)^2(x-3)^3$

37. $y = \frac{1}{2}(x+2)(x-1)(x-3)$

39. $y = -(x+1)^2(x-2)$

41. $y = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

43. $y = \frac{1}{24}(x+4)(x+2)(x-3)^2$

45. $y = \frac{1}{12}(x+2)^2(x-3)^2$

47. $y = \frac{1}{6}(x+3)(x+2)(x-1)^3$

49. $y = -\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

51. Base 2.58, Height 3.336

Section 3.4

1. $4x^2 + 3x - 1 = (x-3)(4x+15) + 44$
3. $5x^4 - 3x^3 + 2x^2 - 1 = (x^2+4)(5x^2 - 3x - 18) + (12x + 71)$
5. $9x^3 + 5 = (2x-3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$
7. $(3x^2 - 2x + 1) = (x-1)(3x+1) + 2$
9. $(3 - 4x - 2x^2) = (x+1)(-2x-2) + 5$
11. $(x^3 + 8) = (x+2)(x^2 - 2x + 4) + 0$
13. $(18x^2 - 15x - 25) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
15. $(2x^3 + x^2 + 2x + 1) = \left(x + \frac{1}{2}\right)(2x^2 + 2) + 0$
17. $(2x^3 - 3x + 1) = \left(x - \frac{1}{2}\right)\left(2x^2 + x - \frac{5}{2}\right) - \frac{1}{4}$
19. $(x^4 - 6x^2 + 9) = (x - \sqrt{3})(x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3}) + 0$
21. $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$
23. $3x^3 + 4x^2 - x - 2 = 3\left(x - \frac{2}{3}\right)(x+1)^2$
25. $x^3 + 2x^2 - 3x - 6 = (x+2)(x+\sqrt{3})(x-\sqrt{3})$
27. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2(x-3)^2$

Chapter 4

Section 4.1

1. Linear 3. Exponential 5. Neither

7. $P(t)=11,000(1.085)^t$ 9. 47622 Fox

11. $17561.70 13. $y=6(5)^x$ 15. $y=2000(0.1)^x$

17. $y=3(2)^x$ 19. $y=\left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} = 2.93(0.699)^x$ 21. $y=\frac{1}{8}(2)^x$

23. $34.32\,mg$ 25. 1.39%; $155,368.09 27. $4,813.55

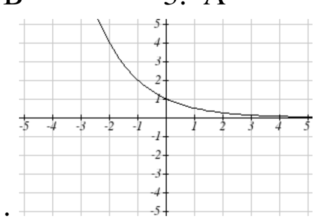
29. Annual $\approx \$7353.84$ Quarterly $\approx \$7,469.63$ Monthly $\approx \$7,496.71$
Continuously $\approx \$7,510.44$

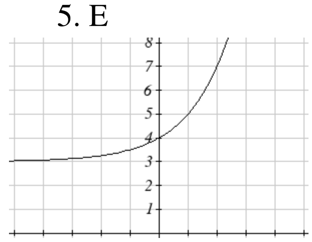
31. 3.03% 33. 7.4 years

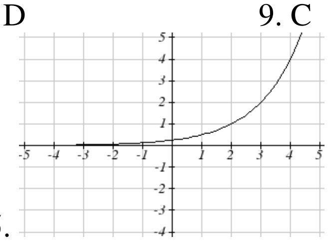
35a. $w(t)=(1.113)(1.046)^t$ b. $1.11 c. Below what the model predicts $\approx \$5.70$

Section 4.2

1. B 3. A 5. E 7. D 9. C

11. 

13. 

15. 

17. $y=4^x+4$ 19. $y=4^{x+2}$ 21. $y=-4^x$

23. As $x\to\infty$ $f(x)\to-\infty$. As $x\to-\infty$ $f(x)\to-1$

25. As $x\to\infty$ $f(x)\to-2$ As $x\to-\infty$ $f(x)\to\infty$

27. As $x\to\infty$ $f(x)\to 2$ As $x\to-\infty$ $f(x)\to\infty$

29. $y=-2^{x+2}+1=-4(2)^x+1$ 31. $y=-2(2)^{-x}+3$

33. $y=-2(3)^x+7$ 35. $y=2\left(\frac{1}{2}\right)^x-4$

Section 4.3

1. $4^m=q$ 3. $a^c=b$ 5. $10^t=v$

7. $e^n=w$ 9. $\log_4(y)=x$ 11. $\log_c(k)=d$

13. $\log(b)=a$ 15. $\ln(h)=k$ 17. 9

19. 1/8 21. 1000 23. e^2

25. 2 27. -3 29. ½

31. 4
33. -3
35. -2
37. -1.398
39. 2.708
41. $\frac{\log(14)}{\log(5)} \approx 1.6397$
43. $\frac{\log\left(\frac{1}{15}\right)}{\log(7)} \approx -1.392$
45. $\frac{\ln(17)}{5} \approx 0.567$
47. $\frac{\frac{\log(38)}{\log(3)}+5}{4} \approx 2.078$
49. $\frac{\log(5)}{\log(1.03)} \approx 54.449$
51. $\frac{\log\left(\frac{8}{3}\right)}{3\log(1.04)} \approx 8.335$
53. $\frac{\ln\left(\frac{1}{5}\right)}{-0.12} \approx 13.412$
55. $\frac{\log\left(\frac{5}{8}\right)}{\log\left(\frac{1}{2}\right)} \approx 0.678$
57. $f(t) = 300e^{-0.0943t}$
59. $f(t) = 10e^{0.03922t}$
61. $f(t) = 150(1.0618)^t$
63. $f(t) = 50(0.98807)^t$
65. During the year 2013
67. During the year 2074
69. $\approx 34\, hours$
71. 13.532 years

Section 4.4

1. $\log_3(4)$
3. $\log_3(7)$
5. $\log_3(5)$
7. $\log_7(2)$
9. $\log(6x^9)$
11. $\ln(2x^7)$
13. $\log\left(x^2(x+1)^3\right)$
15. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$
17. $15\log(x) + 13\log(y) - 19\log(z)$
19. $-2\ln(a) + 4\ln(b) - 5\ln(c)$
21. $\frac{3}{2}\log(x) - 2\log(y)$
23. $\ln(y) + \frac{1}{2}\left(\ln(y) - \ln(1-y)\right)$
25. $\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$
27. $x \approx -0.717$
29. $x \approx -6.395$
31. $t \approx 17.329$
33. $x = \frac{2}{7}$
35. $x \approx 0.123$
37. $x \approx 4.642$
39. $x \approx 30.158$
41. $x \approx -2.889$.
43. $x \approx 6.873$ or $x \approx -0.873$
45. $x = \frac{12}{11} \approx 1.091$
47. $x = 10$

Solutions Manual
for
Precalculus

An Investigation of Functions

David Lippman, Melonie Rasmussen

2nd Edition

Solutions created at The Evergreen State College and Shoreline Community College

1.1 Solutions to Exercises

1. (a) $f(40) = 13$, because the input 40 (in thousands of people) gives the output 13 (in tons of garbage)

   (b) $f(5) = 2$, means that 5000 people produce 2 tons of garbage per week.

3. (a) In 1995 (5 years after 1990) there were 30 ducks in the lake.

   (b) In 2000 (10 years after 1990) there were 40 ducks in the lake.

5. Graphs (a) (b) (d) and (e) represent y as a function of x because for every value of x there is only one value for y. Graphs (c) and (f) are not functions because they contain points that have more than one output for a given input, or values for x that have 2 or more values for y.

7. Tables (a) and (b) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input x=10, there are two different outputs for y.

9. Tables (a) (b) and (d) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input x=3, there are two different outputs for y.

11. Table (b) represents y as a function of x and is one-to-one because there is a unique output for every input, and a unique input for every output. Table (a) is not one-to-one because two different inputs give the same output, and table (c) is not a function because there are two different outputs for the same input x=8.

13. Graphs (b) (c) (e) and (f) are one-to-one functions because there is a unique input for every output. Graph (a) is not a function, and graph (d) is not one-to-one because it contains points which have the same output for two different inputs.

15. (a) $f(1) = 1$ (b) $f(3) = 1$

17. (a) $g(2) = 4$ (b) $g(-3) = 2$

19. (a) $f(3) = 53$ (b) $f(2) = 1$

21. $f(-2) = 4 - 2(-2) = 4 + 4 = 8, f(-1) = 6, f(0) = 4, f(1) = 4 - 2(1) = 4 - 2 = 2, f(2) = 0$

23. $f(-2) = 8(-2)^2 - 7(-2) + 3 = 8(4) + 14 + 3 = 32 + 14 + 3 = 49, f(-1) = 18, f(0) = 3, f(1) = 8(1)^2 - 7(1) + 3 = 8 - 7 + 3 = 4, f(2) = 21$

25. $f(-2) = -(-2)^3 + 2(-2) = -(-8) - 4 = 8 - 4 = 4, f(-1) = -(-1)^3 + 2(-1) = -(-1) - 2 = -1, f(0) = 0, f(1) = -(1)^3 + 2(1) = 1, f(2) = -4$

27. $f(-2) = 3 + \sqrt{(-2) + 3} = 3 + \sqrt{1} = 3 + 1 = 4, f(-1) = \sqrt{2} + 3 \approx 4.41, f(0) = \sqrt{3} + 3 \approx 4.73, f(1) = 3 + \sqrt{(1) + 3} = 3 + \sqrt{4} = 3 + 2 = 5, f(2) = \sqrt{5} + 3 \approx 5.23$

29. $f(-2) = \left((-2) - 2\right)\left((-2) + 3\right) = (-4)(1) = -4, f(-1) = -6, f(0) = -6, f(1) = \left((1) - 2\right)\left((1) + 3\right) = (-1)(4) = -4, f(2) = 0$

31. $f(-2) = \frac{(-2)-3}{(-2)+1} = \frac{-5}{-1} = 5, f(-1) = undefined, f(0) = -3, f(1) = -1, f(2) = -1/3$

33. $f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}, f(-1) = \frac{1}{2}, f(0) = 1, f(1) = 2, f(2) = 4$

35. Using $f(x) = x^2 + 8x - 4$: $f(-1) = (-1)^2 + 8(-1) - 4 = 1 - 8 - 4 = -11$; $f(1) = 1^2 + 8(1) - 4 = 1 + 8 - 4 = 5$.

(a) $f(-1) + f(1) = -11 + 5 = -6$

(b) $f(-1) - f(1) = -11 - 5 = -16$

37. Using $f(t) = 3t + 5$:

(a) $f(0) = 3(0) + 5 = 5$

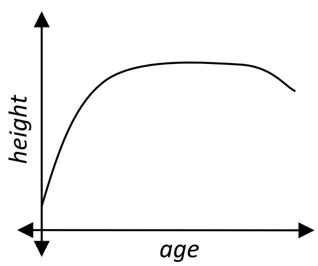
(b) $3t + 5 = 0$
$t = -\frac{5}{3}$

39. (a) $y = x$ (iii. Linear)
(b) $y = x^3$ (viii. Cubic)
(c) $y = \sqrt[3]{x}$ (i. Cube Root)
(d) $y = \frac{1}{x}$ (ii. Reciprocal)
(e) $y = x^2$ (vi. Quadratic)
(f) $y = \sqrt{x}$ (iv. Square Root)
(g) $y = |x|$ (v. Absolute Value)
(h) $y = \frac{1}{x^2}$ (vii. Reciprocal Squared)

41. (a) $y = x^2$ (iv.)
(b) $y = x$ (ii.)
(c) $y = \sqrt{x}$ (v.)
(d) $y = \frac{1}{x}$ (i.)
(e) $y = |x|$ (vi.)
(f) $y = x^3$ (iii.)

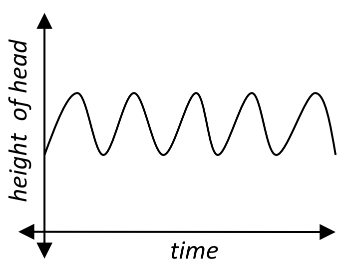
43. $(x-3)^2+(y+9)^2=(6)^2$ or $(x-3)^2+(y+9)^2=36$

45. (a)



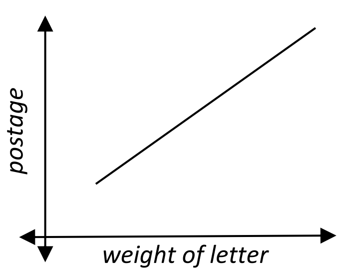
Graph (a)
At the beginning, as age increases, height increases. At some point, height stops increasing (as a person stops growing) and height stays the same as age increases. Then, when a person has aged, their height decreases slightly.

(b)



Graph (b)
As time elapses, the height of a person's head while jumping on a pogo stick as observed from a fixed point will go up and down in a periodic manner.

(c)



Graph (c)
The graph does not pass through the origin because you cannot mail a letter with zero postage or a letter with zero weight. The graph begins at the minimum postage and weight, and as the weight increases, the postage increases.

47. (a) t (b) $x=a$ (c) $f(b)=0$ so $z=0$. Then $f(z)=f(0)=r$.
(d) $L=(c,t), K=(a,p)$

1.2 Solutions to Exercises

1. The domain is $[-5,3)$; the range is $[0,2]$

3. The domain is $2 < x \leq 8$; the range is $6 \leq y < 8$

5. The domain is $0 \leq x \leq 4$; the range is $0 \leq y \leq -3$

7. Since the function is not defined when there is a negative number under the square root, x cannot be less than 2 (it can be equal to 2, because $\sqrt{0}$ is defined). So the domain is $x \geq 2$. Because the inputs are limited to all numbers greater than 2, the number under the square root will always be positive, so the outputs will be limited to positive numbers. So the range is $f(x) \geq 0$.

9. Since the function is not defined when there is a negative number under the square root, x cannot be greater than 3 (it can be equal to 3, because $\sqrt{0}$ is defined). So the domain is $x \leq 3$. Because the inputs are limited to all numbers less than 3, the number under the square root will always be positive, and there is no way for 3 minus a positive number to equal more than three, so the outputs can be any number less than 3. So the range is $f(x) \leq 3$.

11. Since the function is not defined when there is division by zero, x cannot equal 6. So the domain is all real numbers except 6, or $\{x|x \in \mathbb{R}, x \neq 6\}$. The outputs are not limited, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

13. Since the function is not defined when there is division by zero, x cannot equal $-1/2$. So the domain is all real numbers except $-1/2$, or $\{x|x \in \mathbb{R}, x \neq -1/2\}$. The outputs are not limited, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

15. Since the function is not defined when there is a negative number under the square root, x cannot be less than -4 (it can be equal to -4, because $\sqrt{0}$ is defined). Since the function is also not defined when there is division by zero, x also cannot equal 4. So the domain is all real numbers less than -4 excluding 4, or $\{x|x \geq -4, x \neq 4\}$. There are no limitations for the outputs, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

17. It is easier to see where this function is undefined after factoring the denominator. This gives $f(x) = \frac{x-3}{(x+11)(x-2)}$. It then becomes clear that the denominator is undefined when $x = -11$ and when $x = 2$ because they cause division by zero. Therefore, the domain is $\{x|x \in \mathbb{R}, x \neq -11, x \neq 2\}$. There are no restrictions on the outputs, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

19. $f(-1) = -4; f(0) = 6; f(2) = 20; f(4) = 24$

21. $f(-1) = -1; f(0) = -2; f(2) = 7; f(4) = 5$

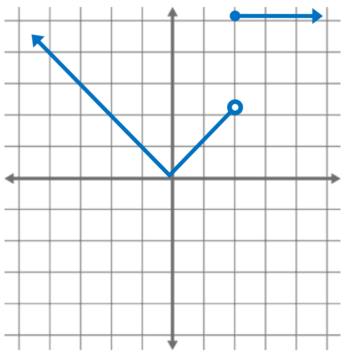
23. $f(-1) = -5; f(0) = 3; f(2) = 3; f(4) = 16$

25. $f(x) = \begin{cases} 2 & if \quad -6 \leq x \leq -1 \\ -2 & if \quad -1 < x \leq 2 \\ -4 & if \quad 2 < x \leq 4 \end{cases}$

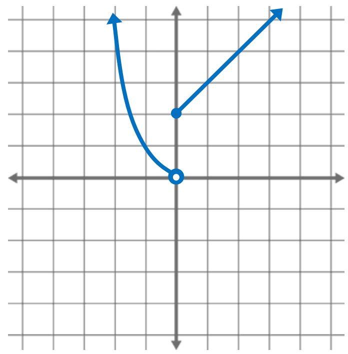
27. $f(x) = \begin{cases} 3 & if \quad x \leq 0 \\ x^2 & if \quad x > 0 \end{cases}$

29.. $f(x) = \begin{cases} \frac{1}{x} & if \quad x < 0 \\ \sqrt{x} & if \quad x \geq 0 \end{cases}$

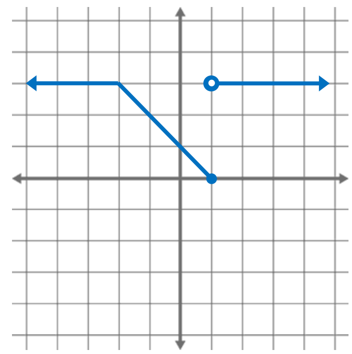
31.



33.



35.



1.3 Solutions to Exercises

1. (a) $\frac{249-243}{2002-2001} = \frac{6}{1} = 6$ million dollars per year

   (b) $\frac{249-243}{2004-2001} = \frac{6}{3} = 2$ million dollars per year

3. The inputs $x = 1$ and $x = 4$ produce the points on the graph: (4,4) and (1,5). The average rate of change between these two points is $\frac{5-4}{1-4} = \frac{1}{-3} = -\frac{1}{3}$.

5. The inputs $x = 1$ and $x = 5$ when put into the function $f(x)$ produce the points (1,1) and (5,25). The average rate of change between these two points is $\frac{25-1}{5-1} = \frac{24}{4} = 6$.

7. The inputs $x = -3$ and $x = 3$ when put into the function $g(x)$ produce the points (-3, -82) and (3,80). The average rate of change between these two points is $\frac{80-(-82)}{3-(-3)} = \frac{162}{6} = 27$.

9. The inputs $t = -1$ and $t = 3$ when put into the function $k(t)$ produce the points (-1,2) and (3,54.14$\overline{8}$). The average rate of change between these two points is $\frac{54.148-2}{3-(-1)} = \frac{52.148}{4} \approx 13$.

11. The inputs $x = 1$ and $x = b$ when put into the function $f(x)$ produce the points (1,-3) and $(b, 4b^2 - 7)$. $Explanation$: $f(1) = 4(1)^2 - 7 = -3, f(b) = 4(b)^2 - 7$. The average rate of

change between these two points is $\frac{(4b^2-7)-(-3)}{b-1} = \frac{4b^2-7+3}{b-1} = \frac{4b^2-4}{b-1} = \frac{4(b^2-1)}{b-1} = \frac{4(b+1)(b-1)}{(b-1)} = 4(b+1)$.

13. The inputs $x = 2$ and $x = 2 + h$ when put into the function $h(x)$ produce the points (2,10) and $(2 + h, 3h + 10)$. $Explanation$: $h(2) = 3(2) + 4 = 10, h(2 + h) = 3(2 + h) + 4 = 6 + 3h + 4 = 3h + 10$. The average rate of change between these two points is $\frac{(3h+10)-10}{(2+h)-2} = \frac{3h}{h} = 3$.

15. The inputs $t = 9$ and $t = 9 + h$ when put into the function $a(t)$ produce the points $(9, \frac{1}{13})$ and $\left(9 + h, \frac{1}{h+13}\right)$. $Explanation$: $a(9) = \frac{1}{9+4} = \frac{1}{13}, a(9 + h) = \frac{1}{(9+h)+4} = \frac{1}{h+13}$. The average rate of change between these two points is $\frac{\frac{1}{h+13}-\frac{1}{13}}{(9+h)-9} = \frac{\frac{1}{h+13}-\frac{1}{13}}{h} = \left(\frac{1}{h+13} - \frac{1}{13}\right)\left(\frac{1}{h}\right) = \frac{1}{h(h+13)} - \frac{1}{13h} = \frac{1}{h^2+13h} - \frac{1}{13h}\left(\frac{\frac{h}{13}+1}{\frac{h}{13}+1}\right)$ $(to\ make\ a\ common\ denominator)$ $= \frac{1}{h^2+13h} - \left(\frac{\frac{h}{13}+1}{h^2+13h}\right) = \frac{1-\frac{h}{13}-1}{h^2+13h} = \frac{\frac{h}{13}}{h^2+13h} = \frac{h}{13}\left(\frac{1}{h^2+13h}\right) = \frac{h}{13(h^2+13h)} = \frac{h}{13h(h+13)} = \frac{h}{13(h+13)}$.

17. The inputs $x = 1$ and $x = 1 + h$ when put into the function $j(x)$ produce the points (1,3) and $(1 + h, 3(1 + h)^3)$. The average rate of change between these two points is $\frac{3(1+h)^3-3}{(1+h)-1} = \frac{3(1+h)^3-3}{h} = \frac{3(h^3+3h^2+3h+1)-3}{h} = \frac{3h^3+9h^2+9h+3-3}{h} = \frac{3h^3+9h^2+9h}{h} = 3h^2 + 9h + 9 = 3(h^2 + 3h + 3)$.

19. The inputs $x = x$ and $x = x + h$ when put into the function $f(x)$ produce the points $(x, 2x^2 + 1)$ and $(x + h, 2(x + h)^2 + 1)$. The average rate of change between these two points is $\frac{(2(x+h)^2+1)-(2x^2+1)}{(x+h)-x} = \frac{(2(x+h)^2+1)-(2x^2+1)}{h} = \frac{2(x+h)^2+1-2x^2-1}{h} = \frac{2(x+h)^2-2x^2}{h} = \frac{2(x^2+2hx+h^2)-2x^2}{h} = \frac{2x^2+4hx+2h^2-2x^2}{h} = \frac{4hx+2h^2}{h} = 4x + 2h = 2(2x + h)$.

21. The function is increasing (has a positive slope) on the interval $(-1.5,2)$, and decreasing (has a negative slope) on the intervals $(-\infty, -1.5)$ and $(2, \infty)$.

23. The function is increasing (has a positive slope) on the intervals $(-\infty, 1)$ *and* $(3.25,4)$ and decreasing (has a negative slope) on the intervals $(1,2.75)$ *and* $(4, \infty)$.

25. The function is increasing because as x increases, $f(x)$ also increases, and it is concave up because the rate at which $f(x)$ is changing is also increasing.

27. The function is decreasing because as x increases, $h(x)$ decreases. It is concave down because the rate of change is becoming more negative and thus it is decreasing.

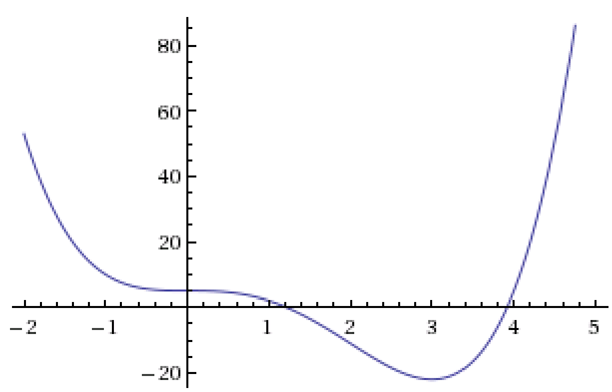
29. The function is decreasing because as x increases, $f(x)$ decreases. It is concave up because the rate at which $f(x)$ is changing is increasing (becoming less negative).

31. The function is increasing because as x increases, $h(x)$ also increases (becomes less negative). It is concave down because the rate at which $h(x)$ is changing is decreasing (adding larger and larger negative numbers).

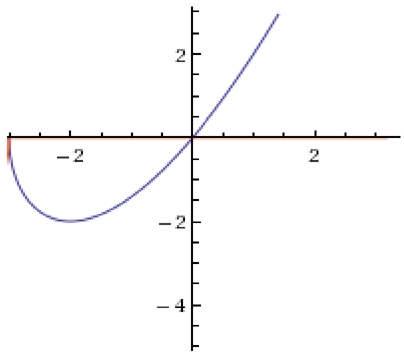
33. The function is concave up on the interval $(-\infty, 1)$, and concave down on the interval $(1, \infty)$. This means that $x = 1$ is a point of inflection (where the graph changes concavity).

35. The function is concave down on all intervals except where there is an asymptote at $x \approx 3$.

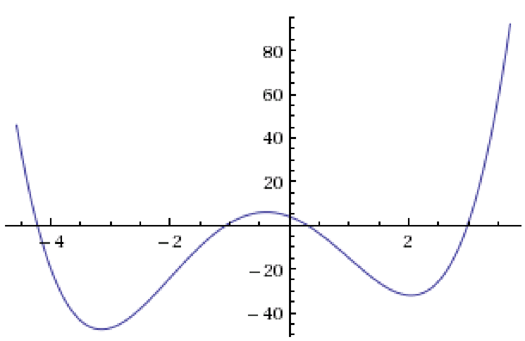
37. From the graph, we can see that the function is decreasing on the interval $(-\infty, 3)$, and increasing on the interval $(3, \infty)$. This means that the function has a local minimum at $x = 3$. We can estimate that the function is concave down on the interval $(0,2)$, and concave up on the intervals $(2, \infty)$and $(-\infty, 0)$. This means there are inflection points at $x = 2$ and $x = 0$.



39. From the graph, we can see that the function is decreasing on the interval $(-3,-2)$, and increasing on the interval $(-2,\infty)$. This means that the function has a local minimum at $x = -2$. The function is always concave up on its domain, (-3,∞). This means there are no points of inflection.



41. From the graph, we can see that the function is decreasing on the intervals $(-\infty,-3.15)$ *and* $(-0.38, 2.04)$, and increasing on the intervals $(-3.15,-0.38)$ *and* $(2.04,\infty)$. This means that the function has local minimums at $x = -3.15$ *and* $x = 2.04$ and a local maximum at $x = -0.38$. We can estimate that the function is concave down on the interval $(-2,1)$, and concave up on the intervals $(-\infty,-2)$ *and* $(1,\infty)$. This means there are inflection points at $x = -2$ *and* $x = 1$.



1.4 Solutions to Exercises

1. $f(g(0)) = 4(7) + 8 = 26,\ g(f(0)) = 7 - (8)^2 = -57$

3. $f(g(0)) = \sqrt{(12) + 4} = 4,\ g(f(0)) = 12 - (2)^3 = 4$

5. $f(g(8)) = 4$

7. $g(f(5)) = 9$

9. $f(f(4)) = 4$

11. $g(g(2)) = 7$

13. $f(g(3)) = 0$

15. $g(f(1)) = 4$

17. $f(f(5)) = 3$

19. $g(g(2)) = 2$

21. $f(g(x)) = \frac{1}{\left(\frac{7}{x}+6\right)-6} = \frac{x}{7},\ g\big(f(x)\big) = \frac{7}{\left(\frac{1}{x-6}\right)} + 6 = 7x - 36$

23. $f(g(x)) = (\sqrt{x+2})^2 + 1 = x + 3, g(f(x)) = \sqrt{(x^2+1)+2} = \sqrt{(x^2+3)}$

25. $f(g(x)) = |5x+1|,\ g(f(x)) = 5|x| + 1$

27. $f(g(h(x))) = \left(\left(\sqrt{x}\right) - 6\right)^4 + 6$

29. b

31. (a) $r\big(V(t)\big) = \sqrt[3]{\frac{3(10+20t)}{4\pi}}$

(b) To find the radius after 20 seconds, we evaluate the composition from part (a) at $t = 20$.

$r\big(V(t)\big) = \sqrt[3]{\frac{3(10+20\cdot 20)}{4\pi}} \approx 4.609\ inches.$

33. $m\big(p(x)\big) = \left(\frac{1}{\sqrt{x}}\right)^2 - 4 = \frac{1}{x} - 4$. This function is undefined when the denominator is zero, or when $x = 0$. The inside function $p(x)$ is defined for $x > 0$. The domain of the composition is the most restrictive combination of the two: $\{x | x > 0\}$.

35. The domain of the inside function, $g(x)$, is $x \neq 1$. The composition is $f\big(g(x)\big) = \frac{1}{\frac{2}{x-1}+3}$.

Simplifying that, $f\big(g(x)\big) = \frac{1}{\frac{2}{x-1}+3} = \frac{1}{\frac{2}{x-1}+\frac{3(x-1)}{x-1}} = \frac{1}{\frac{2}{x-1}+\frac{3x-3}{x-1}} = \frac{1}{\frac{3x-1}{x-1}} = \frac{x-1}{3x-1}$. This function is undefined when the denominator is zero, giving domain $x \neq \frac{1}{3}$. Combining the two restrictions gives the domain of the composition: $\left\{x \middle| x \neq 1, x \neq \frac{1}{3}\right\}$.

In interval notation, $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right) \cup (1, \infty)$.

37. The inside function $f(x)$ requires $x - 2 \geq 0$, giving domain $x \geq 2$. The composition is $g(f(x)) = \frac{2}{\left(\sqrt{x-2}\right)^2 - 3} = \frac{2}{x-2-3} = \frac{2}{x-5}$, which has the restriction $x \neq 5$. The domain of the composition is the combination of these, so values larger than or equal to 2, not including 5: $\{x|2 \leq x < 5 \text{ or } x > 5\}$, or $[2,5) \cup (5, \infty)$.

39. $f(x) = x^2,\ g(x) = x + 2$

41. $f(x) = \frac{3}{x},\ g(x) = x - 5$

43. $f(x) = 3 + x,\ g(x) = \sqrt{x - 2}$

45. (a) $f(x) = ax + b$, so $f(f(x)) = a(ax + b) + b$, which simplifies to $\mathrm{a^2x + 2b}$. a and b are constants, so a^2 and $2b$ are also constants, so the equation still has the form of a linear function.

(b) If we let $g(x)$ be a linear function, it has the form $g(x) = ax + b$. This means that $g(g(x)) = a(ax + b) + b$. This simplifies to $g(g(x)) = a^2x + ab + b$. We want $g(g(x))$ to equal $6x - 8$, so we can set the two equations equal to each other: $a^2x + ab + b = 6x - 8$. Looking at the right side of this equation, we see that the thing in front of the x has to equal 6. Looking at the left side of the equation, this means that $a^2 = 6$. Using the same logic, $ab + b = -8$. We can solve for , $a = \sqrt{6}$. We can substitute this value for a into the second equation to solve for b: $(\sqrt{6})b + b = -8 \rightarrow b(\sqrt{6} + 1) = -8 \rightarrow b = -\frac{8}{\sqrt{6}+1}$. So, since $g(x) = ax + b$, $g(x) = \sqrt{6}x - \frac{8}{\sqrt{6}+1}$. Evaluating $g(g(x))$ for this function gives us 6x-8, so that confirms the answer.

47. (a) A function that converts seconds s into minutes m is $m = f(s) = \frac{s}{60}$. $C(f(s)) = \frac{70(\frac{s}{60})^2}{10 + \left(\frac{s}{60}\right)^2}$; this function calculates the speed of the car in mph after *s* seconds.

(b) A function that converts hours h into minutes m is $m = g(h) = 60h$. $C(g(h)) =$

$\frac{70(60h)^2}{10+(60h)^2}$; this function calculates the speed of the car in mph after h hours.

(c) A function that converts mph s into ft/sec z is $z = v(s) = \left(\frac{5280}{3600}\right)s$ which can be reduced to $v(s) = \left(\frac{22}{15}\right)s$. $v(C(m)) = \left(\frac{22}{15}\right)\left(\frac{70m^2}{10+m^2}\right)$; this function converts the speed of the car in mph to ft/sec.

1.5 Solutions to Exercises

1. Horizontal shift 49 units to the right

3. Horizontal shift 3 units to the left

5. Vertical shift 5 units up

7. Vertical shift 2 units down

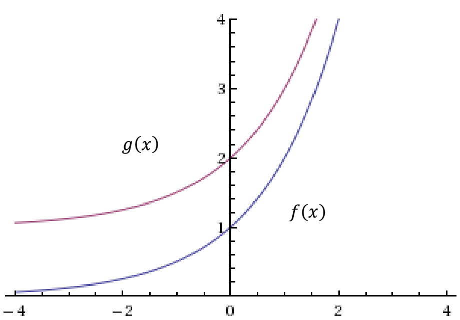
9. Horizontal shift 2 units to the right and vertical shift 3 units up

11. $f(x) = \sqrt{(x+2)} + 1$

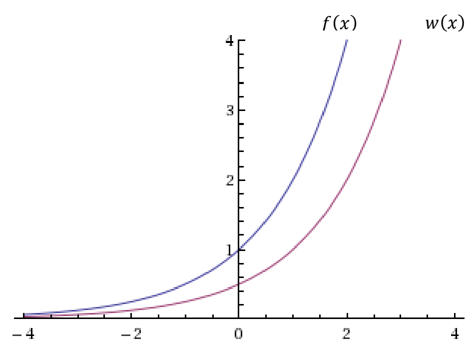
13. $f(x) = \frac{1}{(x-3)} - 4$

15. $g(x) = f(x-1),\ h(x) = f(x) + 1$

17.

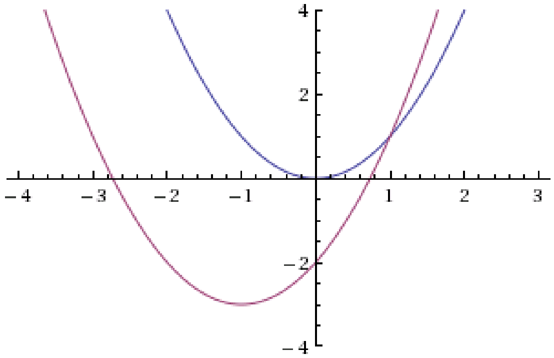


19.

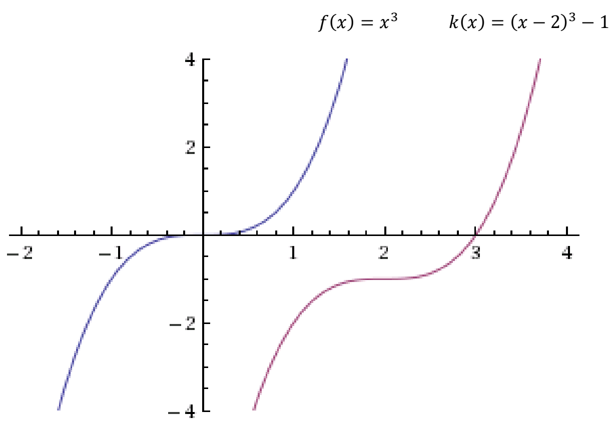


21. $f(t) = (t+1)^2 - 3$ as a transformation of $g(t) = t^2$

$f(t) = (t+1)^2 - 3 \quad g(t) = t^2$



23. $k(x) = (x-2)^3 - 1$ as a transformation of $f(x) = x^3$

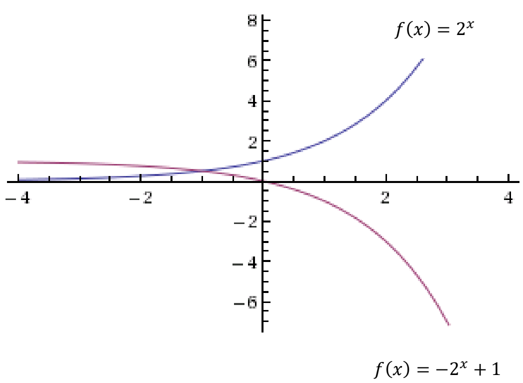


25. $f(x) = |x-3| - 2$

27. $f(x) = \sqrt{x+3} - 1$

29. $f(x) = -\sqrt{x}$

31.



33. (a) $f(x) = -6^{-x}$

(b) $f(x) = -6^{x+2} - 3$

35. $f(x) = -(x+1)^2 + 2$ 37. $f(x) = \sqrt{-x} + 1$

39. (a) even (b) neither (c) odd

41. the function will be reflected over the x-axis

43. the function will be vertically stretched by a factor of 4

45. the function will be horizontally compressed by a factor of $\frac{1}{5}$

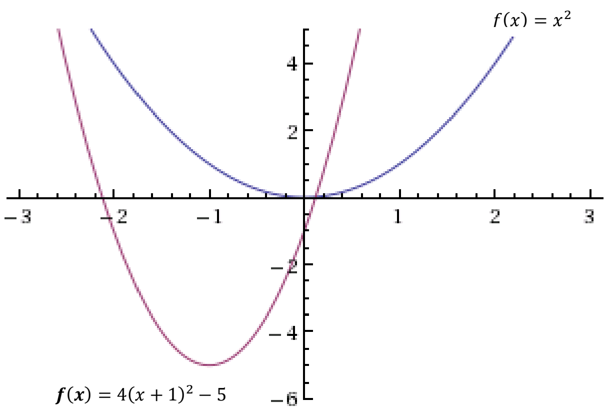
47. the function will be horizontally stretched by a factor of 3

49. the function will be reflected about the y-axis and vertically stretched by a factor of 3

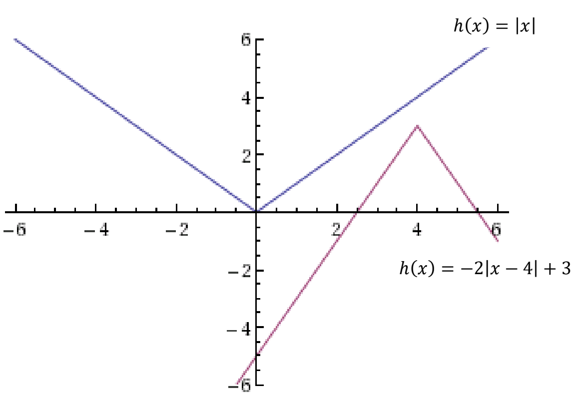
51. $f(x) = |-4x|$ 53. $f(x) = \frac{1}{3(x+2)^2} - 3$

55. $f(x) = (2[x-5])^2 + 1 = (2x-10)^2 + 1$

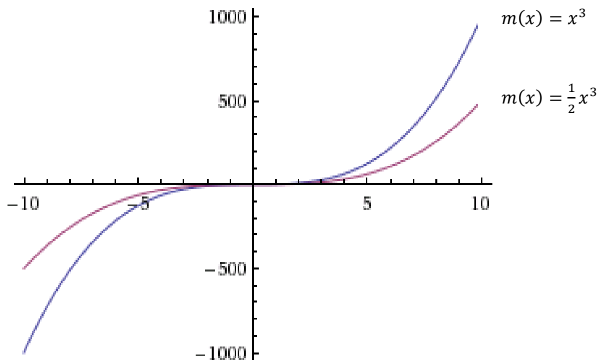
57. $f(x) = x^2$ will be shifted to the left 1 unit, vertically stretched by a factor of 4, and shifted down 5 units.



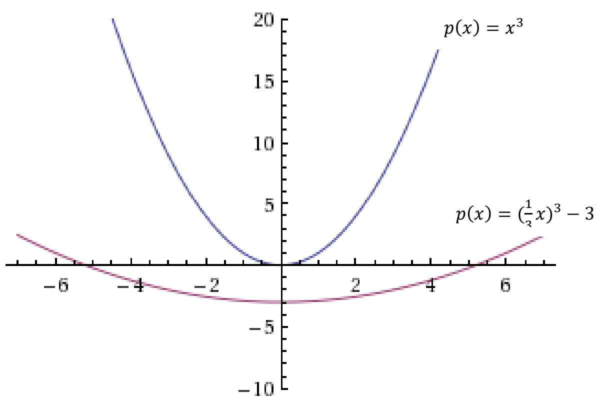
59. $h(x) = |x|$ will be shifted right 4 units vertically stretched by a factor of 2, reflected about the x-axis, and shifted up 3 units.



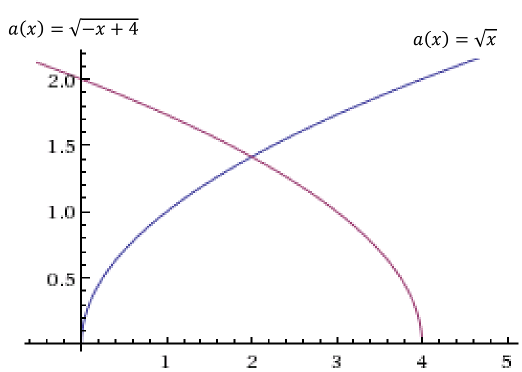
61. $m(x) = x^3$ will be vertically compressed by a factor of $\frac{1}{2}$.



63. $p(x) = x^2$ will be stretched horizontally by a factor of 3, and shifted down 3 units.



65. $a(x) = \sqrt{x}$ will be shifted left 4 units and then reflected about the y-axis.



67. the function is decreasing on the interval $x < -1$ and increasing on the interval $x > -1$

69. the function is decreasing on the interval $x \leq 4$

71. the function is concave up on the interval $x < -1$ and concave down on the interval $x > -1$

73. the function is always concave up.

75. $f(-x)$

77. $3f(x)$

79. $2f(-x)$

81. $2f\left(\frac{1}{2}x\right)$

83. $2f(x) - 2$

85. $-f(x) + 2$

87. $f(x) = -(x+2)^2 + 3$

89. $f(x) = \frac{1}{2}(x+1)^3 + 2$

91. $f(x) = \sqrt{2(x+2)} + 1$

93. $f(x) = -\frac{1}{(x-2)^2} + 3$

95. $f(x) = -|x+1| + 3$

97. $f(x) = -\sqrt[3]{x-2} + 1$

99. $f(x) = \begin{cases} (x+3)^2 + 1 & if\ x \leq -2 \\ -\frac{1}{2}|x-2| + 3 & if\ x > -2 \end{cases}$

101. $f(x) = \begin{cases} 1 & if\ x < -2 \\ -2(x+1)^2 + 4 & if -2 \leq x \leq 1 \\ \sqrt[3]{x-2} + 1 & if\ x > 1 \end{cases}$

103. (a) With the input in factored form, we first apply the horizontal compression by a factor of ½, followed by a shift to the right by three units. After applying the horizontal compression, the domain becomes $\frac{1}{2} \leq x \leq 3$. Then we apply the shift, to get a domain of $\left\{x \middle| 3\frac{1}{2} \leq x \leq 6\right\}$.

(b) Since these are horizontal transformations, the range is unchanged.

(c) These are vertical transformations, so the domain is unchanged.

(d) We first apply the vertical stretch by a factor of 2, followed by a downward shift of three units. After the vertical stretch, the range becomes $-6 \leq y \leq 10$. Next, we apply the shift to get the final domain $\{y|-9 \leq y \leq 7\}$.

(e) The simplest solution uses a positive value of *B*. The new domain is an interval of length one. Before, it was an interval of length 5, so there has been a horizontal compression by a factor of 1/5. Therefore, *B* = 5. If we apply this horizontal compression to the original domain, we get $\frac{1}{5} \leq x \leq \frac{6}{5}$. To transform this interval into one that starts at 8, we must add $7\frac{4}{5} = \frac{39}{5}$. This is our rightward shift, so $c = \frac{39}{5}$.

(f) The simplest solution uses a positive value of A. The new range is an interval of length one. The original range was an interval of length 8, so there has been a vertical compression by a factor of 1/8. Thus, we have $A = \frac{1}{8}$. If we apply this vertical compression to the original range we get $= \frac{3}{8} \leq y \leq \frac{5}{8}$. Now, in order to get an interval that begins at 0, we must add 3/8. This is a vertical shift upward, and we have $D = \frac{3}{8}$.

3.1 Solutions to Exercises

1. (a) $f(x)$ will approach $+\infty$ as x approaches ∞.
   (b) $f(x)$ will still approach $+\infty$ as x approaches $-\infty$, because any negative integer x will become positive if it is raised to an even exponent, in this case, x^4

3. (a) $f(x)$ will approach $+\infty$ as x approaches ∞.
   (b) $f(x)$ will approach $-\infty$ as x approaches $-\infty$, because x is raised to an odd power, in this case, x^3.

5. (a) $f(x)$ will approach $-\infty$ as x approaches ∞, because every number is multiplied by -1.
   (b) $f(x)$ will approach $-\infty$ as x approaches $-\infty$, since any negative number raised to an even power (in this case 2) is positive, but when it's multiplied by -1, it becomes negative.

7. (a) $f(x)$ will approach $-\infty$ as x approaches ∞, because any positive number raised to any power will remain positive, but when it's multiplied by -1, it becomes negative.
   (b) $f(x)$ will approach ∞ as x approaches $-\infty$, because any negative number raised to an odd power will remain negative, but when it's multiplied by -1, it becomes positive.

9. (a) The degree is 7.
   (b) The leading coefficient is 4.

11. (a) The degree is 2.
    (b) The leading coefficient is -1.

13. (a) The degree is 4.
    (b) The leading coefficient is -2.

15. (a) $(2x+3)(x-4)(3x+1) = (2x^2-5x-12)(3x+1) = 6x^3-13x^2-41x-12$
    (b) The leading coefficient is 6.
    (c) The degree is 3.

17. (a) The leading coefficient is negative, so as $x \to +\infty$ the function will approach $-\infty$.
    (b) The leading coefficient is negative, and the polynomial has even degree so as $x \to -\infty$ the function will approach $-\infty$.

19. (a) The leading coefficient is positive, so as $x \to +\infty$, the function will approach $+\infty$.
    (b) The leading coefficient is positive, and the polynomial has even degree so as $x \to -\infty$, the function will approach $+\infty$.

21. (a) Every polynomial of degree n has a maximum of n x-intercepts. In this case $n = 5$ so we get a maximum of five x-intercepts.
    (b) The number of turning points of a polynomial of degree n is $n-1$. In this case $n = 5$ so we get four turning points.

23. Knowing that an n^{th} degree polynomial can have a maximum of $n-1$ turning points we get that this function with two turning points could have a minimum possible degree of three.

25. Knowing that an n^{th} degree polynomial can have a maximum of $n-1$ turning points we get that this function with four turning points could have a minimum possible degree of five.

27. Knowing that an n^{th} degree polynomial can have a maximum of $n-1$ turning points we get that this function with two turning points could have a minimum possible degree of three.

29. Knowing that an n^{th} degree polynomial can have a maximum of $n-1$ turning points we get that this function with four turning points could have a minimum possible degree of five.

31. (a) To get our vertical intercept of our function we plug in zero for t we get $f(0) = 2((0)-1)((0)+2)((0)-3)$ = 12. Therefore our vertical intercept is $(0,12)$

(b) To get our horizontal intercepts when our function is a series of products we look for when we can any of the products equal to zero. For $f(t)$ we get $t = -2, 1, 3$. Therefore our horizontal intercepts are $(-2,0)$, $(1,0)$ and $(3,0)$.

33. (a) To get our vertical intercept of our function we plug in zero for n we get $g(0) = -2((3(0)-1)(2(0)+1) = 2$. Therefore our vertical intercept is $(0,2)$

(b) To get our horizontal intercepts when our function is a series of products we look for when we can any of the products equal to zero. For $g(n)$ we get $n = \frac{1}{3}, \frac{-1}{2}$. Therefore our horizontal intercepts are $(\frac{1}{3}, 0)$ and $(\frac{-1}{2}, 0)$.

3.3 Solutions to Exercises

1 - 5 To find the C intercept, evaluate $c(t)$. To find the t-intercept, solve $C(t) = 0$.

1. (a) C intercept at (0, 48)
   (b) t intercepts at (4,0), (-1,0), (6,0)

3. (a) C intercept at (0,0)
   (b) t intercepts at (2,0), (-1,0), (0,0)

5. $C(t) = 2t^4 - 8t^3 + 6t^2 = 2t^2(t^2 - 4t + 3) = 2t^2(t-1)(t-3)$.
   (a) C intercept at (0,0)
   (b) t intercepts at (0,0), (3,0) (-1,0)

7. Zeros: $x \approx -1.65,\ x \approx 3.64,\ x \approx 5.$

9. (a) as $t \to \infty, h(t) \to \infty$.
   (b) as $t \to -\infty, h(t) \to -\infty$

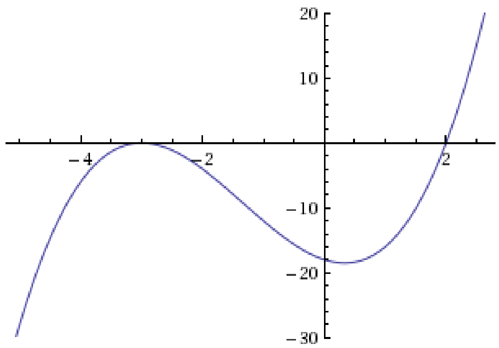
For part a of problem 9, we see that as soon as t becomes greater than 5, the function $\mathrm{h(t)} = 3(\mathrm{t}-5)^3(\mathrm{t}-3)^3(\mathrm{t}-2)$ will increase positively as it approaches infinity, because as soon as t is greater than 5, the numbers within each parentheses will always be positive. In b, notice as t approaches -∞, any negative number cubed will stay negative. If you multiply first three terms: $[3 * (\mathrm{t}-5)^3 * (\mathrm{t}-3)^3]$, as t approaches -∞, it will always create a positive number. When you then multiply that by the final number: $(t-2)$, you will be multiplying a negative: $(t-2)$, by a positive: $[3 * (\mathrm{t}-5)^3 * (\mathrm{t}-3)^3]$, which will be a negative number.

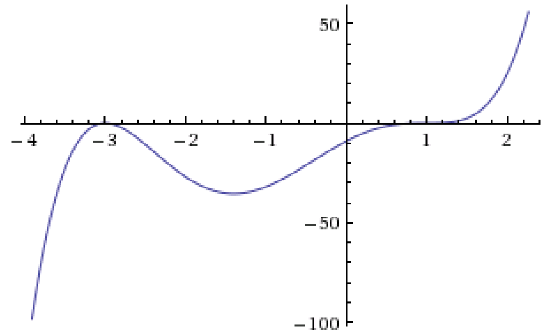
11. (a) as $t \to \infty, p(t) \to -\infty$
    (b) as $t \to -\infty, p(t) \to -\infty$

For part a of this problem as t approaches positive infinity, you will always have two parts of the equation $p(t) = -2t(t-1)(3-t)^2$, that are positive, once t is greater than 1: $[(t-1) * (3-t)^2]$, when multiplied together they stay positive. They are then multiplied by a number that will always be negative: -2t. A negative multiplied by a positive is always negative, so p(t) approaches -∞. For part b of this problem, as t approaches negative infinity, you will always have two parts of the equation that are always positive: $[-2t * (3-t)^2]$, when multiplied together stay positive. They are then multiplied by a number that will always be negative: (t-1). A negative multiplied by a positive is always negative, so p(t) approaches -∞.

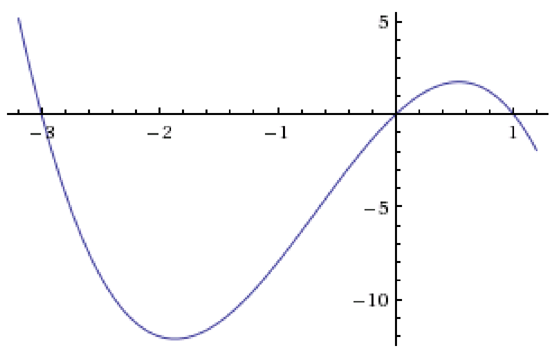
13. $f(x) = (x+3)^2(x-2)$

15. $h(x) = (x-1)^3(x+3)^2$





17. $m(x) = -2x(x-1)(x+3)$



19. $(x-3)(x-2)^2 > 0$ when $x > 3$

To solve the inequality $(x-3)(x-2)^2 > 0$, you first want to solve for x, when the function would be equal to zero. In this case, once you've solved for x, you know that when $f(x) = 0$, $x = 3,$ and $x = 2$. You want to test numbers greater than, less than, and in-between these points, to see if these intervals are positive or negative. If an interval is positive it is part of your solution, and if it's negative it's not part of your solution. You test the intervals by plugging any number greater than 3, less than 2, or in between 2 and 3 into your inequality. For this problem, $(x-3)(x-2)^2 > 0$ is only positive when x is greater than 3. So your solution is: $(x-3)(x-2)^2 > 0$, when $x > 3$.

21. $(x-1)(x+2)(x-3) < 0$ when -2 < x < 1, and when $x > 3$

To solve the inequality$(x-1)(x+2)(x-3)<0$, you first want to solve for x, when the function would be equal to zero. In this case, once you've solved for x, you know that when $f(x)=0,\ x=1,\ x=-2,$ and $x=3$. You want to test numbers greater than, less than, and in-between these points, to see if these intervals are positive or negative. If an interval is positive it is part of your solution, and if it's negative it's not part of your solution. You test the intervals by plugging any number greater than 3, less than -2, or in between -2 and 1, and in between 1 and 3 into your inequality. For this problem, $(x-1)(x+2)(x-3)<0$ is positive when x is greater than 3, and when it's in between -2 and 1. So your solution is: $(x-1)(x+2)(x-3)<0$ when $-2<x<1,$ and when $x>3$.

23. The domain is the values of *x* for which the expression under the radical is nonnegative:

$-42+19x-2x^2\geq 0$

$-(2x^2-19x+42)\geq 0$

$-(2x-7)(x-6)\geq 0$

Recall that this graph is a parabola which opens down, so the nonnegative portion is the interval between (and including) the *x*-intercepts: $\frac{7}{2}<x<6$.

25. The domain is the values of *x* for which the expression under the radical is nonnegative:

$$4-5x-x^2\geq 0$$

$$(x-4)(x-1)\geq 0$$

Recall that this graph is a parabola which opens up, so the nonnegative portions are the intervals outside of (and including) the *x*-intercepts: $x\leq 1$ and $x\geq 4$.

27. The domain is the values of *x* for which the expression under the radical is nonnegative, and since $(x+2)^2$ is always nonnegative, we need only consider where $x-3>0$, so the domain is $x\geq 3$.

29. The domain can be any numbers for which the denominator of $p(t)$ is nonzero, because you can't have a zero in the denominator of a fraction. So find what values of *t* make $t^2+2t-8=0$, and those values are not in the domain of $p(t)$. $t^2+2t-8=(t+4)(t-2)$, so the domain is ℝ where $x\neq -4\ and\ x\neq 2$.

31. $f(x)=-\frac{2}{3}(x+2)(x-1)(x-3)$

For problem 31, you can use the x intercepts you're given to get to the point $f(x)=a(x+2)(x-1)(x-3)$, because you know that if you solved for each of the x values you would end up with the horizontal intercepts given to you in the problem. Since your equation is of degree three, you don't need to raise any of your x values to a power, because if you foiled $(x+2)(x-1)(x-3)$ there will be an x^3, which is degree three. To solve for a, (your stretch factor, in this case $-\frac{2}{3}$) , you can plug the point your given, (in this case it's the y intercept $(0,-4)$) into your equation: $-4=(0+2)(0-1)(0-3)$, to solve for a.

33. $f(x)=\frac{1}{3}(x-3)^2(x-1)^2(x+3)$

For problem 33, you can use the x intercepts you're given to get to the point $f(x)=a(x-3)^2(x-1)^2(x+3)$, because you know that if you solved for each of the x values you would end up with the horizontal intercepts given to you in the problem. The problem tells you at what intercepts has what roots of multiplicity to give a degree of 5, which is why $(x-2)$ and $(x-1)$ are squared. To solve for a, (your stretch factor, in this case, $\frac{1}{3}$), you can plug the point your given, (in this case it's the y intercept (0,9)) into your equation: $9=(0-3)^2(0-1)^2(0+3)$, to solve for a.

35. $f(x)=-15(x-1)^2(x-3)^3$

For problem 35, you can use the x intercepts you're given to get to the point $f(x)=a(x-1)^2(x-3)^3$, because you know that if you solved for each of the x values you would end up with the horizontal intercepts given to you in the problem. The problem tells you at what intercepts has what roots of multiplicity to give a degree of 5, which is why $(x-1)$ is squared, and $(x-3)$ is cubed. To solve for a, (your stretch factor, in this case, -15), you can plug the point your given, (in this case it's (2,15)) into your equation: $15=(2-1)^2(2-3)^3$, to solve for a.

37. The *x*-intercepts of the graph are (-2, 0), (1, 0), and (3, 0). Then $f(x)$ must include the factors $(x+2)$, $(x-1)$, and $(x-3)$ to ensure that these points are on the graph of $f(x)$, and there cannot be any other factors since the graph has no other *x*-intercepts. The graph passes through these three *x*-intercepts without any flattening behavior, so they are single zeros. Filling in what we know so far about the function: $f(x)=a(x+2)(x-1)(x-3)$. To find the value of a, we can use the *y*-intercept, (0, 3):

$$3=a(0+2)(0-1)(0-3)$$

$$3=6a$$

$$a = \frac{1}{2}$$

Then we conclude that $f(x) = \frac{1}{2}(x+2)(x-1)(x-3)$.

39. $f(x) = -(x+1)^2(x-2)$

41. $f(x) = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

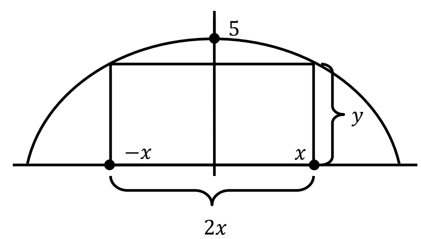
43. $f(x) = \frac{1}{24}(x+4)(x+2)(x-3)^2$

45. $f(x) = \frac{3}{32}(x+2)^2(x-3)^2$

47. $f(x) = \frac{1}{6}(x+3)(x+2)(x-1)^3$

49. $f(x) = -\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

51. See the diagram below. The area of the rectangle is $A = 2xy$, and $y = 5 - x^2$, so $A = 2x(5-x^2) = 10x - 2x^3$. Using technology, evaluate the maximum of $10x - 2x^3$. The y-value will be maximum area, and the x-value will be half of base length. Dividing the y-value by the x-value gives us the height of the rectangle. The maximum is at $x = 1.29, y = 8.61$. So, Base $= 2.58$, Height $= 6.67$.



problem 51

3.4 Solutions to Exercises

1. $4x^2 + 3x - 1 = (x-3)(4x+15) + 44$

3. $5x^4 - 3x^3 + 2x^2 - 1 = (x^2+4)(5x^2 - 3x - 18) + (12x + 71)$

5. $9x^3 + 5 = (2x-3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

1	3	-2	1
	↓	3	1
	3	1	2

7.

$(3x^2 - 2x + 1) = (x-1)(3x+1) + 2$

9.

-1	-2	-4	3
	↓	2	2
	-2	-2	5

$(3-4x-2x^2)=(x+1)(-2x-2)+5$

11.

-2	1	0	0	8
	↓	-2	4	-8
	1	-2	4	0

$(x^3+8)=(x+2)(x^2-2x+4)+0$

13.

5/3	18	-15	-25
	↓	30	25
	18	15	0

$(18x^2-15x-25)=\left(x-\frac{5}{3}\right)(18x+15)+0$

15.

-1/2	2	1	2	1
	↓	-1	0	-1
	2	0	2	0

$(2x^3+x^2+2x+1)=\left(x+\frac{1}{2}\right)(2x^2+2)+0$

17.

1/2	2	0	-3	1
	↓	1	1/2	-5/4
	2	1	-5/2	-1/4

$(2x^3-3x+1)=\left(x-\frac{1}{2}\right)\left(2x^2+x-\frac{5}{2}\right)-\frac{1}{4}$

19.

$\sqrt{3}$	1	0	-6	0	9
	↓	$\sqrt{3}$	3	$-3\sqrt{3}$	-9
	1	$\sqrt{3}$	-3	$-3\sqrt{3}$	0

$(x^4-6x^2+9)=(x-\sqrt{3})(x^3+\sqrt{3}\,x^2-3x-3\sqrt{3})+0$

21. Dividing by $x-1$ leaves $x^2-5x-6=(x-2)(x-3)$.
$x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$

23. Dividing by $x-\frac{2}{3}$ leaves $3x^2+6x+3=3(x^2+2x+1)=3(x+1)^2$
$3x^3+4x^2-x-2=3\left(x-\frac{2}{3}\right)(x+1)^2$

25. Dividing by $x + 2$ leaves $x^2 - 3 = \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$

$x^3 + 2x^2 - 3x - 6 = (x + 2)(x + \sqrt{3})(x - \sqrt{3})$

27. Dividing by $x - \frac{1}{2}$ twice leaves $4x^2 - 24x + 36 = 4(x^2 - 6x + 9) = 4(x - 3)^2$.

$4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2 (x - 3)^2$

4.1 Solutions to Exercises

1. Linear, because the average rate of change between any pair of points is constant.

3. Exponential, because the difference of consecutive inputs is constant and the ratio of consecutive outputs is constant.

5. Neither, because the average rate of change is not constant nor is the difference of consecutive inputs constant while the ratio of consecutive outputs is constant.

7. $f(x) = 11{,}000(1.085)^x$ You want to use your exponential formula f(x)= ab^x You know the initial value a is 11,000. Since b, your growth factor, is $b = 1 \pm r$, where r is the percent (written as a decimal) of growth/decay, $b = 1.085$. This gives you every component of your exponential function to plug in.

9. $f(x) = 23{,}900(1.09)^x$ $f(8) = 47{,}622$. You know the fox population is 23,900, in 2010, so that's your initial value. Since b, your growth factor is $b = 1 \pm r$, where r is the percent (written as a decimal) of growth/decay, $b = 1.09$. This gives you every component of your exponential function and produces the function $f(x) = 23{,}900(1.09)^x$. You're trying to evaluate the fox population in 2018, which is 8 years after 2010, the time of your initial value. So if you evaluate your function when $x = 8$, because $2018 - 2010 = 8$, you can estimate the population in 2018.

11. $f(x) = 32{,}500(.95)^x$ $f(12) = \$17{,}561.70$. You know the value of the car when purchased is 32,500, so that's your initial value. Since your growth factor is $b = 1 \pm r$, where r is the percent (written as a decimal) of growth/decay, $b = .95$ This gives you every component of your exponential function produces the function $f(x) = 32{,}500(.95)^x$. You're trying to evaluate the value of the car 12 years after it's purchased. So if you evaluate your function when $x = 12$, you can estimate the value of the car after 12 years.

13. We want a function in the form $f(x) = ab^x$. Note that $f(0) = ab^0 = a$; since (0, 6) is a given point, $f(0) = 6$, so we conclude $a = 6$. We can plug the other point (3, 750), into $f(x) = 6b^x$ to solve for b: $750 = 6(b)^3$. Solving gives $b = 5$, so $f(x) = 6(5)^x$.

15. We want a function in the form $f(x) = ab^x$. Note that $f(0) = ab^0 = a$; since (0, 2000) is a given point, $f(0) = 2000$, so we conclude $a = 2000$. We can plug the other point (2, 20) into $f(x) = 2000b^x$, giving $20 = 2000(b)^2$. Solving for *b*, we get $b = 0.1$, so $f(x) = 2000(.1)^x$.

17. $f(x) = 3(2)^x$ For this problem, you are not given an initial value, so using the coordinate points your given, $\left(-1, \frac{3}{2}\right), (3, 24)$ you can solve for b and then a. You know for the first coordinate point, $\left(\frac{3}{2}\right) = a(b)^{-1}$. You can now solve for a in terms of b: $\left(\frac{3}{2}\right) = \frac{a}{b} \rightarrow \left(\frac{3b}{2}\right) = a$. Once you know this, you can substitute $\left(\frac{3b}{2}\right) = a$, into your general equation, with your other coordinate point, to solve for b: $24 = \left(\frac{3b}{2}\right)(b)^3 \rightarrow 48 = 3b^4 \rightarrow 16 = b^4 \rightarrow b = 2$. So you have now solved for b. Once you have done that you can solve for a, by using what you calculated for b, and one of the coordinate points your given: $24 = a(2)^3 \rightarrow 24 = 8a \rightarrow a = 3$. So now that you've solved for a and b, you can come up with your general equation: $f(x) = 3(2)^x$.

19. $f(x) = 2.93(.699)^x$ For this problem, you are not given an initial value, so using the coordinate points you're given, $(-2,6), (3, 1)$ you can solve for b and then a. You know for the first coordinate point, $1 = a(b)^3$. You can now solve for a in terms of b: $\frac{1}{b^3} = a$. Once you know this, you can substitute $\frac{1}{b^3} = a$, into your general equation, with your other coordinate point, to solve for b: $6 = \frac{1}{b^3}(b)^{-2} \rightarrow 6b^5 = 1 \rightarrow b^5 = \frac{1}{6} \rightarrow b = .699$. So you have now solved for b. Once you have done that you can solve for a, by using what you calculated for b, and one of the coordinate points you're given: $6 = a(.699)^{-2} \rightarrow$ 6=2.047a $\rightarrow a = 2.93$. So now that you've solved for a and b, you can come up with your general equation: $f(x) = 2.93(.699)^x$

21. $f(x) = \frac{1}{8}(2)^x$ For this problem, you are not given an initial value, so using the coordinate points you're given, $(3,1), (5,4)$ you can solve for b and then a. You know for the first coordinate point, $1 = a(b)^3$. You can now solve for a in terms of b: $1/b^3 = a$. Once you know this, you can substitute $\frac{1}{b^3} = a$, into your general equation, with your other coordinate point, to solve for b: $4 = \frac{1}{b^3}(b)^5 \rightarrow 4 = b^2 \rightarrow b = 2$. So you have now solved for b. Once you have done that you can solve for a, by using what you calculated for b, and one of the coordinate points your given: $1 = a(2)^3 \rightarrow 1 = 8a \rightarrow a = 1/8$. So now that you've solved for a and b, you can come up with your general equation: $f(x) = \frac{1}{8}(2)^x$

23. 33.58 milligrams. To solve this problem, you want to use the exponential growth/decay formula , $f(x) = a(b)^x$, to solve for b, your growth factor. Your starting amount is a, so a=100 mg. You are given a coordinate, (35,50), which you can plug into the formula to solve for b, your effective growth rate giving you your exponential formula $f(x) = 100(0.98031)^x$ Then you can plug in your $x = 54$, to solve for your substance.

25. $1,555,368.09 Annual growth rate: 1.39% To solve this problem, you want to use the exponential growth/decay formula f(x)=ab^x First create an equation using the initial conditions, the price of the house in 1985, to solve for a. You can then use the coordinate point you're given to solve for b. Once you've found a, and b, you can use your equation f(x)=$110{,}000(1.0139)^x$ to predict the value for the given year.

27. $4,813.55 To solve this problem, you want to use the exponential growth/decay formula f(x)=ab^x First create an equation using the initial conditions, the value of the car in 2003, to solve for a. You can then use the coordinate point you're given to solve for b. Once you've found a, and b, you can use your equation f(x)=$38{,}000(.81333)^x$ to predict the value for the given year.

29. Annually: $7353.84 Quarterly: $47469.63 Monthly: $7496.71 Continuously: $7,501.44. Using the compound interest formula A(t)=$a(1 + \frac{r}{K})^{Kt}$ you can plug in your starting amount,

\$4000 to solve for each of the three conditions, annually—$k = 1$, quarterly—$k = 4$, and monthly—$k = 12$. You then need to plug your starting amount, \$4000 into the continuous growth equation f(x)=ae^{rx} to solve for continuous compounding.

31. APY= .03034 ≈ 3.03% You want to use the APY formula $f(x) = (1 + \frac{r}{K})^K$-1 you are given a rate of 3% to find your r and since you are compounding quarterly K=4

33. $t = 7.4$ years To find out when the population of bacteria will exceed 7569 you can plug that number into the given equation as P(t) and solve for t. To solve for t, first isolate the exponential expression by dividing both sides of the equation by 1600, then take the ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for t.

35. (a) $w(t) = 1.1130(1.0464)^t$ For this problem, you are not given an initial value, since 1960 corresponds to 0, 1968 would correspond to 8 and so on, giving you the points $(8,1.60)$ $(16,2.30)$ you can use these points to solve for b and then a. You know for the first coordinate point, $1.60 = ab^8$. You can now solve for a in terms of b: $\frac{1.60}{b^8} = a$. Once you know this, you can substitute $\frac{1.60}{b^8} = a$, into your general equation, with your other coordinate point, to solve for b: $2.30 = \frac{1.60}{b^8}(b)^{16} \rightarrow 1.60b^8 = 2.30 \rightarrow b^8 = \frac{2.30}{1.60} \rightarrow b = 1.0464$. So you have now solved for b. Once you have done that you can solve for a, by using what you calculated for b, and one of the coordinate points you're given: $2.30 = a(1.0464)^{16} \rightarrow$ $2.30 = 2.0664a \rightarrow a = 1.1130$. So now that you've solved for a and b, you can come up with your general equation: $w(t) = 1.1130(1.0464)^t$

(b) \$1.11 using the equation you found in part a you can find w(0)

(c) The actual minimum wage is less than the model predicted, using the equation you found in part a you can find w(36) which would correspond to the year 1996

37. (a) 512 dimes the first square would have 1 dime which is 2^0 the second would have 2 dimes which is 2^1 and so on, so the tenth square would have 2^9 or 512 dimes

(b) 2^{n-1} if n is the number of the square you are on the first square would have 1 dime which is 2^{1-1} the second would have 2 dimes which is 2^{2-1} the fifteenth square would have 16384 dimes which is 2^{15-1}

(c) 2^{63}, 2^{64-1}

(d) 9,223,372,036,854,775,808 mm

(e) There are 1 million millimeters in a kilometer, so the stack of dimes is about 9,223,372,036,855 km high, or about 9,223,372 million km. This is approximately 61,489 times greater than the distance of the earth to the sun.

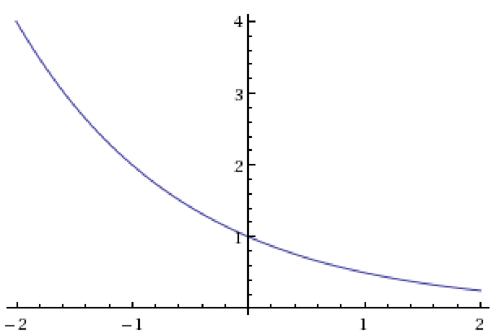
4.2 Solutions to Exercises

1. b 3. a 5. e

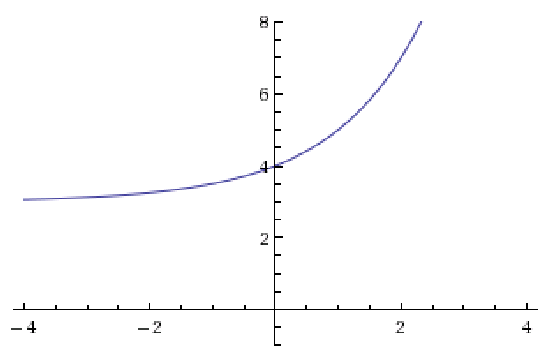
7. The value of b affects the steepness of the slope, and graph D has the highest positive slope it has the largest value for b.

9. The value of a is your initial value, when your $x = 0$. Graph C has the largest value for a.

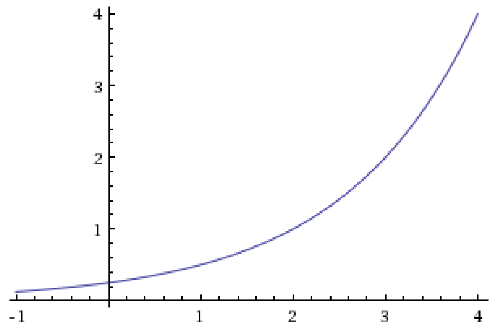
11. The function changes x to $-x$, which will reflect the graph across the y-axis.



13. The function will shift the function three units up.



15. The function will shift the function two units to the right.



17. $f(x) = 4^x + 4$ 19. $f(x) = 4^{(x+2)}$ 21. $f(x) = -4^x$

23. as $x \to \infty, f(x) \to -\infty$. When x is approaching $+\infty$, $f(x)$ becomes negative because 4^x is multiplied by a negative number.

as $x \to -\infty, f(x) = -1$. As x approaches$-\infty$, $f(x)$ approaches 1, because $-5(4^{-x})$ will approach 0, which means $f(x)$ approaches -1 as it's shifted down one.

25. as $x \to \infty$, $f(x) \to -2$ As x approaches $+\infty$, $f(x)$ approaches -2, because $3\left(\frac{1}{2}\right)^x$ will approach 0, which means $f(x)$ approaches -2 as it's shifted down 2.

as $x \to -\infty$, $f(x) \to +\infty$ because $\left(\frac{1}{2}\right)^{-x} = (2)^x$ so $f(x) \to \infty$.

27. as $x \to \infty$, $f(x) \to 2$ As x approaches $+\infty$, $f(x)$ approaches 2, because $3(4)^{-x}$ will approach 0, which means $f(x)$ approaches 2 as it's shifted up 2.

as $x \to -\infty$, $f(x) \to \infty$ because $(4)^{-x} = \left(\frac{1}{4}\right)^x$ so $f(x) \to \infty$.

29. $f(x) = -2^{x+2} + 1$ flipped about the x-axis, horizontal shift 2 units to the left, vertical shift 1 unit up

31. $f(x) = -2^{-x} + 2$ flipped about the x-axis, flipped about the y-axis, vertical shift 2 units up

33. $f(x) = -2(3)^x + 7$ The form of an exponential function is $y = ab^x + c$. This equation has a horizontal asymptote at $x = 7$ so we know $c = 7$, you can also now solve for a and b by choosing two other points on the graph, in this case (0,5) an (1,1), you can then plug (0,5) into your general equation and solve for a algebraically, and then use your second point to solve for b.

35. $f(x) = 2\left(\frac{1}{2}\right)^x - 4$ The form of an exponential function is $y = ab^x + c$. This equation has a horizontal asymptote at $x = -4$ so we know $c = -4$, you can also now solve for a and b by choosing two other points on the graph, in this case (0,-2) an (-1,0), you can then plug (0,-2) into your general equation and solve for a algebraically, and then use your second point to solve for b.

4.3 Solutions to Exercises

1. $4^m = q$ use the inverse property of logs $\log_b c$=a is equivalent to b^a=c

3. $a^c = b$ use the inverse property of logs $\log_b c$=a is equivalent to b^a=c

5. $10^t = v$ use the inverse property of logs $\log_b c$=a is equivalent to b^a=c

7. $e^n = w$ use the inverse property of logs $\log_b c$=a is equivalent to b^a=c

9. $\log_4 y = x$ use the inverse property of logs b^a=c is equivalent to $\log_b c$=a

11. $\log_c k = d$ use the inverse property of logs b^a=c is equivalent to $\log_b c$=a

13. $\log b = a$ use the inverse property of logs b^a=c is equivalent to $\log_b c$=a

15. $\ln h = k$ use the inverse property of logs b^a=c is equivalent to $\log_b c$=a

17. $x = 9$ solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $3^2 = x$ then solve for x

19. $x = \frac{1}{8}$ solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $2^{-3} = x$ then solve for x

21. $x = 1000$ solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $10^3 = x$ then solve for x

23. $x = e^2$ solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $e^2 = x$

25. 2 solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $5^x = 25$ then solve for x

27. -3 solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $3^x = \frac{1}{27}$ then solve for x

29. $\frac{1}{2}$ solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $6^x = \sqrt{6}$ then solve for x

31. 4 solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $10^x = 10{,}000$ then solve for x

33. -3 solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $10^x = 0.001$ then solve for x

35. -2 solve using the inverse properties of logs to rewrite the logarithmic expression as the exponential expression $e^x = e^{-2}$ then solve for x

37. $x = -1.398$ use calculator

39. $x = 2.708$ use calculator

41. $x \approx 1.639$ Take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

43. $x \approx -1.392$ Take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

45. $x \approx 0.567$ Take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

47. $x \approx 2.078$ Take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

49. $x \approx 54.449$ First isolate the exponential expression by dividing both sides of the equation by 1000 to get it into b^a=c form, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

51. $x \approx 8.314$ First isolate the exponential expression by dividing both sides of the equation by 3 to get it into b^a=c form, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

53. $x \approx 13.412$ First isolate the exponential expression by dividing both sides of the equation by 50 to get it into b^a=c form, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

55. $x \approx .678$ First isolate the exponential expression by subtracting 10 from both sides of the equation and then dividing both sides by -8 to get it into b^a=c form, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for x.

57. $f(t) = 300e^{-.094t}$ You want to change from the form $f(t) = a(1+r)^t$ *to* $f(t) = ae^{kt}$. From your initial conditions, you can solve for k by recognizing that, by using algebra, $(1 + r) = e^k$. In this case $e^k = 0.91$ Then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, and then use algebra to solve for k. You then have all the pieces to plug into your continuous growth equation.

59. $f(t) = 10e^{.0392t}$ You want to change from the form $f(t) = a(1+r)^t$ *to* $f(t) = ae^{kt}$. From your initial conditions, you can solve for k by recognizing that, by using algebra, $(1 + r) = e^k$. In this case $e^k = 1.04$ Then take the log or ln of both sides of the equation, utilizing

the exponent property for logs to pull the variable out of the exponent, and then use algebra to solve for x. You then have all the pieces to plug into your continuous growth equation.

61. $f(t) = 150(1.062)^t$ You want to change from the form $f(t) = ae^{kt} to\ f(t) = a(1+r)^t$. You can recognize that, by using algebra, $(1+r) = e^k$. You can then solve for b, because you are given k, and you know that $b = (1+r)$. Once you've calculated $b = 1.06184$, you have solved for all your variables, and can now put your equation into annual growth form.

63. $f(t) = 50(.988)^t$ You want to change from the form $f(t) = ae^{kt} to\ f(t) = a(1+r)^t$. You can recognize that, by using algebra, $(1+r) = e^k$. You can then solve for b, because you are given k, and you know that $b = (1+r)$. Once you've calculated $b = .988072$, you have solved for all your variables, and can now put your equation into annual growth form.

65. 4.78404 years You want to use your exponential growth formula $y = ab^t$ and solve for t, time. You are given your initial value a= 39.8 million and we know that $b = (1+r)$ you can solve for b using your rate, r=2.6% so b=1.026. You want to solve for t when f(t)=45 million so your formula is 45=39.8$(1.026)^t$. To solve for t, first isolate the exponential expression by dividing both sides of the equation by 39.8, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for t.

67. 74.2313 years You want to use your exponential growth formula $y = ab^t$ and first solve for b. You are given your initial value a=563,374 and you know that after 10 years the population grew to 608, 660 so you can write your equation 608,660=563,374$(b)^{10}$ and solve for b getting 1.00776. Now you want to find t when f(t)=1,000,000 so you can set up the equation 1,000,000=563,364$(1.00776)^t$. To solve for t, first isolate the exponential expression by dividing both sides of the equation by 563,364, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for t.

69. 34.0074 hrs You want to use your exponential decay formula $y = ab^t$ and first solve for b. You are given your initial value a=100mg and you know that after 4 hours the substance decayed

to 80mg so you can write your equation 80=100$(b)^4$ and solve for b getting .945742. Now you want to find t when f(t)=15 so you can set up the equation 15=100$(.945742)^t$. To solve for t, first isolate the exponential expression by dividing both sides of the equation by 100, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for t.

71. 13.5324 months You want to use your compound interest formula A(t)= $a(1+\frac{r}{k})^{kt}$ to solve for t when f(t)=1500. You are given your initial value a=1000, a rate of r=.03, and it compounds monthly so k=12. You can then write your equation as 1500=$1000(1+\frac{.03}{12})^{12t}$ and solve for t. To solve for t, first isolate the exponential expression by dividing both sides of the equation by 1000, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, then use algebra to solve for t.

4.4 Solutions to Exercises

1. $\log_3 4$ simplify using difference of logs property

3. $\log_3 7$ the -1 can be pulled inside the log by the exponential property to raise $\frac{1}{7}$ to the -1

5. $\log_3 5$ simplify using sum of logs property

7. $\log_7 2$ the $\frac{1}{3}$ can be pulled inside the log by the exponential property to raise 8 to the $\frac{1}{3}$

9. $\log(6x^9)$ simplify using sum of logs property

11. $\ln(2x^7)$ simplify using difference of logs property

13. $\log(x^2(x+1)^3)$ x can be raised to the 2nd power, and $(x+1)$ can be raised to the 3rd power via the exponential property, these two arguments can be multiplied in a single log via the sum of logs property

15. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$ y can be raised to the -$\frac{1}{2}$ power, and z to the 3rd power via the exponential property, then these three arguments can be multiplied in a single log via the sum of logs property

17. $15\log(x) + 13\log(y) - 19\log(z)$ expand the logarithm by adding $\log(x^{15})$ and $\log(y^{13})$ (sum property) and subtracting $\log(z^{19})$ (difference property) then pull the exponent of each logarithm in front of the logs (exponential property)

19. $4\ln(b) - 2\ln(a) - 5\ln(c)$ expand the logarithm by adding $\ln(b^{-4})$ and $\ln(c^5)$ (sum property) and subtracting that from $\ln(a^{-2})$ (difference property) then pull the exponent of each logarithm in front of the logs (exponential property)

21. $\frac{3}{2}\log(x) - 2\log(y)$ expand the logarithm by adding $\log\left(x^{\frac{3}{2}}\right)$ and $\log\left(y^{\frac{-4}{2}}\right)$ (sum property) then pull the exponent of each logarithm in front of the logs (exponential property)

23. $\ln(y) + (\frac{1}{2}\ln(y) - \frac{1}{2}\ln(1-y))$ expand the logarithm by subtracting $\ln\left(y^{\frac{1}{2}}\right)$ and $\ln\left((1-y)^{\frac{1}{2}}\right)$ (difference property) and adding $\ln(y)$(sum property) then pull the exponent of each logarithm in front of the logs (exponential property)

25. $2\log(x)$+$3\log(y)$+$\frac{2}{3}\log(x)$+$\frac{5}{3}\log(y)$ expand the logarithm by adding $\log(x^2)$, $\log(y^3)$, $\log\left(x^{\frac{2}{3}}\right)$ and $\log\left(y^{\frac{5}{3}}\right)$ then pull the exponent of each logarithm in from of the logs (exponential property)

27. $x \approx -.7167$Take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, remembering to keep parenthesis on (4x-7) and (9x-6), and then use algebra to solve for x.

29. $x \approx -6.395$ divide both sides by 17 and $(1.16)^x$ using properties of exponents, then take the log or ln of both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent and then use algebra to solve for x

31. $t \approx 17.329$ divide both sides by 10 and $e^{(.12t)}$ using properties of exponents, then ln both sides of the equation, utilizing the exponent property for logs to pull the variable out of the exponent, remembering that ln(e)=1, and then use algebra to solve for t

33. $x = \frac{2}{7}$ rewrite as an exponential expression using the inverse property of logs and a base of 2 and then use algebra to solve for x

35. $x = \frac{1}{3e} \approx 0.1226$ subtract 3 from both sides of the equation and then divide both sides by 2, then rewrite as an exponential expression using the inverse property of logs and a base of e and then use algebra to solve for x

37. $x = \sqrt[3]{100} \approx 4.642$ rewrite as an exponential expression using the inverse property of logs and a base of 10 and then use algebra to solve for x

39. $x \approx 30.158$ combine the expression into a single logarithmic expression using the sum of logs property, then rewrite as an exponential expression using the inverse property of logs and a base of 10 and then use algebra to solve for x

41. $x = -\frac{26}{9} \approx -2.8889$ combine the expression into a single logarithmic expression using the difference of logs property, then rewrite as an exponential expression using the inverse property of logs and a base of 10 and then use algebra to solve for x

43. $x \approx -.872983$ combine the expression into a single logarithmic expression using the difference of logs property, then rewrite as an exponential expression using the inverse property of logs and a base of 6 and then use algebra to solve for x

45. $x = \frac{12}{11}$ combine the expression into a single logarithmic expression using the difference of logs property and the sum of logs property, then rewrite as an exponential expression using the inverse property of logs and a base of 10 and then use algebra to solve for x

47. $x = 10$ combine the expression into a single logarithmic expression using the difference of logs property and the sum of logs property, then rewrite as an exponential expression using the inverse property of logs and a base of 10 and then use algebra to solve for x

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